Stochastic center manifold analysis in scalar nonlinear systems involving distributed delays and additive noise

Axel Hutt, Jérémie Lefebvre

Stochastic center manifold analysis in scalar nonlinear systems involving distributed delays and additive noise

Axel Hutt[1] and Jeremie Lefebvre[2]

[1] INRIA CR Nancy - Grand Est, Team Neurosys, 615 Rue due Jardin Botanique, Villers-les-Nancy, France
[2] Laboratory for Investigative Neurophysiology (The LINE), Department of Radiology and Department of Clinical Neurosciences, University Hospital Center and University of Lausanne, 1011 Lausanne, Switzerland

Abstract

This study reviews and extends a recent center manifold analysis technique developed to characterize stochastic bifurcations in delayed systems induced by additive noise. Motivated by the dynamics of spatially extended neural field models with finite propagation velocity, we revealed and fully characterized codimension 1 stochastic bifurcations induced by additive white noise. In contrast to previous studies, we here extended our analysis to the case of distributed delays while applying our results to the stochastic Hopf bifurcation. Taken together, our results provide further insight on the conjugate role of noise and delays in the genesis non-linear phenomena.

1 Introduction

Complex natural systems are rife with constraints and limitations that shape their dynamics. Interaction delays are inherent temporal latencies that play

a crucial role in understanding and modelling complex non-linear systems across fields, ranging from optics [36, 16], computer science [50] to neuroscience [1, 13, 17]. Delays not only constrain interactions amongst constituents of a given system, they control and even fully regulate regimes in which complex systems evolve, immensely extending the range of realizable dynamics. In parallel to the consideration of delays, noise in the form of stochastic, random fluctuations, has also been identified as an essential ingredient of complex systems [14, 39, 21, 43, 20]. It has further been shown in several instances that both delays and noise conspire together to sustain the intricate mechanisms at play in many complex systems, such as in genetics [9] or postural control [8].
Despite this, the conjugated action of both noise and delays in shaping non-linear dynamics is poorly understood. Due in great part to the limitation of existing analysis tools, there are few ways to access and expose the combined influence of stochasticity and timing constraints on the behavior of non-linear systems, especially near dynamic instabilities. As such, a general theory of stochastic delay-differential equations has yet to be devised. While some progress has been made with respect to parametric forms of noise [22], the additive stochastic case has yet to be fully characterized.

To overcome such limitations, previous studies have recently developed and applied a delayed center manifold approach, properly adapted to the presence of additive noise. This previous work is based on a center manifold technique developed for the analysis of spatially extended nonlinear systems subjected to additive noise [32, 31, 28]. Using this method, it was demonstrated that noise changes significantly the stability of delayed nonlinear systems and further, that noise intensity has to be considered as a bifurcation parameter on its own [37, 38, 30]. Surprisingly, while the technique was previously limited to the case of constant and unique delays, it may easily be extended to the more general case of distributed delays, thus shedding light on a broader range of problems. The present work reviews briefly the literature on center manifold theory in the presence of distributed delays and shows how this well-established technique can be extended to the stochastic case.

1.1 Model

As an illustrative application, we will here consider an integro-differential neural field equation that mimics the dynamics of large-scale neural ensem-

ble, and serves as powerful tool to model pattern formation in cortical networks. Here we consider a a spatially extended population of neurons which exhibits delayed interactions, which may result from delayed feedback connections [27, 34, 33] or from delayed interactions of single neurons [47]. Since the delay in neural populations may either be unknown or heterogeneous between single neurons, it is thus more realistic to assume a distribution of delays. This distribution also results from the presence of space-dependent delays of axonal connections [5, 6] which represents a distributed delay in Fourier space [29].
The activity of such a neural population obeys the stochastic neural field equation

$$\begin{aligned} dV(x,t) &= \left(-\gamma V(x,t) + \int_0^{\tau_{max}} g(\tau) \int_\Omega K(|x-y|) S\left[V\left(y, t-\tau\right)\right] \, dy \, d\tau + I_0\right) dt \\ &\quad + \kappa_0 d\xi(x,t), \end{aligned} \tag{1}$$

where $K(\cdot)$ is a symmetric spatial connectivity kernel, Ω is the one-dimensional circular spatial domain with length L, $g \in \mathcal{R}_0^+$ is the distribution function of constant delays with $\int_0^{\tau_{max}} g(\tau) = 1$ and $\tau_{max} > 0$ is the maximum delay present in the system. Moreover, $S[\cdot]$ is a non-linear activation function that takes the form of a sigmoid and $1/\gamma$ represents the time scale of synaptic responses of neurons in the population. The term I_0 is an external input constant in space and time and $d\xi(x,t)$ denotes the external additive random fluctuations with fluctuation strength κ_0 and $\xi(x,t)$ is a space-time Wiener process with $\langle d\xi(x,t)\rangle = 0$, $\langle d\xi(x,t) d\xi(y,T)\rangle = 2\delta(x-y)\delta(t-T)$, for the delta-distribution $\delta(\cdot)$. Hence the additive fluctuations are uncorrelated in space and time. The symbol $\langle\cdot\rangle$ denotes the ensemble average. In the absence of random fluctuations, Eq. (1) has a stationary state V_0 which implicitly satisfies $\gamma V_0 = \bar{K} S(V_0) + I_0$ where $\bar{K} = \int_\Omega K(x) dx$ and Ω is a circular one-dimensional spatial domain with length L. The stability of the state V_0 is exposed by close investigation of small deviations from equilibrium i.e., $v(x,t) = V(x,t) - V_0$, which satisfies, to third order,

$$\begin{aligned} & dv(x,t) = \left(-\gamma v(x,t) + \int_0^{\tau_{max}} g(\tau) \int_\Omega K(|x-y|) \right. \\ & \left. \times \left(s' v\left(y, t-\tau\right) + \frac{s''}{2} v^2\left(y, t-\tau\right) + \frac{s'''}{3!} v^3\left(y, t-\tau\right) \right) \, dy \, d\tau \right) dt \\ & + \kappa_0 d\xi(x,t), \end{aligned} \tag{2}$$

with $s' = (\partial S(V)/\partial V)|_{V_0}$, $s'' = (\partial^2 S(V)/\partial V^2)|_{V_0}$ and $s''' = (\partial^3 S(V)/\partial V^3)|_{V_0}$. Although the terms of second and third order in $v(x,t)$ do not play any role in stability of the stationary state, we will need them later in bifurcation analysis. The deviations $v(x,t)$ have the mode expansion in Fourier space [32]

$$v(x,t) = \frac{1}{\sqrt{L}} \sum_{l=-\infty}^{\infty} \tilde{v}_l(t) e^{ik_l x} \ , \ k_l = \frac{2\pi}{L} l \ , \ l \in \mathcal{Z}_0,$$

and individual modes obey

$$\begin{aligned} d\tilde{v}_l(t) \;\; = \;\; & \Bigg(-\gamma \tilde{v}_l(t) + \sqrt{L} s' \int_0^{\tau_{max}} \tilde{L}_l(\tau) \tilde{v}_l(t-\tau) \, d\tau \\ & + \frac{s''}{2\sqrt{L}} \int_0^{\tau_{max}} \tilde{L}_l(\tau) \sum_{n=-\infty}^{\infty} \tilde{v}_n(t-\tau) \tilde{v}_{l-n}(t-\tau) \, d\tau \\ & + \frac{s'''}{6L} \int_0^{\tau_{max}} \tilde{L}_l(\tau) \sum_{n,m=-\infty}^{\infty} \tilde{v}_m(t-\tau) \tilde{v}_n(t-\tau) \tilde{v}_{l-n-m}(t-\tau) \, d\tau \Bigg) \, dt \\ & + \kappa_0 d\tilde{\xi}_l(t), \end{aligned} \tag{3}$$

with the distribution

$$\tilde{L}_l(t) = \tilde{K}_l g(\tau) \ ,$$

$\tilde{K}_l = \tilde{K}(k_l)$ is the spatial Fourier transform of the spatial kernel $K(x)$ and $\tilde{\xi}_l(t)$ is the spatial Fourier transform of the space-time Wiener process $\xi(x,t)$ with $\langle d\tilde{\xi}_l(t) \rangle = 0$, $\langle d\tilde{\xi}^*_l(t) d\tilde{\xi}_n(T) \rangle = 2\delta(t-T)\delta_{l,n}$ where $\delta_{l,n}$ is the Kronecker symbol. This shows that the fluctuations are also uncorrelated in Fourier space.

In the absence of noise, linear dynamics of the system obey

$$\frac{d\tilde{v}_l(t)}{dt} \;\; = \;\; -\gamma \tilde{v}_l(t) + \sqrt{L} s' \int_0^{\tau_{max}} \tilde{L}_l(\tau) \tilde{v}_l(t-\tau) \, d\tau \tag{4}$$

and consequently the linear stability of the stationary state V_0 is read off the root of the characteristic equation

$$\lambda + \gamma = s' \sqrt{L} \mathcal{L}(\rho). \tag{5}$$

Here $\mathcal{L}(\rho) = \int_0^{\tau_{max}} \tilde{L}_l(\tau) e^{-\lambda\tau} \, d\tau$ is the Laplace transform and the complex number λ is an eigenvalue of the corresponding linear eigenvalue problem.

The stability of the deterministic linear system (4) depends on the statistical moments of the delay distribution $\tilde{L}_l(\tau)$ [7, 3].
The characteristic equation (5) defines the stability of the deterministic linear system. First of all, we point out that $\lambda = \lambda_l$ in Eq. (5) since $\mathcal{L}(\rho)$ is different for each mode l. Hence each mode has its own eigenvalue λ_l. If $Re(\lambda_l) < 0$ for all l, then the stationary state V_0 is exponentially stable, whereas it is sufficient that there is at least one eigenvalue $Re(\lambda_l) > 0$ to render the stationary state V_0 unstable. If the stationary state V_0 is exponentially stable in the absence of noise, a stationary probability distribution of the solution of the stochastic linear dynamics of Eq. (3) exists as shown for single delays [35] and for distributed delays [19, 53].
The influence of stochastic fluctuations on delay-dependent stability of Eq. (1) is exposed by closer investigation of bifurcating modes. Specifically, we consider the case where modes $l = \pm k$ are meta-stable or unstable and thus evolve on a large time scale, whereas all others ($l \neq \pm k$) are stable and evolving on a much shorter time scale. This implies that eigenvalues of modes $\tilde{v}_{\pm k}$ are located either on the imaginary axis or on its right hand side whereas the stable modes $\tilde{v}_{l \neq \pm k}$ are associated to eigenvalues bounded to the left of the imaginary axis. As such, we assume that the stable modes are rapidly damped compared to the unstable modes $\tilde{v}_{\pm k}$ and become negligible. Consequently, on a large time scale, the dynamics of the system is fully captured by the complex-valued stochastic distributed delay differential equation.

$$
\begin{aligned}
d\tilde{v}_k(t) &= \Bigg(-\gamma \tilde{v}_k(t) + \sqrt{L} s' \int_0^{\tau_{max}} \tilde{L}_k(\tau) \tilde{v}_k(t-\tau)\, d\tau \\
&\quad + \frac{s'''}{2L} \int_0^{\tau_{max}} \tilde{L}_k(\tau) |\tilde{v}_k(t-\tau)|^2 \tilde{v}_k(t-\tau)\, d\tau \Bigg) dt + \kappa_0 d\tilde{\xi}_k(t).
\end{aligned}
$$

After re-scaling of time by $t \rightarrow t\tau_{max}$ and assuming $\tilde{v}_k = u(t)\exp(i\phi(t))$ with amplitude $u \in \mathcal{R}$ and phase angle $\phi \in \mathcal{R}$, we obtain

$$
\begin{aligned}
du(t) &= \left(-\mu u + \alpha \rho \circ u + \beta \rho \circ u^3\right) dt + \kappa_0 \tau_{max} (\cos(\phi) d\eta_r - \sin(\phi) d\eta_i) \\
d\phi &= \frac{\kappa_0 \tau_{max}}{u} (-\sin(\phi) d\eta_r + \cos(\phi) d\eta_i)
\end{aligned}
$$

where $d\tilde{\xi}_k(t) = d\eta_r + i d\eta_i$, $\rho = \tilde{L}_k(\tau)$ and $\rho \circ u = \int_0^1 \rho(\tau) u(t-\tau)\, d\tau$. Coefficients of the above system are given by $\mu = \tau_{max}\gamma$, $\alpha = \tau_{max}^2 \sqrt{L} s'$ and $\beta = \tau_{max}^2 s'''/2L$. For simplicity, we first neglect the stochastic dynamics of the phase angle ϕ since its discussion would exceed by far the major aim of the present work. Choosing $\phi = 0$ without loss of generality yields the stochastic order parameter equation

$$du = \left(-\mu u + \alpha\rho \circ u + \beta\rho \circ u^3\right) dt + \kappa dW, \tag{6}$$

with $\kappa = \kappa_0 \tau_{max}$ and $dW(t) = d\eta_r$. Equation (6) approximates the dynamic evolution of the neural field and becomes our prime concern for the rest of the analysis. As shown in the above derivation, the stochastic fluctuations $d\xi(x,t)$ in Eq. (1), and $dW(t)$ in Eq. (6), exert an additive effect on the non-linear and delayed dynamics of the unstable modes. Previous studies have shown that additive noise shapes the stability of those modes in the presence of single delays [38, 30]. We shall extend this result to distributed delays. Figure 1 shows the stationary probability density function of the solution

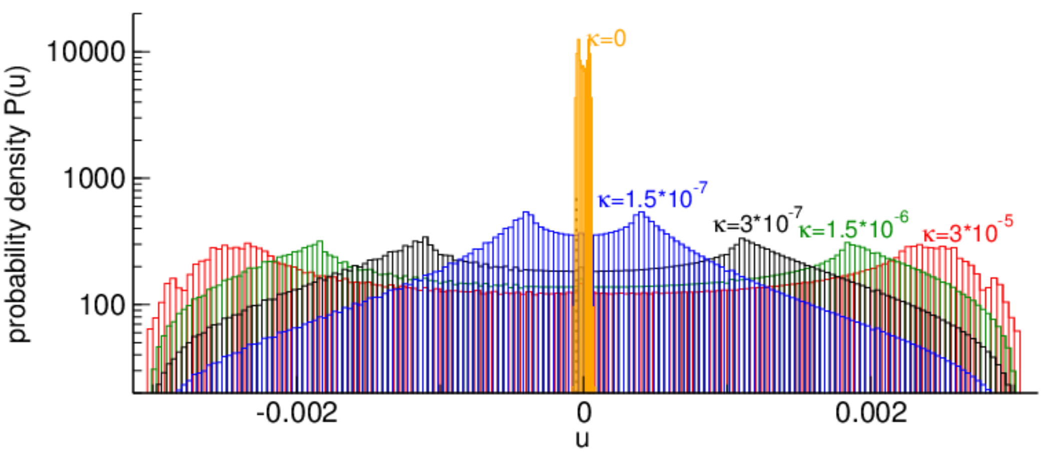

Figure 1: Noise shapes delay-induced oscillations. The stationary probability density P of u which obeys the dynamics of Eq. (6) for a discrete distribution of two delays. The colors encode different noise levels κ. The delay distribution is $\rho(\tau) = 0.5 \sum_{n=1}^{2} \delta(\tau - \tau_n),\ \tau_1 = 0.97,\ \tau_2 = 1.0$, other parameters $\mu = 30.38,\ \alpha = -31.92,\ \beta = -15.19$. For numerical simulation, we have applied the Euler-Maruyama integration scheme [10] with the discrete time step $\Delta t = 0.01$ and 10^5 integration steps.

of Eq. (6), which evolves above a supercritical Hopf bifurcation. For a discrete distribution of two delays, additive noise moves the non-zero maxima of the stationary probability density function to larger values of u indicating

a stochastic bifurcation [4] in the scalar stochastic systems involving two discrete delays similar to the case of a single delay [37]. Moreover, we observe that the effect of additive noise is strong since the noise level κ is smaller by several orders than the oscillation amplitude u. Hence a rather low noise level has a large effect on the corresponding stationary probability density.
To understand this, we investigate center manifold projections of Eq. (6), appropriately adapted to the stochastic and delayed case close to a codimension-1 Hopf bifurcation. This strategy has previously been used in several instances to study stochastic bifurcations in delayed systems [30], and is further shown here, for the first time, to apply to distributed delay problems as well. Moreover, the analysis focusses on the dynamics close to a Hopf bifurcation. The subsequent analysis assume a general distribution of delays. However, where necessary, we will assume that this otherwise arbitrary distribution corresponds to a sum of constant delays.
The manuscript is structured as follows. Section 2 reviews briefly the deterministic center manifold analysis technique and shows the most important computational steps. Then section 3 introduces into the major ideas of a recently developed analysis method of stochastic delayed center manifolds for single delays and extends previous results to distributed delays.

2 Deterministic analysis

Let us first consider the dynamics close to the steady state $u = 0$ of Eq. (6) in the absence of noise, i.e., $\kappa = 0$. We note that the fixed point $u = 0$ corresponds to states of vanishing linear deviations i.e. $v(x, t) = 0$ and $V(x, t) = V_0$. The stability of this equilibrium is fully determined by the roots of the characteristic equation (5). An instability occurs whenever roots of this equation cross the imaginary axis, i.e., whenever $Re(\lambda) = 0|\lambda \in \mathbb{C}$. We assume that this occurs once the parameter α reaches the critical value α_c, for a given delay distribution ρ and fixed γ and β.
Introducing the real unfolding parameter $\varepsilon = \alpha - \alpha_c$ which parameterizes the distance from the bifurcation, Eq. (6) can be unfolded around the instability and written as

$$\begin{aligned} \frac{du}{dt} &= L[u] + F[u, \epsilon] \qquad (7) \\ \frac{d\varepsilon}{dt} &= 0. \end{aligned}$$

where the linear operator $L[\cdot]$ reads

$$L[u] = -\mu u + \alpha_c \rho \circ u, \tag{8}$$

and the non-linear terms are collected in

$$F[u, \epsilon] = \epsilon\rho \circ u + \beta\rho \circ |u|^2 u. \tag{9}$$

For reasons discussed elsewhere [23, 51, 18], the phase space $\mathcal{W}$ in which the delayed dynamics of Eq. (6) are well formulated can be shown to be the Banach space of continuous maps, i.e., $\mathcal{W} \equiv C([-1;0], \mathbb{C}\times\mathbb{C})$. Accordingly, taking into account the history dependence of the state variable $u(t)$ on an interval on negative real line, one introduces the parameter $\theta \in [-1;0]$ so that $u(t+\theta) = u_t(\theta) \in \mathcal{W}$. With such a formulation, the system (7) reads

$$\frac{du_t(\theta)}{dt} = \mathcal{A}[u_t] + X_0 F[u_t] \tag{10}$$

with the linear operator $\mathcal{A} : u_t \to \mathcal{W}$,

$$\begin{aligned} \mathcal{A}[u_t] &= \frac{\partial u_t(\theta)}{\partial\theta} \quad , \quad -1 \leq \theta < 0 \\ &= L[u_t] \quad , \quad \theta = 0 \end{aligned}$$

and $X_0 = 0,\ -1 \leq \theta < 0$ and $X_0 = 1,\ \theta = 0$. The linear operator $L[u]$ is [23, 24]

$$L[u_t] = \int_{-1}^{0} u_t(\theta) d\eta[\theta], \tag{11}$$

with

$$d\eta[\theta] = w(\theta)d\theta \tag{12}$$

and where $w(\theta)$ is the delay density function [51]. For the model described above $w(\theta) = -\mu\delta(\theta) + \alpha_c\rho(\theta+1)$. The Riesz-Markov-Kakutani representation theorem given in Eq. (11) together with the definition (12) represents a powerful formulation of delayed systems involving distributed delays since it provides an analysis framework for both single delays and distributed delays. Accordingly, the non-linear component of Eq. (10) can be written as

$$\begin{aligned} F[u_t] &= \int_{-1}^{0}\int_{-1}^{0} u_t(\theta_1)u_t(\theta_2)w_2(\theta_1,\theta_2)d\theta_1 d\theta_2 \\ &\quad + \int_{-1}^{0}\int_{-1}^{0}\int_{-1}^{0} u_t(\theta_1)u_t(\theta_2)u_t(\theta_3)w_3(\theta_1,\theta_2,\theta_3)d\theta_1 d\theta_2 d\theta_3 + \cdots \end{aligned}$$

with the delay density functions $w_2,\ w_3$ (see [51]).
The linear dynamics in the vicinity of $u = 0$ is governed by

$$\frac{du_t(\theta)}{dt} = \mathcal{A}[u_t] \tag{13}$$

with characteristic equation (5). By construction, characteristic roots which have zero real part define an eigenvector basis $\mathbf{\Phi} \in \mathcal{C}$ associated with the linear operator $\mathcal{A}$ spanning the center subspace $\mathcal{U}$. In the present work, we consider delay-induced Hopf bifurcations for which $\mathbf{\Phi}(\theta)$ is a two-dimensional vector function. The adjoint system to (13) may be formulated [23] with identical characteristic roots and the adjoint basis function $\mathbf{\Psi}$. This basis satisfies $(\mathbf{\Phi}, \mathbf{\Psi}) = \mathbb{I}_2$ where $(a(\theta), b(\theta))$ is a bilinear product of functions, appropriately defined for delayed systems [23] and $\mathbb{I}_2$ is the two-dimensional unit matrix. We note that $(\Phi_i, \Psi_j) = \delta_{ij}$ where δ_{ij} is such that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ otherwise.
For the Hopf bifurcation with critical angle frequency ω_c, the linear unstable subspace has the eigenbasis

$$\begin{aligned}
\mathbf{\Phi} &= \left(e^{i\omega_c\theta}, e^{-i\omega_c\theta}\right)^t, \\
\mathbf{\Psi} &= \left(\frac{e^{-i\omega_c\theta}}{1 - L[\theta e^{i\omega_c\theta}]}, \frac{e^{i\omega_c\theta}}{1 - L[\theta e^{-i\omega_c\theta}]}\right)^t
\end{aligned}$$

where $L[\cdot]$ is defined in Eq. (11). According to the distinction of characteristic roots with zero real part and negative real parts, the phase space can be decomposed into the center and stable subspaces U and S, respectively, such that $W = U \oplus S$. Consequently, the state variable u may also be decomposed into its center and stable projections

$$\begin{aligned}
u_t(\theta) &= \mathbf{\Phi}(\theta)\mathbf{z}_t + s_t(\theta) && (14) \\
&= \mathcal{P}u_t(\theta) + (1 - \mathcal{P})u_t(\theta) && (15)
\end{aligned}$$

where $\mathbf{z} \in \mathcal{R}^2$ are the amplitudes of the center modes of the dynamics, and $s \in S$ are the stable components evolving in the linear stable subspace. The operator $\mathcal{P}$ in Eq. (15) projects u_t into the center subspace and is defined as $\mathcal{P}\cdot = \mathbf{\Phi}(\theta)(\mathbf{\Psi}, \cdot)$ [38].
According to the center manifold theorem [12, 25, 48], the dynamics of Eq. (6) is governed by the center modes $\mathbf{z}$ associated to the instability. Specifically, the stable component of the state variable u obeys

$$s_t(\theta) = \tilde{H}(\mathbf{z}, \epsilon, \theta), \tag{16}$$

for some non-linear function $\tilde{H}(\mathbf{z}, \epsilon, \theta)$. Equation (16) together with Eq. (14) implies that the initial value reads

$$u_0(\theta) = \mathbf{\Phi}(\theta)\mathbf{z}(0) + \tilde{H}(\mathbf{z}(0), \epsilon, \theta) \ . \tag{17}$$

Given a certain initial function $u_0(\theta) = k(\theta),\ -1 \leq \theta \leq 0$, Eq. (17) and the implicit condition $\mathcal{P}\tilde{H} = 0$ define the initial value $\mathbf{z}(0)$ and $\tilde{H}$ implies the initial function $k(\theta)$.
According to this, Eq. (7) reduces to the order parameter equation [23, 26]

$$\frac{d}{dt}\mathbf{z} = \mathbf{B}\mathbf{z} + \mathbf{\Psi}(0)F[\mathbf{\Phi}(\theta)\mathbf{z} + \tilde{H}(\mathbf{z}, \epsilon, \theta))], \tag{18}$$

where $\mathbf{B}$ is a diagonal matrix of characteristic roots of Eq. (5). The functional $\tilde{H}(\mathbf{z}, \epsilon, \theta))$ is not unique, yet may be approximated to low order using polynomial expansions [11, 46].
For our specific model example (6), the resulting order parameter equation (18) reads

$$\begin{aligned} \frac{d}{dt}\mathbf{z}(t) \ &= \ \mathbf{B}\mathbf{z}(t) + \epsilon\mathbf{\Psi}(0)\int_{-1}^{0}\rho(\theta)\mathbf{\Phi}(\theta)d\theta\ \mathbf{z}(t) \ + \ \epsilon\mathbf{\Psi}(0)\int_{-1}^{0}\rho(\theta)\tilde{H}(\mathbf{z}(t), \epsilon, \theta)d\theta \\ &\quad +\mathbf{\Psi}(0)\beta\int_{-1}^{0}\rho(\theta)\left(\mathbf{\Phi}(\theta)\mathbf{z}(t) + \tilde{H}(\mathbf{z}(t), \epsilon, \theta)\right)^3 d\theta \ . \end{aligned} \tag{19}$$

for a given deterministic stable manifold $\tilde{H}(\mathbf{z}, \epsilon, \theta)$. This order parameter equation does not include delay terms in $\mathbf{z}(t)$ anymore and captures the stability and dynamic properties of Eq. (6) and Eq. (1) close to a bifurcation point. Equation (18) includes convolutions over the delay distribution ρ, but which now operates only on the basis components $\Phi(\theta)$ and through higher order terms $\tilde{H}$. This aspect is important: via proper subspace projection and the application of the center manifold theorem, the delayed dependence of the center modes dynamics in Eq. (18) has been conveyed to the coefficients, greatly simplifying the analysis of the distributed delay case.
The following section extends the deterministic center manifold analysis by adding stochasticity and gives the computation steps of the stochastic center manifold analysis in some detail. The major approach has been developed in previous studies for a single delay, whereas the subsequent section shows new results in the presence of distributed delays.

3 Stochastic Analysis

Taking up the notation of the previous section 2, Eq. (6) can be written as

$$du = (L[u] + F[u, \epsilon])dt + \kappa dW(t). \tag{20}$$

The analysis of this stochastic delayed system follows the major steps of the deterministic center manifold technique [38, 37]. Previous work on stochastic differential equations involving a single delay has considered an additive time-dependent correction of the center manifold if the system evolves sufficiently close to instability. This time-dependent correction provides an accurate characterization of noise-induced stability transitions on the driven system for both a pitchfork [30] and Hopf instability [37].
Following the same approach, Eq. (20) may be re-cast to

$$\begin{aligned} d\mathbf{z} &= \left(\mathbf{Bz} + \mathbf{\Psi}_{\mathcal{U}}(0)F\left[\mathbf{\Phi}_{\mathcal{U}}\mathbf{z}(t) + \tilde{\mathbf{H}}(u,\varepsilon,t)\right]\right)dt + \mathbf{\Psi}_{\mathcal{U}}(0)\kappa dW(t) && (21)\\ ds_t &= \left(\mathcal{A}\left[\tilde{H}(\mathbf{z},\varepsilon,t)\right] + (X_o - \mathbf{\Phi}_{\mathcal{U}}(\theta)\mathbf{\Psi}_{\mathcal{U}}(0))F[\mathbf{\Phi}_{\mathcal{U}}\mathbf{z}(t) + \tilde{H}(\mathbf{z},\varepsilon,t)]\right)dt \\ &\quad +\kappa\left(X_o - \mathbf{\Phi}_{\mathcal{U}}(\theta)\mathbf{\Psi}_{\mathcal{U}}(0)\right)dW(t). && (22) \end{aligned}$$

where the stable manifold $\tilde{\mathbf{H}}(u, \varepsilon, t)$ depends on time now.

3.1 Stochastic center manifold theory

The system may evolve close to its stability threshold, i.e., $\varepsilon \ll 1$, assuming small noise levels $\kappa \sim \mathcal{O}(\varepsilon^2)$ and hence small amplitudes $u \sim \mathcal{O}(\varepsilon)$. According to the theorem for delayed stochastic center manifolds [40, 41] the stochastic stable manifold obeys

$$s_t = \tilde{H}(\theta, \mathbf{z}, \varepsilon, t) \tag{23}$$

similar to the deterministic case.
By virtue of Ito's rule

$$ds_t = \left(\frac{\partial \tilde{H}}{\partial t} + \frac{1}{2}\kappa^2 \mathbf{\Psi}^t_{\mathcal{U}}(0)\left(\nabla^2_z \tilde{H}\right)\mathbf{\Psi}_{\mathcal{U}}(0)\right)dt + \left(\nabla_z \tilde{H}\right)d\mathbf{z}$$

with the differential vector operator $\nabla_z = (\partial/\partial z_1, \partial/\partial z_2)$ and the Hessian matrix $\nabla_z^2 \tilde{H}$, the functional $\tilde{H}$ satisfies the implicit equation

$$\begin{aligned}
&\nabla_z \tilde{H}(\theta, \mathbf{z}, \varepsilon, t) \left(\mathbf{B}\mathbf{z} + \mathbf{\Psi}_{\mathcal{U}}(0) F \left[\mathbf{\Phi}_{\mathcal{U}} \mathbf{z}(t) + \tilde{H}(\mathbf{z}, \varepsilon, t) \right] dt + \mathbf{\Psi}_{\mathcal{U}}(0) \kappa dW(t) \right) \\
&+ \frac{\partial \tilde{H}(\theta, z, \varepsilon, t)}{\partial t} dt + \frac{1}{2} \kappa^2 \mathbf{\Psi}_{\mathcal{U}}^t(0) \left(\nabla_z^2 \tilde{H} \right) \mathbf{\Psi}_{\mathcal{U}}(0) dt \\
&= \mathcal{A}(\tilde{H}(z, \varepsilon, t)) dt + (X_o - \Phi_{\mathcal{U}}(\theta) \Psi_{\mathcal{U}}(0)) \left(F[\Phi_{\mathcal{U}} z(t) + \tilde{H}(z, \varepsilon, t)] dt + \kappa dW(t) \right).
\end{aligned} \tag{24}$$

To lowest orders $\mathcal{O}(\varepsilon^2)$, i.e. for small delays and noise levels, we assume that the stable manifold represents a linear superposition of the deterministic and the stochastic stable manifold $H(\mathbf{z}, \theta)$ and $h(\theta, t)$, respectively [38]

$$\tilde{H}(\mathbf{z}, \theta), t) \approx H(\mathbf{z}, \epsilon, \theta) + h(\theta, t) \tag{25}$$

where H and h are distinct non-linear functions of the center modes $\mathbf{z}$ and time t, respectively. The deterministic part H is not subject to noise and hence autonomous and may be computed by applying standard techniques for time-independent center manifolds, e.g. by polynomial expansion [51, 45, 44, 38]. However, the stochastic correction h depends on the additive noise but is independent of the center mode $\mathbf{z}$. Inserting (25) into Eq. (24), we obtain at order $\mathcal{O}(\varepsilon^2)$ [38]

$$dh(\theta, t) = \mathcal{A}\left[h(\theta, t)\right] dt + \kappa(X_o - \mathbf{\Phi}_{\mathcal{U}}(\theta) \mathbf{\Psi}_{\mathcal{U}}(0)) dW(t) \tag{26}$$

whose solution is discussed in some detail in section 3.3.
After computation of the deterministic part $H(\mathbf{z}, \theta)$ and the solution of Eq. (26), insertion into Eq. (21) yields the non-delayed stochastic delay differential equation

$$d\mathbf{z} = (\mathbf{B}\mathbf{z} + \mathbf{\Psi}(0)\mathbf{F}[\mathbf{\Phi}(\theta)\mathbf{z} + H(\mathbf{z}, \theta) + h(\theta, t)])\, dt + \kappa \mathbf{\Psi}(0) dW(t). \tag{27}$$

One recalls that the only approximation made to gain this equation is the separation of deterministic and stochastic stable manifold.
Note how Eq. (27) involves multiplicative noise via the stochastic process $h(\theta, t)$ and additive noise as the Wiener process $W(t)$. Loosely speaking speaking, the stable manifold couples to the center manifold dynamics and generates the multiplicative noise in the center manifold dynamics through the backdoor. This coupling of a stochastic stable manifold into the center manifold also occurs in non-delayed high-dimensional systems [32, 31, 52, 4, 49] where one observes stochastic bifurcations induced by additive noise.

3.2 Adiabatic approximation

Now let us assume a discrete delay distribution $g(\tau)$ and, consequently, $\rho \circ u = \sum_{m=1}^{M} \rho_m u(t-\tau_m)$ comprising time delays τ_m of number M. Consequently, θ takes M values in the nonlinear function $F[\cdot]$ and there are M stochastic terms $h(\theta,t) = h(\{\theta_m\},t),\ 1,\ldots,M$ with $\theta_m = -\tau_m$. Introducing the vector $\mathbf{h}(t) = (h(\theta_1,t), h(\theta_2,t),\ldots,h(\theta_M,t))^t$, the Fokker Planck equation for Eq. (27) reads

$$\frac{\partial p(\mathbf{z},\mathbf{h},t)}{\partial t} = -\nabla_z\{\mathbf{Bz}+\mathbf{\Phi}(0)\mathbf{F}[\mathbf{\Phi z}+H+h]\}p(\mathbf{z},\mathbf{h},t)+\frac{1}{2}\mathbf{\Psi}(0)\mathbf{\Psi}(0)^T\nabla_z^2 p(\mathbf{z},\mathbf{h},t), \tag{28}$$

with the joint probability density $p(\mathbf{z},\mathbf{h},t)$ and where the ε dependence was dropped for readability. Stochastic shifts in stability are revealed by performing the adiabatic elimination of the fast stochastic components in Eq. (28) [32, 31, 15, 30]. The characteristic time scale separation involved near non-hyperbolic fixed points implies that the stochastic perturbations $dW(t)$ and h can be seen as transient perturbations around the slow center modes $\mathbf{z}$ associated to the unperturbed system. Given the separable ansatz in Eq. (25), the probability density $p(\mathbf{z},\mathbf{h})$ factorizes as

$$p(\mathbf{z},\mathbf{h},t) = w(\mathbf{z},t)q(\mathbf{h},t).$$

The elinimation scheme is exposed by performing an ensemble average of Eq. (28) over $\mathbf{h}$. While technical, it is nonetheless illustrative to go through the calculations. In particular, $\mathbf{h} = \mathbf{h_r}+i\mathbf{h_i}$ and $q(\mathbf{h},t) = \Pi_{m=1}^{M} q_r(h_{r,m},t)q_i(h_{i,m},t)$ while $g(\mathbf{z},\epsilon,\theta) \equiv \mathbf{\Phi}(\theta)\mathbf{z} + H(\mathbf{z},\epsilon,\theta)$ with $g = g_r + ig_i$. Then the ensemble average of the drift component of Eq. (28) over the stochastic fluctuations of the stable manifold is

$$\begin{aligned}
\mathbf{I}_{drift} \;=\; & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\{\mathbf{Bz}+\mathbf{\Psi}(0)\epsilon\sum_{m=1}^{M}\rho_m(g_r(\mathbf{z},\theta_m)+ig_i(\mathbf{z},\theta_m)+h_r(\theta_m)+ih_i(\theta_m)) \\
& +\mathbf{\Psi}(0)\beta\sum_{m=1}^{M}\rho_m[(g_r(\mathbf{z},\theta_m)+h_r(\theta_m))^2+(g_i(\mathbf{z},\theta_m)+h_i(\theta_m))^2] \\
& \times(g_r(\mathbf{z},\theta_m)+ig_i(\mathbf{z},\theta_m)+h_r(\theta_m)+ih_i(\theta_m))\} \\
& \times\Pi_{m=1}^{M} q(h_{r,m},t)q(h_{i,m},t)dh_{r,m}dh_{i,m}\ .
\end{aligned} \tag{29}$$

One can show that $h_{r,k},\ h(i,k)\ \forall k = 1, \ldots M$ have zero mean. Assuming symmetric probability densities of the stochastic fluctuations, i.e., $\int_{-\infty}^{\infty} h_r^n q_r(h_r) dh_r = \int_{-\infty}^{\infty} h_i^n q_i(h_i) dh_i = 0$ for odd n, the integration over $\mathbf{h}$ yields

$$
\begin{aligned}
\mathbf{I}_{drift} &= \mathbf{B}\mathbf{z} + \mathbf{\Psi}(0)\epsilon\rho \circ (g^r + ig^i) + \mathbf{\Psi}(0)\beta\rho \circ [(g^r)^2 + (g^i)^2 + 3\Delta](g^r + ig^i) \\
&= \mathbf{B}\mathbf{z} + \mathbf{C}\mathbf{z} + \mathbf{\Psi}(0)\mathbf{F}[\mathbf{\Phi}(\theta)\mathbf{z} + H(\mathbf{z},\theta) + d(H(\mathbf{z},\theta))]
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(\theta) &= \langle (h^r(\theta))^2 + (h^i(\theta))^2 \rangle = \langle |h(\theta)|^2 \rangle \qquad (30) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{s=1}^{M} \left(h_{r,s}^2(\theta) + h_{i,s}^2(\theta) \right) \Pi_{m=1}^{M} q(h_{r,m}(\theta)) q(h_{i,m}(\theta)) dh_{r,m}(\theta) dh_{i,m}(\theta) \\
\mathbf{C}\mathbf{z}(t) &= 3\beta\mathbf{\Psi}(0)\rho \circ \Delta\mathbf{\Phi}\mathbf{z}(t) \\
&= \beta\mathbf{\Psi}(0) \sum_{m=1}^{M} \rho_m \Delta(\theta_m)(\Phi_1(\theta_m) z_1(t) + \Phi_2(\theta_m) z_2(t)) \qquad (31) \\
d(H) &= 3\beta\mathbf{\Psi}(0)\rho \circ \Delta H(\mathbf{z}(t),\theta) \\
&= 3\beta\mathbf{\Psi}(0) \sum_{m=1}^{M} \rho_m \Delta(\theta_m) H(\mathbf{z}(t),\theta_m) \qquad (32)
\end{aligned}
$$

are the noise correction terms. With the above adiabatic expression for the drift term I_{drift}, the adiabatic Fokker Planck representations of the center modes dynamics is

$$
\begin{aligned}
\frac{\partial w(\mathbf{z},t)}{\partial t} &\approx -\nabla_z \{ (\mathbf{B}+\mathbf{C})\mathbf{z} + \mathbf{\Phi}(0)F[\mathbf{\Phi}\mathbf{z} + H(\mathbf{z}) + d(H(\mathbf{z}))] \} w(\mathbf{z},t) \\
&\quad + \frac{1}{2}\mathbf{\Psi}(0)\mathbf{\Psi}(0)^T \nabla_z^2 w(\mathbf{z},t), \qquad (33)
\end{aligned}
$$

which may be interpreted as the Fokker-Planck equation for the stochastic order parameter equation

$$
d\mathbf{z} = ((\mathbf{B}+\mathbf{C})\mathbf{z} + \mathbf{\Psi}(0)F[\mathbf{\Phi}\mathbf{z} + H(\mathbf{z}) + d(H(\mathbf{z}))])\, dt + \kappa\mathbf{\Psi}(0) dW(t) \qquad (34)
$$

This equation does not include any delay in $\mathbf{z}$ anymore and describes the slow dynamics of the system close to the Hopf bifurcation including the noise correction terms $\mathbf{C}$ and $\mathbf{d}$.

A first look at the noise correction $\mathbf{C}$ reveals that noise ($\Delta \sim \kappa$) and propagation delays (τ_{max}) shift the linear gain proportional to $\Delta\tau_m$ and hence, depending on the value of the system's parameters, may provoke or prevent the occurence of instabilities [37, 30], i.e., induces a stochastic bifurcation.

3.3 The stochastic stable manifold

In order to learn more about the noise correction terms (31) and (32), it is necessary to compute $\Delta = \langle |h|^2 \rangle$ and, hence, compute the solution of Eq. (26). Since $s_t = H + h$ evolves on the linear stable manifold S,

$$h(\theta, t) = (1 - \mathcal{P})H(t + \theta) \tag{35}$$

with $H(t+\theta) \in \mathcal{C}$. For $-1 \le \theta < 0$, inserting this ansatz into Eq. (26), applying $\mathcal{P}$ to both sides of the resulting equation and utilizing the property $\mathcal{P}h = 0$ leads to

$$\begin{aligned} dH(t+\theta) &= \left(\mathcal{P}\frac{\partial}{\partial\theta}\mathcal{P}H(t+\theta)\right)dt + \kappa\mathbf{\Phi}_{\mathcal{U}}(\theta)\mathbf{\Psi}_{\mathcal{U}}(0)dW(t) \\ &= \left(\mathbf{\Phi}_{\mathcal{U}}(\theta)\left(\mathbf{\Psi}_{\mathcal{U}}, \frac{\partial}{\partial\theta}\mathbf{\Phi}_{\mathcal{U}}\right)(\mathbf{\Psi}_{\mathcal{U}}, H)\right)dt + \kappa\mathbf{\Phi}_{\mathcal{U}}(\theta)\mathbf{\Psi}_{\mathcal{U}}(0)dW(t) \\ &= \left(\mathbf{\Phi}_{\mathcal{U}}(\theta)\mathbf{B}\left(\mathbf{\Psi}_{\mathcal{U}}, H\right)\right)dt + \kappa\mathbf{\Phi}_{\mathcal{U}}(\theta)\mathbf{\Psi}_{\mathcal{U}}(0)dW(t) \ . \end{aligned}$$

For $\theta = 0$, inserting Eq. (35) into Eq. (26) yields

$$dH(t) = L[H]dt + \kappa dW(t)\ ,\ t \ge 0\ . \tag{36}$$

For both cases initial values are given by $H(\tau) = H_o(\tau),\ -1 \le \tau \le 0$. Equation (36) is a delayed stochastic delay differential equation whose solution is well-studied for single delays [35, 2] and multiple delays [53, 42]. We find

$$H(t) = \kappa\sum_{n=0}^{\infty} H_n(t) = \kappa\sum_{n=0}^{\infty}\underbrace{\int_0^t e^{-\lambda_n(t-t')}dW(t')}_{H_n(t)} \tag{37}$$

assuming initial conditions $H_o(\tau) = 0$, i.e. the system rests on the deterministic stable manifold before the stochastic fluctuations set in at $t = 0$. The terms λ_n are the roots of the characteristic equation with $\lambda_0 = i\omega_c,\ \lambda_1 = -i\omega_c$ and $Re(\lambda_n) < 0,\ n \ge 2$. Since Eq. (36) is linear, the full solution (37) is a linear superposition of single solutions (or modes) $H_n(t)$ corresponding to the characteristic roots. These modes are independent to each other and the probability density function of $H(t)$ is the infinite product of the probability density functions of single modes $H_n(t)$. It is well known that the stable modes $H_{n\ge2}(t)$ with $Re(\lambda_{n\ge2}) < 0$ are stationary in time [35] and hence their corresponding probability density function remains finite for large time. In contrast, for the center modes $H_0,\ H_1$ with purely imaginary roots $\lambda_0,\ \lambda_1$ the variance

$$\sigma^2(t) = \langle |H_{0,1}(t)|^2 \rangle = 2\kappa^2 t$$

diverges for large times. Consequently, the variance of $H(t)$ and $h(\theta, t) \to \infty$ and $\Delta \to \infty$ defined in (30) diverge for $t \to \infty$ at the low order $\mathcal{O}(\varepsilon^2)$. As a consequence the bifurcation shift $\mathbf{Cz}$ diverges after long time. For finite time, $\Delta \sim \kappa^2$ and hence the noise effect in Eqs. (31),(32) is proportional to $\tau_m \kappa^2$. However, we point out that $h(\theta, t)$ may not diverge at higher orders of ε due to nonlinear saturation effects. The finite shift of stationary numerical solutions shown in Fig. 1 supports this view as the system probability density function is stationary.

4 Concluding Remarks

The present work shows how to describe the stochastic dynamics of a scalar nonlinear stochastic differential equation involving distributed delays by a stochastic non-delayed reduced evolution equation on the corresponding center manifold. This is made possible by allowing an explicit time dependence of the stable manifold. The present work reviews the major approach developed in previous studies on a single delay and yet can easily be extended to distributed delays. As shown in our analysis, additive noise translates into multiplicative noise due to the nonlinear coupling of a center and stable manifold dynamics. Hence additive noise induces a stochastic bifurcation [4]. Such an effect is expected from previous studies on high-dimensional yet non-delayed systems. In addition, we have shown for the first time that the separation of deterministic and stochastic dynamics on the stable manifold yields, in close vicinity to the origin, the divergence of the noise-induced correction. Future work will consider the effect of initial conditions in more detail and extend the analysis to higher orders of ε in order to gain deeper insight into the dynamics of the system far from the equilibrium point.

Acknowledgements

AH acknowledges funding from the European Research Council for support under the European Union's Seventh Framework Programme (FP7/2007-

2013) ERC grant agreement No.257253. JL acknowledges financial support from the National Science and Engineering Research Council of Canada (NSERC).

References

[1] H. Agmon-Sir and I. Segev. Signal delay and input synchronisation in passive dendritic structures. *J. Neurophysiol.*, 70:2066–2085, 1993.

[2] A. Amann, E. Schoell, and W. Just. Some basic remarks on eigenmode expansions of time-delay dynamics. *Physica A*, 373:191–202, 2007.

[3] R.F.V. Anderson. Intrinsic parameters and stability of differential-delay equations. *J. Math. Anal. Appl.*, 163:184–199, 1992.

[4] L. Arnold. *Random Dynamical Systems*. Springer-Verlag, Berlin, 1998.

[5] F. M. Atay and A. Hutt. Stability and bifurcations in neural fields with finite propagation speed and general connectivity. *SIAM J. Appl. Math.*, 65(2):644–666, 2005.

[6] F. M. Atay and A. Hutt. Neural fields with distributed transmission speeds and constant feedback delays. *SIAM J. Appl. Dyn. Syst.*, 5(4):670–698, 2006.

[7] S. Bernard, J. Belair, and M. Mackey. Sufficient conditions for stability of linear differential equations with distributed delay. *Discrete Cont. Dyn. Syst. B*, 1(2):233–256, 2001.

[8] J. Boulet, R. Balasubramaniam, A. Daffertshofer, and A. Longtin. Stochastic two delay- differential model of delayed visual feedback effects on postural dynamics. *Phil. Trans. R. Soc. A*, 368(1911):423–438, 2010.

[9] D. Bratsun, D. Volfson, L.S. Tsimring, and J. Hasty. Delay-induced stochastic oscillations in gene regulation. *Proc. Natl. Acad. Sci. USA*, 102(41):14593–14598, 2013.

[10] E. Buckwar. Introduction to the numerical analysis of stochastic delay differential equations. *J.Comp. Appl. Math.*, 125(1-2):297–307, 2000.

[11] S.-A. Campbell and J. Belair. Analytical and symbolically-assisted investigations of hopf bifurcations in delay-differential equations. *Canadian Applied Mathematics Quarterly*, 3:137–154, 1995.

[12] O. Diekmann, S. A. van Gils, S. M.V. Lunel, and H. O. Walther. *Delay Equations*. Springer, New York, 1995.

[13] B. Doiron, M.J. Chacron, L.Maler, A. Longtin, and J. Bastian. Inhibitory feedback required for network burst responses to communication but not to prey stimuli. *Nature*, 421:539–543, 2003.

[14] B. Doiron, B. Lindner, A. Longtin, L.Maler, and J. Bastian. Oscillatory activity in electrosensory neurons increases with the spatial correlation of the stochastic input stimulus. *Phys.Rev.Lett.*, 93:048101, 2004.

[15] F. Drolet and J. Vinals. Adiabatic elimination and reduced probability distribution functions in spatially extended systems with a fluctuating control parameter. *Phys. Rev. E*, 64:026120, 2001.

[16] T. Erneux. *Applied Delay Differential Equations. Surveys and Tutorials in the Applied Mathematical Sciences*. Springer, 2009.

[17] C. Eurich, M.C. Mackey, and H. Schwegler. Recurrent inhibitory dynamics: The role of state dependent distributions of conduction delay times. *J. Theor. Biol.*, 216:31–50, 2002.

[18] T. Faria and L. Magalhaes. Normal forms for retarded functional differential equations with parameters and applications to hopf bifurcation. *Journal of Differential Equations*, 122:281, 1995.

[19] B. Ficak and J. Klamka. Stability criteria for a class of stochastic distributed delay systems. *Bull. Pol. Acad. Sci. Techn. Sci.*, 61(1):221–228, 2013.

[20] E. Forgoston, L. Billings, and I.B. Schwartz. Accurate noise projection for reduced stochastic epidemic models. *Chaos*, 19:043110, 2009.

[21] L. Gammaitoni, P. Hanggi, and P. Jung. Stochastic resonance. *Rev.Mod.Phys.*, 70(1):223–287, 1998.

[22] M. Gaudreault, F. Drolet, and J. Vinals. Bifurcation threshold of the delayed van der pol oscillator under stochastic modulation. *Phys. Rev. E*, 85(5):056214, 2012.

[23] J.K. Hale and S.M.V. Lunel. *Introduction to functional differential equations*. Springer, Berlin, 1993.

[24] P. Halmos. *Measure Theory*. Springer Verlag, New York, 1974.

[25] M. Haragus and G. Iooss. *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*. Springer, 2011.

[26] B. D. Hassard, N.D. Kazarinoff, and Y.N. Wan. *Theory and Applications of Hopf bifurcation*. Cambridge University Press, Cambridge, 1981.

[27] A. Hutt. Effects of nonlocal feedback on traveling fronts in neural fields subject to transmission delay. *Phys. Rev. E*, 70:052902, 2004.

[28] A. Hutt. Additive noise may change the stability of nonlinear systems. *Europhys. Lett.*, 84:34003, 2008.

[29] A. Hutt and F.M. Atay. Spontaneous and evoked activity in extended neural populations with gamma-distributed spatial interactions and transmission delay. *Chaos, Solitons and Fractals*, 32:547–560, 2007.

[30] A. Hutt, J. Lefebvre, and A. Longtin. Delay stabilizes stochastic systems near an non-oscillatory instability. *Europhys. Lett.*, 98:20004, 2012.

[31] A. Hutt, A. Longtin, and L. Schimansky-Geier. Additive global noise delays turing bifurcations. *Phys. Rev. Lett.*, 98:230601, 2007.

[32] A. Hutt, A. Longtin, and L. Schimansky-Geier. Additive noise-induced turing transitions in spatial systems with application to neural fields and the swift-hohenberg equation. *Physica D*, 237:755–773, 2008.

[33] A. Hutt, C. Sutherland, and A. Longtin. Driving neural oscillations with correlated spatial input and topographic feedback. *Phys.Rev.E*, 78:021911, 2008.

[34] A. Hutt and L. Zhang. Distributed nonlocal feedback delays may destabilize fronts in neural fields, distributed transmission delays do not. *J. Math. Neurosci.*, 3:9, 2013.

[35] U. Küchler and B. Mensch. Langevin stochastic differential equation extended by a time-delayed term. *Stoch. Stoch. Rep.*, 40:23–42, 1992.

[36] R. Lang and K. Kobayashi. External optical feedback effects on semiconductor injection laser properties. *IEEE J. Quantum Electron.*, 16:347, 1980.

[37] J. Lefebvre and A. Hutt. Additive noise quenches delay-induced oscillations. *Europhys. Lett.*, 102:60003, 2013.

[38] J. Lefebvre, A. Hutt, V. G. LeBlanc, and A. Longtin. Reduced dynamics for delayed systems with harmonic or stochastic forcing. *Chaos*, 22:043121, 2012.

[39] A. Longtin, F. Moss, and A. Bulsara. Time interval sequences in bistable systems and noise induced transmission of neural information. *Phys. Rev. Lett.*, 67:656–659, 1991.

[40] S. A. Mohammed and M. K. R. Scheutzow. The stable manifold theorem for non-linear stochastic systems with memory: I. existence of the semiflow. *J. Func. Analysis*, 205(2):271–305, 2003.

[41] S. A. Mohammed and M. K. R. Scheutzow. The stable manifold theorem for non-linear stochastic systems with memory: Ii. the local stable manifold theorem. *J. Func. Analysis*, 260(2):253–306, 2004.

[42] S.E.A. Mohammed. Lyapunov exponents and stochastic flows of linear and affine hereditary system. In M. Pinsky and V. Wihstutz, editors, *Diffus. Proc. and Rel. Probl. of Anal., II, Stochastic Flows*, pages 141–169. Birkhausser, 1992.

[43] A. Nitzan, P. Ortoleva, J. Deutch, and J. Ross. Fluctuations and transitions at chemical instabilities: The analogy to phase transition. *J. Chem. Physics*, 61:1056–1074, 1974.

[44] R. Quesmi, M. Ait Babram, and M.L. Hbid. A maple program for computing a terms of a center manifold, and element of bifurcations for

a class of retarded functional differential equations with hopf singularity. *Applied Mathematics and Computation*, 175:932–968, 2006.

[45] B. Redmond, V.G. LeBlanc, and A . Longtin. Bifurcation analysis of a class of first-order nonlinear delay-differential equations with reflectional symmetry. *Physica D*, 166:131–146, 2002.

[46] A. J. Roberts. Simple exmaples of the derivation of amplitude equations for systems possessing bifurcations. *Austral. Math. Soc. Ser. B*, 27:48–65, 1985.

[47] A. Roxin, N. Brunel, and D. Hansel. Rate models with delays and the dynamics of large networks of spiking models. *Prog. Theor. Phys.*, 161:68–85, 2006.

[48] A. Vanderbauwhede and G. Iooss. Center manifold theory in infinite dimensions. *Dyn. Rep.*, 1:125–163, 1992.

[49] W. Wang and A.J. Roberts. Slow manifold and averaging for slow-fast stochastic differential system. *J. Math. Ana. Appl.*, 398:822–839, 2012.

[50] T. Weller and B. Hajek. Scheduling nonuniform traffic in a packet-switching system with small propagation delay. *IEEE/ACM Trans. Networking*, 5:813–823, 1997.

[51] W. Wischert, A. Wunderlin, and A. Pelster. Delay-induced instabilities in nonlinear feedback systems. *Physical Review E*, 49(1), 1994.

[52] C. Xu and A.J. Roberts. On the low-dimensional modelling of Stratonovich stochastic differential equations. *Physica A*, 225:62–80, 1996.

[53] W. Zhen, L. Xiong, and L. Jinzhi. Moment boundedness of linear stochastic delay differential equations with distributed delay. *Stoch. Proc. Appl.*, 125(1):586–612, 2014.