Making a voting system depend only on orders of preference reduces its manipulability rate

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MAKING A VOTING SYSTEM DEPEND ONLY ON ORDERS OF PREFERENCE REDUCES ITS MANIPULABILITY RATE*

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For any non-trivial voting system, there exists manipulable situations where a coalition of voters, by casting an insincere ballot, may secure an outcome that is better from their point of view. In this paper, we investigate how it is possible to reduce the *manipulability rate*, which is the probability of such situations in a given *culture*, i.e. a probabilistic structure of the population.

We prove that when electors are independent, the culture meets a condition that we call *decomposability*. And when this condition is met, for any voting system that uses more complex ballots than orders of preferences (for example grades), there exists a "reasonable" voting system that depends only on orders of preference and whose manipulability rate is at most as high.

Combining this result with Condorcification theorem from Durand et al. (2014) and Green-Armytage et al. (2014), we conclude that when searching for a "reasonable" voting system whose manipulability is minimal, one can restrict to those that depend only on orders of preference and meet the Condorcet criterion.

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I Introduction

I.A *Motivation*

A desirable property of a voting system would be that any voter, after having determined her opinions, has at her disposal a ballot that best defends her views, whatever the other voters do. If it is not the case, as C. L. Dodgson noticed (cited by Black, 1958), voting becomes "more of a game of skill than a real test of the wishes of the voters."

Unfortunately, Gibbard (1973) proved that any non-dictatorial voting system with three eligible candidates or more is manipulable; that is, at least one voter does not always have an undominated strategy. Although this result is frequently cited under the form of Gibbard-Satterthwaite theorem (Satterthwaite, 1975), which deals only with voting systems depending on orders of preference, it is worth remembering that Gibbard's fundamental theorem applies to any *game form*, where available strategies may be objects of any kind, including grades for example.

Despite this negative result, all voting systems are not equal when it comes to manipulation. Several indicators of coalitional manipulability have been defined and studied, for example by Saari (1990), Smith (1999), Slinko (2004) and Tideman (2006). A very common one is the *manipulability rate*, which is the probability that a situation is coalitionaly manipulable, under a given assumption about the probabilistic structure of the population, or *culture*. It is an important indicator because it is an upper bound for most of the others: if we could identify voting systems with close-to-zero manipulability rates in realistic cultures, then the impact of manipulability would be tolerable.

Several authors have used a theoretical approach (Lepelley and Mbih, 1987, 1994; Lepelley and Valognes, 1999; Smith, 1999; Favardin et al., 2002; Lepelley and Valognes, 2003; Favardin and Lepelley, 2006; Lepelley et al., 2008), computer simulations (Lepelley and Mbih, 1987; Pritchard and Wilson, 2007; Green-Armytage, 2011, 2014; Green-Armytage et al., 2014) or experimental results (Chamberlin et al., 1984; Tideman, 2006; Green-Armytage et al., 2014) to evaluate the manipulability rates of several voting systems, according to various assumptions about the structure of the population.

To the best of our knowledge, the first general theoretical result that allows to reduce the manipulability rate has been shown independently by Durand et al. (2012, 2014) and Green-Armytage et al. (2014). It states that if a voting system meets the "informed majority coalition criterion", then forcing it to respect Condorcet criterion makes its less manipulable, in the sense of the inclusion of manipulable situations. Hence, for any culture, the "condorcified" voting system has a lower manipulability rate than the original one.

Recently, special attention has been paid to voting systems that are not based on orders of preference, such as Majority Judgment. According to Balinski and Laraki (2010), one of their objectives is a kind of resistance to manipulation. However, simulations results by Green-Armytage et al. (2014) suggest that voting systems based on grades perform quite badly in terms of manipulation.

In this paper, we investigate this question from a theoretical point of view: given a voting system based on more information than orders of preferences, for example grades, is it always outperformed by a well-chosen voting system that depends only on orders of preference?

I.B Contributions and plan

In section II, we present our general framework and recall previous results.

In II.A, we recall the definition of an *electoral space* from Durand et al. (2014). This framework allows to study any kind of voting system, including those that may depend on grades, approval values or other objects. Then we complete this framework by defining the notion of *culture*, a probabilistic measure over the possible states of opinion for the population of voters.

In II.B, we recall the definition of *state-based voting system*, also from Durand et al. (2014). Is has been shown by Moulin (1978) and Durand et al. (2014) that, for the purpose of reducing manipulability, we can restrict our study to such voting systems. Then we define *manipulability* in general, and the *manipulability rate* of a voting system in a given culture.

In II.C, we recall the Condorcification theorem from Durand et al. (2014) and Green-Armytage et al. (2014): if a voting system meets the *informed majority coalition criterion*, then its *Condorcification* is at most as manipulable as the original system.

Section III contains the contributions of this paper.

In III.A, for any voting system, we define its *slices*, each of them being a voting system that depends only on binary relations of preference. Furthermore, if the original system meets the Condorcet criterion, then so does any of its slices.

In III.B, we define a criterion for the culture of the population: *decomposability*. We prove, in particular, that it is met when voters are independent.

In III.C, we prove the main result of this paper, the slicing theorem: when the probabilized electoral space is decomposable, then for any voting system f, at least one of its slices has a manipulability rate that is at most as high as f.

In III.D, we conclude by combining both results, Condorcification and slicing theorems. In order to minimize manipulability, we show that we can restrict our search to voting systems that:

- Depend only on binary relations of preference,
- And meet the Condorcet criterion.

Finally, we remark that for any decomposable culture, such an "optimal" voting system exists.

In appendix A, the reader will find a glossary of the main notations used in this paper. Other appendices contains some developments that can be skipped at first reading: appendix B gives theoretical foundations for the notion of decomposability and appendix C investigates whether some generalizations of the main theorem are possible.

II General framework and previous results

In this section, we present our general framework. For greater details about the notions of electoral space and state-based voting system, see Durand et al. (2014).

II.A Electoral space

For $n \in \mathbb{N} \setminus \{0\}$, we note $\mathcal{V}_n = \{1, \ldots, n\}$ the set of the indexes of *voters*, called simply *voters* in the following. For $m \in \mathbb{N} \setminus \{0\}$, we note $\mathcal{C}_m = \{1, \ldots, m\}$ the set of the indexes of the *candidates*[1].

In order to model preferences, we need to recall a few definitions about binary relations. Most of the ones in this paper represent the strict preferences of a voter and are generally supposed to be antisymmetric, so we will use the "strict" notation $\succ$, even when relaxing the assumption of antisymmetry for the sake of mathematical generality.

Let E be a set, $r \in \mathcal{P}(E^2)$ a binary relation and $(c, d) \in E^2$. We note:

- $c \succ_r d$ if and only if (iff) $(c, d) \in r$;
- $c \succcurlyeq\!\!\!\preccurlyeq_r d$ iff $c \succ_r d$ and $d \succ_r c$;
- $c \not\succcurlyeq\!\!\!\preccurlyeq_r d$ iff not$(c \succ_r d)$ and not$(d \succ_r c)$;
- $c \ggcurly_r d$ iff $c \succ_r d$ and not$(d \succ_r c)$.

Let us remark that if relation r is antisymmetric, which will be assumed in most models, then there are only three mutually exclusive possibilities: $c \succ_r d$ (which is equivalent to $c \ggcurly_r d$ in this case), $d \succ_r c$ and $c \not\succcurlyeq\!\!\!\preccurlyeq_r d$.

We say that relation r is:

- *Irreflexive* iff $\forall c \in E, \text{not}(c \succ_r c)$;
- *Antisymmetric* iff $\forall (c, d) \in E^2, c \neq d$ and $c \succ_r d \Rightarrow \text{not}(d \succ_r c)$;
- *Complete*[2] iff $\forall (c, d) \in E^2, c \neq d$ and $\text{not}(d \succ_r c) \Rightarrow c \succ_r d$;
- *Transitive* iff $\forall (c, d, e) \in E^3, c \succ_r d$ and $d \succ_r e \Rightarrow c \succ_r e$;
- *Negatively transitive* iff $\forall (c, d, e) \in E^3, \text{not}(c \succ_r d)$ and $\text{not}(d \succ_r e) \Rightarrow \text{not}(c \succ_r e)$;
- A *strict weak order* iff it is negatively transitive, irreflexive and antisymmetric;
- A *strict total order* iff it is transitive, irreflexive (hence antisymmetric) and complete.

For $m \in \mathbb{N} \setminus \{0\}$, we note $\mathcal{R}_m = \mathcal{P}({\mathcal{C}_m}^2)$ the set of the binary relations over $\mathcal{C}_m$: an element of this set allows to represent a voter's binary relation of preference over the candidates. For $(n, m) \in (\mathbb{N} \setminus \{0\})^2$, we note $\mathcal{R} = {\mathcal{R}_m}^n$: an element of $\mathcal{R}$ represents the binary relations of preference of the whole population of voters.

Throughout this paper, we will illustrate our results with an example of model that allows to study most classical voting systems. In this specific model, each voter i is capable of mentally establishing:

1. Candidates can be or not be voters themselves, without impact on our results.
2. We could say *weakly complete* because this definition does not require $c \succ_r c$.

- A strict weak order of preference r_i over the candidates,
- A vector $u_i \in [0, 1]^m$ of grades over the candidates[3],
- A vector $a_i \in \{0, 1\}^m$ of approval values over the candidates.

The triple $\omega_i = (r_i, u_i, a_i)$ will be called her *state* and we will note R_i the function that extracts the first element of this triple: $R_i(\omega_i) = R_i(r_i, u_i, a_i) = r_i$.

To be more general, we now define a class of models, called *electoral spaces*, that allow to study any kind of voting system, with virtually any assumptions about the structure of each voter's opinions.

Definition II.1 (ES, electoral space)**.** An *electoral space*, or *ES*, is given by:

- A number of voters $n \in \mathbb{N} \setminus \{0\}$ and a number of candidates $m \in \mathbb{N} \setminus \{0\}$,
- For each voter $i \in \mathcal{V}_n$, a non-empty set Ω_i of her possible states,
- For each voter $i \in \mathcal{V}_n$, a function $R_i : \Omega_i \to \mathcal{R}_m$, which allows to know the binary relation of preference associated to her state.

We note $\Omega = \prod_{i \in \mathcal{V}_n} \Omega_i$: it is the set of possible states for the whole population. We note $R = (R_1, \ldots, R_n)$ the n-tuple of all functions R_i; by a slight abuse of notation, R will also denote the function that, to any situation $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$, associates the n-tuple $(R_1(\omega_1), \ldots, R_n(\omega_n))$, which gives each voter's binary relation of preference.

Such an electoral space is denoted (n, m, Ω, R), or just Ω when there is no ambiguity.

Back to our example, we might want to assume that a voter's state has some form of inner coherence: for example, if voter i strictly prefers candidate c to d, i.e. if $c \succ_{r_i} d$, we could demand that her grade (resp. her approval value) for c is no lower than her one for d, i.e. $u_i(c) \geq u_i(d)$ (resp. $a_i(c) \geq a_i(d)$). To embed this assumption in our model, we simply have to define Ω_i as the set of triples (r_i, u_i, a_i) that meet this assumption[4].

Here is another example: in some studies about voting, a voter's opinion is only represented by a strict total order over the candidates. This model simply corresponds to the choice of the *electoral space of strict total orders (for n and m)*, where each Ω_i is the set $\mathfrak{S}_m$ of strict total orders and each R_i is the identity function.

In definition II.1, binary relations of preference $R_i(\omega_i)$ are not supposed to be antisymmetric: this is discussed in Durand et al. (2014). If the reader feels uncomfortable with this, she may read the following with an additional assumption of antisymmetry in mind.

For probabilistic notions, such as the manipulability rate, we will have to deal with measurable sets, which are constituted by a set E and a sigma-algebra τ_E over E. Such a measurable set is denoted (E, τ_E), or just E when there is no ambiguity.

The set $\mathcal{R}_m$ of binary relations over $\mathcal{C}_m$ will always be endowed with its discrete sigma-algebra, which we will denote $\tau_{\mathcal{R}_m}$.

3. We could demand that grades belong to a finite subset of $[0, 1]$ for practical applications, but our results hold true with or without this assumption.

4. In fact, our results hold true in both models: with or without this additional assumption.

When we consider a Cartesian product E of measurable spaces (E_i, τ_{E_i}), we will always endow it with its product sigma-algebra. For example, the set $\mathcal{R}$ will be endowed with the product sigma-algebra $\tau_{\mathcal{R}} = \prod_{i \in \mathcal{V}_n} \tau_{\mathcal{R}_m} = (\tau_{\mathcal{R}_m})^n$, which is simply its discrete sigma-algebra.

Definition II.2 (MES, measurable electoral space)**.** A *measurable electoral space*, or *MES*, is given by an electoral space (n, m, Ω, R) and, for each voter $i \in \mathcal{V}_n$, a sigma-algebra τ_{Ω_i} over Ω_i, such that the function R_i is measurable.

We will note $\tau_\Omega = \prod_{i \in \mathcal{V}_n} \tau_{\Omega_i}$ the product sigma-algebra over Ω.

Such an MES is denoted $(n, m, \Omega, \tau_\Omega, R)$, or just Ω.

Definition II.3 (PES, probabilized electoral space)**.** A *probabilized electoral space*, or *PES*, is given by a measurable electoral space $(n, m, \Omega, \tau_\Omega, R)$ and a probability measure $\mathbb{P}$ over Ω, called *culture*.

Such a PES is denoted $(n, m, \Omega, \tau_\Omega, R, \mathbb{P})$, or just $(\Omega, \mathbb{P})$.

When considering a PES, we will note μ the probability law of R.

For example, let us consider our reference electoral space where $\omega_i = (r_i, u_i, a_i)$. Let us endow each Ω_i with the product sigma-algebra of the discrete one on $\mathcal{R}_m$, Lebesgue sigma-algebra on $[0, 1]^m$ and the discrete one on $\{0, 1\}^m$. Then each function R_i is obviously measurable. Hence, Ω is a measurable electoral space.

Independently for each voter, let us draw uniformly a point u_i in $[0, 1]^m$. Let us define r_i as the strict weak order of preference such that $c \succ_{r_i} d$ iff $u_i(c) > u_i(d)$. And for each candidate c, let us take $a_i(c)$ to be the rounding of $u_i(c)$ to the closest integer (rounded up). Then we have defined an example of culture $\mathbb{P}$, i.e. a probability measure over the electoral space Ω.

II.B Voting system and manipulability

Generally, a voting system can be quite complex: for example, it can involve a multi-round process. But it has been shown by Moulin (1978) and Durand et al. (2014) that, in order to reduce manipulability, we can restrict our study to *state-based voting systems*.

Definition II.4 (state-based voting system, SBVS)**.** Let Ω be an ES.

A *state-based voting system* over Ω, or *SBVS*, is a function:

$$f : \left| \begin{array}{ccl} \Omega & \to & \mathcal{C}_m \\ (\omega_1, \ldots, \omega_n) & \to & f(\omega_1, \ldots, \omega_n). \end{array} \right.$$

For example, let us consider one of the possible variants for the voting system called *Range voting*, in our reference electoral space where $\omega_i = (r_i, u_i, a_i)$.

- Each voter i communicates a state belonging to Ω_i;
- We say that she votes *sincerely* iff she communicates her true state ω_i;
- Function f takes into account only the vectors of grades communicated by the voters, then returns the candidate with highest total grade (and resolves ties in an arbitrary deterministic way).

Definition II.5 (manipulability)**.** As is usually done in the literature, we say that a voting system is *manipulable* in a given situation iff there exists a coalition of voters who, by misrepresenting their states, may secure an outcome that is different from the sincere winner and that they all prefer to her, while assuming that other voters still vote sincerely.

Formally, let Ω be an ES and f an SBVS.

For $(\omega, \psi) \in \Omega^2$, we say that f is *manipulable in situation* ω *towards* ψ iff:

$$\begin{cases} f(\psi) \neq f(\omega), \\ \forall i \in \mathcal{V}_n, \psi_i \neq \omega_i \Rightarrow f(\psi) \succ_{R_i(\omega_i)} f(\omega). \end{cases}$$

For $\omega \in \Omega$, we say that f is *manipulable in situation* ω iff $\exists \psi \in \Omega$ s.t. f is manipulable in situation ω towards ψ.

The *manipulability indicator of* f is the function:

$$M_f : \left| \begin{array}{ccl} \Omega & \to & \{0,1\} \\ \omega & \to & \left| \begin{array}{l} 1 \text{ if } f \text{ is manipulable in } \omega, \\ 0 \text{ otherwise.} \end{array} \right. \end{array} \right.$$

When $(\Omega, \mathbb{P})$ is a PES, we call *manipulability rate of* f *for* $\mathbb{P}$ (provided M_f is measurable):

$$\begin{aligned} \rho_{\mathbb{P}}(f) &= \mathbb{P}(f \text{ is manipulable in } \omega) \\ &= \int_{\omega \in \Omega} M_f(\omega) \mathbb{P}(d\omega). \end{aligned}$$

II.C Condorcification

Now, we recall the Condorcification theorem from Durand et al. (2014), which gives a first way to reduce manipulability for a large class of voting systems.

Definition II.6 (Condorcet winner)**.** Let Ω be an ES, $\omega \in \Omega$ and $z \in \mathcal{C}_m$.

We say that z is *Condorcet winner* in ω iff z has a *strict victory*[5] against any other candidate c; that is:

$$\forall c \in \mathcal{C}_m \setminus \{z\}, \begin{cases} \text{card}\{i \in \mathcal{V}_n \text{ s.t. } z \succ_{R_i(\omega_i)} c\} > \dfrac{n}{2}, & (1) \\ \text{card}\{i \in \mathcal{V}_n \text{ s.t. } c \succ_{R_i(\omega_i)} z\} \leq \dfrac{n}{2}. & (2) \end{cases}$$

If relations $R_i(\omega_i)$ are antisymmetric, which is the case in most models, then this definition amounts to condition (1) only.

5. Since we demand victories based on more than half voters, and not only more voters than the opponent, our definition of Condorcet winner coincides with the one commonly found in the literature when preferences are strict total orders, but may differ otherwise. See Durand et al. (2014) for more details.

Definition II.7 (two criteria for an SBVS)**.** Let Ω be an ES and f an SBVS.

We say that f *meets the Condorcet criterion* iff, for any $\omega \in \Omega$ and $z \in \mathcal{C}_m$, if z is a Condorcet winner in ω, then $f(\omega) = z$.

We say that f *meets the informed majority coalition criterion* iff any majority coalition, that is informed of what the other voters do, may decide of the outcome. That is, $\forall \omega \in \Omega, \forall \mathcal{M} \in \mathcal{P}(\mathcal{V}_n)$, if $\text{card}(\mathcal{M}) > \frac{n}{2}$ then $\forall c \in \mathcal{C}_m$:

$$\exists \psi \in \Omega \text{ s.t } \begin{cases} f(\psi) = c, \\ \forall i \in \mathcal{V}_n \setminus \mathcal{M}, \psi_i = \omega_i. \end{cases}$$

Durand et al. (2014) show that most classical voting systems from the literature meet the informed majority coalition criterion: all those meeting the Condorcet criterion, plurality, two-round system, Instant-Runoff Voting, Bucklin method, Borda count, Coombs method, approval voting, range voting.

Definition II.8 (Condorcification)**.** Let Ω be an ES and f an SBVS.

We call *Condorcification of* f the state-based voting system:

$$f^c : \left| \begin{array}{lll} \Omega & \to & \mathcal{C}_m \\ \omega & \to & \left| \begin{array}{l} \text{if } \omega \text{ has a Condorcet winner } z, \text{then } z, \\ \text{otherwise, } f(\omega). \end{array} \right. \end{array} \right.$$

Now, we recall the main result from Durand et al. (2014).

Theorem II.9 (Condorcification)**.** *Let Ω be an ES and f an SBVS. We suppose that f meets the informed majority coalition criterion.*

Then its Condorcification f^c is at most as manipulable as f:

$$\{\omega \in \Omega \textit{ s.t. } f^c \textit{ is manipulable in } \omega\} \subseteq \{\omega \in \Omega \textit{ s.t. } f \textit{ is manipulable in } \omega\}.$$

As a consequence, for any probability measure $\mathbb{P}$ over Ω, we have (provided M_f and M_{f^c} are measurable):

$$\rho_{\mathbb{P}}(f^c) \leq \rho_{\mathbb{P}}(f).$$

III Slicing

In this section, for any voting system, we define its *slices*, each of them being a voting system that depends only on binary relations of preference. Then we introduce the notion of *decomposable* culture, which is proven to be a more general condition than probabilistic independence of voters. We prove our main result: when the culture is decomposable, for any voting system, one of its slices is at most as manipulable as the original system. Finally, we combine this result with the Condorcification theorem.

III.A Slices of a voting system

Here, we define what is a *slice* of a voting system f. The idea is the following: when voter i communicates her binary relation of preference r_i, we use a predefined rule, denoted y_i, to reconstitute a fictional state ω'_i that is coherent with r_i. Then we apply f to these fictional states $(\omega'_1, \ldots, \omega'_n)$.

For example, in our reference electoral space where $\omega_i = (r_i, u_i, a_i)$, let us consider $y_i(r_i) = (r_i, u'_i, a'_i)$, where u'_i is the vector of Borda scores[6] associated to r_i and a'_i is an approval vector with 1 for each candidate.

In the *slice* of Range voting by $y = (y_1, \ldots, y_n)$, once voters have communicated binary relations of preference $r = (r_1, \ldots, r_n)$, we use $y = (y_1, \ldots, y_n)$ in order to reconstitute fictional states, then we apply Range Voting to these states. Finally, the winner is obviously the candidate with highest total Borda score. Hence, the *slice* of Range voting by y is simply the Borda method.

Of course, Range voting has many other slices, depending on the choice of y.

We now give the formal definitions.

Notations III.1 (space $\mathcal{Y}$)**.** When Ω is an ES, for each voter i, we note:

$$\mathcal{Y}_i = \{y_i : R_i(\Omega_i) \to \Omega_i \text{ s.t. } R_i \circ y_i = \text{Id}\}.$$

It is the set of right inverses of R_i (corestricted to its image), i.e. functions y_i that, to each possible r_i, associate a state $\omega'_i = y_i(r_i)$ that is an antecedent of r_i by R_i.

We note $\mathcal{Y} = \prod_{i=1}^{n} \mathcal{Y}_i$. For $r = (r_1, \ldots, r_n) \in \prod_{i \in \mathcal{V}_n} R_i(\Omega_i)$ and $y = (y_1, \ldots, y_n) \in \mathcal{Y}$, we note $y(r) = (y_1(r_1), \ldots, y_n(r_n)) \in \Omega$.

When Ω is an MES, the set ${\Omega_i}^{R_i(\Omega_i)}$ of functions from $R_i(\Omega_i)$ to Ω_i has a canonical sigma-algebra: associating each function to the list of its values, consider the product sigma-algebra $\tau_{\Omega_i} \times \ldots \times \tau_{\Omega_i}$, with a number of factors equal to the cardinal of $R_i(\Omega_i)$. Space $\mathcal{Y}_i$, as a subset of ${\Omega_i}^{R_i(\Omega_i)}$, inherits from this sigma-algebra.

Definition III.2 (slice)**.** Let Ω be an ES, f an SBVS and $y \in \mathcal{Y}$.

We call *slice of f by y* the voting system f_y defined as:

$$f_y : \left| \begin{array}{ccc} \Omega & \to & \mathcal{C}_m \\ \omega & \to & f(y(R(\omega))). \end{array} \right.$$

It is easily proved, but worth noticing, that if f meets the Condorcet criterion, then so does f_y. Indeed, if there is a Condorcet winner in ω, then she is also Condorcet winner in $y(R(\omega))$, since voters have the same binary relations of preference in both situations.

6. Borda score of candidate c (for voter i): c gets one point for each candidate d such that i prefers c to d, and half a point for each candidate that i judges incomparable of mutually preferable to c. Then this score is divided by $m - 1$, in order to have a value in $[0, 1]$.

III.B Decomposable electoral space

Now, we introduce the notion of decomposability. The idea is the following: by independently drawing r and y with suitable laws, we would like to reconstitute ω with the correct probability measure $\mathbb{P}$. Interested readers can read appendix B for more details about decomposability.

We will first give the formal definition, then an interpretation and an example. Let us recall that μ denotes the law of R.

Definition III.3 (decomposability)**.** Let $(\Omega, \mathbb{P})$ be a PES.

We say that $(\Omega, \mathbb{P})$ is *R-decomposable*, or just *decomposable*, iff there exists a probability law ν over $\mathcal{Y}$ such that:

$$\forall A \in \tau_\Omega, \mathbb{P}(A) = (\mu \times \nu)\{(r, y) \in R(\Omega) \times \mathcal{Y} \text{ s.t. } y(r) \in A\}.$$

In the following, when $(\Omega, \mathbb{P})$ is decomposable, we will always note ν an arbitrary measure over $\mathcal{Y}$ among those meeting the above condition.

This definition demands that $\mathbb{P}$ is the image measure of $\mu \times \nu$ by the operator that, to r and y, associates $y(r)$. So, by independently drawing r and y (with measures μ and ν), then considering $\omega = y(r)$, we draw ω with the correct probability measure $\mathbb{P}$.

Example III.4 (yellow and blue voters)**.** Let us consider $n = 2$ voters and $m = 2$ candidates named A and B. Let us assume that the state of a voter is the pair of a strict total order of preference and a complementary information, "yellow" or "blue".

In a real study case, these two colors might have specific meanings, like "strongly prefers" and "somehow prefers", but it does not matter for our present purpose.

Let $\mathbb{P}$ be the law that draws with equal probability one of the two following situations:

1. Each voter is in state $\mathcal{A} = (\text{A} \succ \text{B}, \text{yellow})$;
2. Each voter is in state $\mathcal{B} = (\text{B} \succ \text{A}, \text{blue})$.

To prove that this PES is decomposable, let us consider the measure ν that surely draws two identical functions y_1 and y_2 that, to relation of preference $\text{A} \succ \text{B}$, associate $\mathcal{A}$; and to $\text{B} \succ \text{A}$ associate $\mathcal{B}$.

Drawing $r = (r_1, r_2)$ with law μ, we have with equal probabilities $r = (\text{A} \succ \text{B}, \text{A} \succ \text{B})$ or $r = (\text{B} \succ \text{A}, \text{B} \succ \text{A})$. Then, drawing y with the (deterministic) law ν, we have with equal probability $y(r) = (\mathcal{A}, \mathcal{A})$ or $y(r) = (\mathcal{B}, \mathcal{B})$, which is coherent with $\mathbb{P}$.

In short, to "emulate" this PES $(\Omega, \mathbb{P})$, it is sufficient to draw (r_1, r_2) with law μ (which is directly defined by $\mathbb{P}$), draw (y_1, y_2) with law ν (which we exhibited), then glue r and y together.

We conclude that this PES $(\Omega, \mathbb{P})$ is decomposable.

Generally, it is not straightforward to see whether an electoral space is decomposable or not. Hence, we will give some sufficient or necessary conditions.

Proposition III.5 (independence implies decomposability)**.** *Let* $(\Omega, \mathbb{P})$ *be a PES.*

If voters $(\omega_1, \ldots, \omega_n)$ *are independent, then* $(\Omega, \mathbb{P})$ *is decomposable.*

Proof. Cf. proposition B.3 (in appendix). □

However, independence is not necessary for decomposability. Indeed, in example III.4, voters are not independent: either they are both in state $\mathcal{A}$, or in state $\mathcal{B}$. But, as we saw, the PES is decomposable.

Now, we examine another sufficient condition: when each space Ω_i is, by definition, constructed as a product $\mathcal{P}_i \times \mathcal{I}_i$ (where $\mathcal{P}_i \subset \mathcal{R}_m$), with random variables R and I that are independent. We provide an example just after the proposition.

Proposition III.6 (another sufficient condition for decomposability)**.** *Let* $n \in \mathbb{N} \setminus \{0\}$ *and* $m \in \mathbb{N} \setminus \{0\}$.

For each $i \in \mathcal{V}_n$, *let* $\mathcal{P}_i$ *be a subset of* $\mathcal{R}_m$, *let* $(\mathcal{I}_i, \tau_{\mathcal{I}_i})$ *be a measurable set and let* $\Omega_i = \mathcal{P}_i \times \mathcal{I}_i$. *Let* R_i *be the function:*

$$R_i : \left| \begin{array}{lcl} \mathcal{P}_i \times \mathcal{I}_i & \to & \mathcal{R}_m \\ (r_i, I_i) & \to & r_i. \end{array} \right.$$

Let $\mathbb{P}$ *be a probability measure over* $\Omega = \prod_{i \in \mathcal{V}_n} \Omega_i$.

If the two random variables $R = (R_1, \ldots, R_n)$ *and* $I = (I_1, \ldots, I_n)$ *are independent, then* $(\Omega, \mathbb{P})$ *is decomposable.*

Proof. To $I_i \in \mathcal{I}_i$, we associate the function $\pi_i(I_i) \in \mathcal{Y}_i$ that consists of concatenating r_i and I_i in order to reconstitute a state ω_i:

$$\pi_i(I_i) : \left| \begin{array}{lcl} \mathcal{P}_i & \to & \Omega_i \\ r_i & \to & \big(\pi_i(I_i)\big)(r_i) = (r_i, I_i). \end{array} \right.$$

To $I \in \mathcal{I}$, we associate $\pi(I) = (\pi_1(I_1), \ldots, \pi_n(I_n)) \in \mathcal{Y}$.

Then, denoting ξ the law of I, and ν the image measure of ξ by π, measure ν is suitable to prove decomposability. □

However, this condition is not necessary. When the sets Ω_i are defined as products $\mathcal{P}_i \times \mathcal{I}_i$ (where $\mathcal{P}_i \subset \mathcal{R}_m$), it may be the case that random variables R and I are not independent, but the space is decomposable anyway.

Indeed, in example III.4, if $R = (\mathrm{A} \succ \mathrm{B}, \mathrm{A} \succ \mathrm{B})$, then we know for sure that $I = (\text{yellow}, \text{yellow})$, whereas if $R = (\mathrm{B} \succ \mathrm{A}, \mathrm{B} \succ \mathrm{A})$, then $I = (\text{blue}, \text{blue})$; so, R and I are not independent. However, as we saw, the PES is decomposable.

As an example of proposition III.6, let us consider an electoral space where each voter i's state is constituted by a strict total order of preference r_i over the candidates, and an integer $k_i \in \{0, \ldots, m\}$: the voter "approves" the first k_i candidates of her order of preference (whatever "approving" means exactly).

Noting $\mathfrak{S}_m$ the set of strict total orders over $\mathcal{C}_m$, let μ be a probability law over ${\mathfrak{S}_m}^n$: for each voter i, we draw a strict total order of preference. Let ξ be a probability law over $\{0,\ldots,m\}^n$: for each voter i, we draw the number of candidates that she approves.

We draw a population by independently using μ for orders of preference and ξ for approval numbers. Let us note that for μ as well as for ξ, voters may not be independent. But drawings by μ and ξ are independent by assumption.

Then, according to proposition III.6, the PES is decomposable.

Let us finish with a more complex example, where the sufficient conditions from propositions III.5 and III.6 are not met, but the PES is decomposable (as it was already the case in the example III.4 of yellow and blue voters).

Noting $\mathcal{W}_m$ the set of strict weak orders over $\mathcal{C}_m$, let μ be a probability law over ${\mathcal{W}_m}^n$: for each voter i, we draw a strict weak order of preference. Let ξ be a probability law over $(\mathbb{R}^m)^n$: for each voter, we draw m grades. We choose ξ such that for any voter, her m grades are almost surely all different.

Given a strict weak order r_i and a vector of grades $(u_1,\ldots,u_m) \in \mathbb{R}^m$, we build a state in the following way: if r_i is a strict total order, we assign grades to candidates in this order, from the greatest to the lowest; if there are ties (for example between candidates of ranks $k, k+1, k+2$), then we assign them the corresponding average grade (for example from the k-th to the $k+2$-th greatest grades).

By design, this PES is decomposable.

Finally, we give a necessary condition for decomposability.

Proposition III.7 (necessary condition of decomposability)**.** *Let $(\Omega,\mathbb{P})$ be a PES.*

If it is decomposable, then for any subset of indexes $V \in \mathcal{P}(\mathcal{V}_n)$, for any $(A_i)_{i\in V} \in \prod_{i\in V} \tau_{\Omega_i}$, for any $r=(r_1,\ldots,r_n) \in \mathcal{R}$ of positive probability:

$$\mathbb{P}\big((\forall i \in V, \omega_i \in A_i) | R = r\big) = \mathbb{P}\big((\forall i \in V, \omega_i \in A_i) | \forall i \in V, R_i = r_i\big).$$

Proof. Cf. proposition B.4 (in appendix). □

We can emit an intuitive interpretation: the process of reconstituting states ω_i of a subset V of voters from the only knowledge of their relations r_i must be doable independently of relations r_j belonging to the other voters.

However, this condition is not sufficient to ensure decomposability, as shown in appendix B.

III.C Slicing theorem

Now we prove our main result: if the probabilized electoral space is decomposable, then there is at least one slice that is at most as manipulable as the original voting system.

The two following lemmas will get rid of some questions of measurability for the main theorem.

Lemma III.8 (measurability of functions depending on binary relations)**.** *Let Ω be an MES. Let (E, τ_E) be a measurable set and $g : \Omega \to E$.*

We assume that g depends only on binary relations of preference:

$$\forall(\omega, \psi) \in \Omega^2, R(\omega) = R(\psi) \Rightarrow g(\omega) = g(\psi).$$

Then g is measurable.

Proof. Since g depends only on $R(\omega)$, we may define $h : (\mathcal{R}_m)^n \to E$ such that $g = h \circ R$. Since $(\mathcal{R}_m)^n$ is endowed with the discrete measure, h is measurable; and by definition of an electoral space, R is measurable. So, $g = h \circ R$ is measurable. □

Lemma III.9 (measurability of any slice)**.** *Let Ω an MES, f an SBVS and $y \in \mathcal{Y}$.*

Then f_y and M_{f_y} are measurable.

Proof. Immediate consequence of previous lemma. □

Now, we present a lemma that gives a central idea of the theorem: when we are in situation $y(r)$, then voting systems f and f_y give the same result; but for manipulators, their possibility of expression in f_y are included in those they have in f, so they have less power.

For example, let us consider a very specific situation ω where for each voter i, her sincere vector of grades u_i is the vector of Borda scores associated to r_i. In ω, if the Borda method is manipulable, then Range voting also is: manipulators just have to use the same strategies they would use in the Borda method.

Lemma III.10 (manipulability of the slice)**.** *Let $(\Omega, \mathbb{P})$ be a decomposable PES and f an SBVS.*

Then, $\forall(r, y) \in R(\Omega) \times \mathcal{Y}$:

$$f_y \text{ is manipulable in } y(r) \Rightarrow f \text{ is manipulable in } y(r).$$

In other words:

$$M_{f_y}(y(r)) \leq M_f(y(r)).$$

Proof. Suppose that f_y is manipulable in $\omega = y(r)$. By definition, $\exists \psi \in \Omega$ s.t. $f_y(\psi) \neq f_y(\omega)$ and:

$$\forall i \in \mathcal{V}_n, \psi_i \neq \omega_i \Rightarrow f_y(\psi) \succ_{R_i(\omega_i)} f_y(\omega).$$

Expanding the definition of y and f_y and noting $\phi = y(R(\psi))$, we have:

$$\forall i \in \mathcal{V}_n, \psi_i \neq \omega_i \Rightarrow f(\phi) \succ_{R_i(\omega_i)} f(y(R(y(r)))).$$

Since $R \circ y = \text{Id}$, we have:

$$\forall i \in \mathcal{V}_n, \psi_i \neq \omega_i \Rightarrow f(\phi) \succ_{R_i(\omega_i)} f(\omega).$$

Let us remark that if $\psi_i = \omega_i$, then $\phi_i = y_i(R_i(\psi_i)) = y_i(R_i(\omega_i)) = y_i(R_i(y_i(r_i))) = y_i(r_i) = \omega_i$. By contraposition, we have the implication $\phi_i \neq \omega_i \Rightarrow \psi_i \neq \omega_i$, which leads to:

$$\forall i \in \mathcal{V}_n, \phi_i \neq \omega_i \Rightarrow f(\phi) \succ_{R_i(\omega_i)} f(\omega).$$

Hence, f is manipulable in ω towards ϕ. □

Theorem III.11 (slicing)**.** *Let $(\Omega, \mathbb{P})$ be a PES and f an SBVS whose manipulability rate is well defined (i.e. M_f is measurable).*

If $(\Omega, \mathbb{P})$ is decomposable, then there exists $y \in \mathcal{Y}$ such that the slice of f by y has a manipulability rate that is at most as high as f:

$$\rho_{\mathbb{P}}(f_y) \leq \rho_{\mathbb{P}}(f).$$

Proof. For any $y \in \mathcal{Y}$, lemma III.9 ensures that M_{f_y} is measurable. We have:

$$\rho_{\mathbb{P}}(f_y) = \int_{\omega \in \Omega} M_{f_y}(\omega) \mathbb{P}(d\omega).$$

Since $(\Omega, \mathbb{P})$ is decomposable, we have by substitution:

$$\rho_{\mathbb{P}}(f_y) = \int_{(r,z) \in R(\Omega) \times \mathcal{Y}} M_{f_y}(z(r))(\mu \times \nu)(dr, dz).$$

Fubini-Tonelli theorem gives:

$$\rho_{\mathbb{P}}(f_y) = \int_{r \in R(\Omega)} \left(\int_{z \in \mathcal{Y}} M_{f_y}(z(r)) \nu(dz) \right) \mu(dr).$$

And, since $M_{f_y}(z(r))$ does not depend on z:

$$\rho_{\mathbb{P}}(f_y) = \int_{r \in R(\Omega)} M_{f_y}(y(r)) \mu(dr).$$

Now, let us study the manipulability of f. Using decomposability and Fubini-Tonelli theorem again, we have:

$$\rho_{\mathbb{P}}(f) = \int_{y \in \mathcal{Y}} \left(\int_{r \in R(\Omega)} M_f(y(r)) \mu(dr) \right) \nu(dy).$$

Since (lemma III.10) we have $M_f(y(r)) \geq M_{f_y}(y(r))$, we then deduce:

$$\begin{aligned} \rho_{\mathbb{P}}(f) &\geq \int_{y \in \mathcal{Y}} \left(\int_{r \in R(\Omega)} M_{f_y}(y(r)) \mu(dr) \right) \nu(dy) \\ &\geq \int_{y \in \mathcal{Y}} \rho_{\mathbb{P}}(f_y) \nu(dy). \end{aligned}$$

So, the manipulability rate of f is no less than the average of the manipulability rates of the slices f_y. Hence the result.

□

Corollary III.12 (slicing for independent voters)**.** *Let* $(\Omega, \mathbb{P})$ *be a PES and* f *an SBVS whose manipulability rate is well defined (i.e.* M_f *is measurable).*

If the voters $(\omega_1, \ldots, \omega_n)$ *are independent, then there exists* $y \in \mathcal{Y}$ *such that* $\rho_{\mathbb{P}}(f_y) \leq \rho_{\mathbb{P}}(f)$*.*

Corollary III.13 (slicing for Condorcet voting systems)**.** *Let* $(\Omega, \mathbb{P})$ *be a PES and* f *an SBVS whose manipulability rate is well defined (i.e.* M_f *is measurable).*

If $(\Omega, \mathbb{P})$ *is decomposable and if* f *meets the Condorcet criterion, then there exists an SBVS* f' *such that:*
- $\rho_{\mathbb{P}}(f') \leq \rho_{\mathbb{P}}(f)$*;*
- f' *depends only on the binary relations of preference of the voters;*
- f' *meets the Condorcet criterion.*

III.D Combining Condorcification and slicing theorems

Theorem III.14 (Condorcification and slicing)**.** *Let* $(\Omega, \mathbb{P})$ *be a PES and* f *an SBVS (such that* M_f *and* M_{f^c} *are measurable).*

We assume that:
- $(\Omega, \mathbb{P})$ *is decomposable;*
- f *meets the informed majority coalition criterion.*

Then there exists an SBVS f' *such that:*
- $\rho_{\mathbb{P}}(f') \leq \rho_{\mathbb{P}}(f)$*;*
- f' *depends only on the binary relations of preference;*
- f' *meets the Condorcet criterion.*

Proof. By Condorcification theorem II.9, we know that $\rho_{\mathbb{P}}(f^c) \leq \rho_{\mathbb{P}}(f)$. Applying slicing corollary III.13 to f^c, we have a suitable f'. □

So, if the PES is decomposable and if we look for a voting system that is as little manipulable as possible among those meeting the informed majority coalition criterion, we can restrict our research to those of the following kind: each voters gives only her binary relation of preference (not more information), and the voting system meets the Condorcet criterion.

We are going to formalize that now.

Proposition III.15 (existence of an optimal Condorcet voting system)**.** *Let* $\Omega = {\mathcal{R}_m}^n$ *be the electoral space of the binary relations for* n *and* m*. Let* μ *be a probability measure over* Ω*.*

Then, there exists an SBVS g *such that:*
- g *meets the Condorcet criterion;*
- *And for any SBVS* g' *meeting the Condorcet criterion,* $\rho_\mu(g) \leq \rho_\mu(g')$*.*

We say that g *is* ρ_μ-optimal among preferential Condorcet voting systems.

Proof. There are a finite number of functions $g : \Omega \to \mathcal{C}_m$, *a fortiori* if we demand that they meet the Condorcet criterion. So, as least one of them minimizes $\rho_\mu(g)$. □

Theorem III.16 (general optimality of a preferential Condorcet voting system)**.** *Let $(n, m, \Omega, \tau_\Omega, R, \mathbb{P})$ be a PES. As usual, we note μ the law of R.*

In the electoral space of binary relations for n and m, let $g : {\mathcal{R}_m}^n \to \mathcal{C}_m$ be an SBVS that is ρ_μ-optimal among preferential Condorcet voting systems.

If $(\Omega, \mathbb{P})$ is decomposable, then for any SBVS $f : \Omega \to \mathcal{C}_m$ meeting the informed majority coalition criterion (and such that M_f and M_{f^c} are measurable), $g \circ R$ is at most as manipulable as f:

$$\rho_{\mathbb{P}}(g \circ R) \leq \rho_{\mathbb{P}}(f).$$

Proof. From lemma III.8, we know that $M_{g \circ R}$ is measurable. Hence, $\rho_{\mathbb{P}}(g \circ R)$ is well defined.

By theorem III.14 (condorcification and slicing), we know that there exists an SBVS f' that depends only on $R(\omega)$, meets the Condorcet criterion, and such that $\rho_{\mathbb{P}}(f') \leq \rho_{\mathbb{P}}(f)$.

And since g is ρ_μ-optimal, we know that $\rho_{\mathbb{P}}(g \circ R) \leq \rho_{\mathbb{P}}(f')$. □

In other words, g is optimal, not only among voting systems that depend only on binary relations and that meet the Condorcet criterion, but among the larger class of all voting systems that meet the informed majority coalition criterion and that may depend on more information that the binary relations of preference of the voters.

Finally, when the PES is decomposable, there exists, among the voting systems that meet the informed majority coalition criterion, one system or more that minimize the manipulability rate. Among these, there exists at least one that is only based on binary relations of preference and that meets the Condorcet criterion.

As a consequence, in order to find a voting system that minimizes the manipulability rate (among those meeting the informed majority coalition criterion), we can restrict our investigation to those that depend only on the binary relations of preference and that meet the Condorcet criterion.

Conclusion

We have proved that when the probabilized electoral space is *decomposable*, which is in particular true when voters are independent, for any voting system f, at least one of its *slices* is at most as manipulable as f. A slice of f depends only on binary relations of preference and when f meets the Condorcet criterion, it does also.

Combined with Condorcification theorem from Durand et al. (2014) and Green-Armytage et al. (2014), this proves that when searching for a voting system whose manipulability is minimal among those meeting the informed majority coalition criterion, one may restrict to those that depend only on binary relations of preference and that meet the Condorcet criterion.

Since there are a finite number of such systems, this also proves that in any decomposable culture, there exists a voting system whose manipulability is minimal among those meeting the informed majority coalition criterion.

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A Notations

Electoral space	
$c \succ_r d$	Pair (c, d) belongs to binary relation r.
$n \in \mathbb{N} \setminus \{0\}$	Number of voters.
$\mathcal{V}_n$	Set $\{1, \ldots, n\}$ of the indexes of the voters.
$m \in \mathbb{N} \setminus \{0\}$	Number of candidates.
$\mathcal{C}_m$	Set $\{1, \ldots, m\}$ of the indexes of the candidates.
$\mathcal{R}_m$	Set of binary relations over $\mathcal{C}_m$.
$\mathcal{R}$	Set ${\mathcal{R}_m}^n$, used to represent the binary relations of all voters.
Ω_i	Set of possible states for voter i.
$\omega_i \in \Omega_i$	State of voter i.
R_i	Measurable function $\Omega_i \to \mathcal{R}_m$ that extracts, from voter i's state ω_i, her binary relation of preference r_i.
R	Measurable function $\Omega \to \mathcal{R}$ that extracts, from the state ω of the whole population, the n-tuple $(R_1(\omega_1), \ldots, R_n(\omega_n))$.
Ω	Universe $\prod_{i \in \mathcal{V}_n} \Omega_i$ of possible states for the whole population of voters. Also used as a shortcut for electoral space (n, m, Ω, R) or measurable electoral space $(n, m, \Omega, \tau_\Omega, R)$.
Probabilized electoral space	
τ_E	A sigma-algebra over a set E.
$\mathbb{P}$	A probability measure over universe (Ω, τ_Ω).
$\mathbb{P}(A\|B)$	Conditional probability of event A knowing B.
$(\Omega, \mathbb{P})$	Shortcut for probabilized electoral space $(n, m, \Omega, \tau_\Omega, R, \mathbb{P})$.
μ	Probability law of R (under probability measure $\mathbb{P}$).
$\mathcal{Y}_i$	Set $\{y_i : R_i(\Omega_i) \to \Omega_i \text{ s.t. } R_i \circ y_i = \text{Id}\}$ of right inverses of R_i.
$\mathcal{Y}$	Set $\prod_{i \in \mathcal{V}_n} \mathcal{Y}_i$.
ν	A probability measure over space $\mathcal{Y}$.
Voting system	
f	A state-based voting system (SBVS) $f : \Omega \to \mathcal{C}_m$.
f^c	Condorcification of f.
f_y	Slice of f by y.
M_f	Manipulability indicator $\Omega \to \{0, 1\}$ of voting system f.
$\rho_{\mathbb{P}}(f) \in [0, 1]$	Manipulability rate of voting system f in culture $\mathbb{P}$.

B Decomposability

One dimension

We now prove the lemma of the *complementary random variable*, which shows that a decomposition like in definition III.3 is always possible when there is only one voter.

We have a probabilized space $(\Omega, \tau_\Omega, \mathbb{P})$ and a random variable X with values in a finite measurable set $(\mathcal{X}, \tau_\mathcal{X})$ endowed with the discrete measure. We denote μ the law of X.

When we realize the random experience, the state of the system is described by ω. The value $x = X(\omega)$ is a partial information about this state: if we know only x, we generally lack some information about ω. Let us imagine that there exists a space $\mathcal{Y}$ that allows to express this additional piece of information: then, the state can be expressed as the pair $\omega = (x, y)$.

In that case, we may consider y as a function from $\mathcal{X}$ to Ω: if one knows x, then y allows to reconstitute state ω in its whole.

Let us imagine, moreover, that random variables x and y are independent: this would allow to express the universe Ω as the product of probabilized sets $\mathcal{X}$ and $\mathcal{Y}$. It would be a considerable asset, as we have seen in the proof of slicing theorem III.11.

The construction that we will consider is a generalization of this notion of complementary information. Indeed, we have a very important freedom: we can choose the set $\mathcal{Y}$. For the sake of generality, we will always choose the set of functions $y : X(\Omega) \to \Omega$ that are coherent with R, in the sense that $R \circ y = \mathrm{Id}$. Indeed, it is the general framework so that giving an x an a y perfectly defines a state ω that is coherent with r.

Lemma B.1 (complementary random variable)**.** *Let $(\Omega, \tau_\Omega, \mathbb{P})$ be a probabilized set and X a random variable, i.e. a measurable function from Ω to a measurable set $(\mathcal{X}, \tau_\mathcal{X})$. We note μ the law of X.*

We note $\mathcal{Y} = \{y : X(\Omega) \to \Omega \text{ s.t. } X \circ y = Id\}$.

We assume that $(\mathcal{X}, \tau_\mathcal{X})$ is finite and endowed with the discrete measure.

Then there exists a measure ν over $\mathcal{Y}$ such that:

$$\forall A \in \tau_\Omega, \mathbb{P}(A) = (\mu \times \nu)\{(x, y) \in X(\Omega) \times \mathcal{Y} \text{ s.t. } y(x) \in A\}.$$

Proof. For any $x \in X(\Omega)$:

- If $\mathbb{P}(X = x) > 0$, we note $\mathbb{P}_x$ the measure of conditional probability knowing $X = x$ (restricted to $X^{-1}(x)$);
- If $\mathbb{P}(X = x) = 0$, we choose[7] an arbitrary $\omega_x \in X^{-1}(x)$ and we note $\mathbb{P}_x$ the probability measure that surely returns ω_x.

Identifying a function $y \in \mathcal{Y}$ to the list of its values for each possible argument x, we define ν as the product measure of all $\mathbb{P}_x$.

7. There is no "axiom of choice" issue because we do a finite number of choices.

Then for $A \in \tau_\Omega$:

$$\begin{aligned}(\mu \times \nu)\{(x,y) \text{ s.t. } y(x) \in A\} &= \sum_{x \in X(\Omega)} \mu\{x\} \cdot \nu\{y(x) \in A\} \\ &= \sum_{\mathbb{P}(X=x)>0} \mathbb{P}(X=x) \cdot \mathbb{P}(A|X=x) \\ &= \mathbb{P}(A).\end{aligned}$$

We thank Anne-Laure Basdevant and Arvind Singh for fruitful discussions about this lemma. □

Several dimensions

Now, we deal with decomposability in the general case.

Let $n \in \mathbb{N} \setminus \{0\}$. For each $i \in \{1, \dots, n\}$, let $(\Omega_i, \tau_{\Omega_i})$ be a measurable set, $(\mathcal{X}_i, \tau_{\mathcal{X}_i})$ a finite set endowed with the discrete measure and X_i a measurable function:

$$X_i : \left| \begin{array}{lcl} \Omega_i & \to & \mathcal{X}_i \\ \omega_i & \to & X_i(\omega_i). \end{array} \right.$$

Let $(\Omega, \tau_\Omega) = \prod_{i=1}^n (\Omega_i, \tau_{\Omega_i})$ and $(\mathcal{X}, \tau_\mathcal{X}) = \prod_{i=1}^n (\mathcal{X}_i, \tau_{\mathcal{X}_i})$ the product measurable sets, endowed with their product sigma-algebras.

The universe (Ω, τ_Ω) is endowed[8] with a probability measure $\mathbb{P}$. We note X the random variable that, to $\omega = (\omega_1, \dots, \omega_n)$, associates $(X_1(\omega_1)), \dots, X_n(\omega_n))$. We note μ the law of X.

For each $i \in \{1, \dots, n\}$, we note $\mathcal{Y}_i = \{y_i : X_i(\Omega_i) \to \Omega_i \text{ s.t. } X_i \circ y_i = \text{Id}\}$ the set of right inverses of X_i, endowed with its canonical sigma-algebra (defined as in notation III.1). We note $(\mathcal{Y}, \tau_\mathcal{Y}) = \prod_{i=1}^n (\mathcal{Y}_i, \tau_{\mathcal{Y}_i})$ the product measurable space, endowed with the product measure.

For $x = (x_1, \dots, x_n) \in X(\Omega)$ and $y = (y_1, \dots, y_n) \in \mathcal{Y}$, we note $y(x) = (y_1(x_1), \dots, y_n(x_n)) \in \Omega$.

Definition B.2 (decomposability in general)**.** We say that $(\Omega, \mathbb{P})$ is X-*decomposable* iff there exists a measure ν over $\mathcal{Y}$ such that:

$$\forall A \in \tau_\Omega, \mathbb{P}(A) = (\mu \times \nu)\{(x,y) \in X(\Omega) \times \mathcal{Y} \text{ s.t. } y(x) \in A\}.$$

The difficulty comes from our demand for complementary random variables y_i that are *individual*: y may not be any function from $X(\Omega)$ to Ω, but a n-tuple of functions, where each y_i is from $X_i(\Omega_i)$ to Ω_i. Indeed, we need individual random variables for the proof of lemma III.10.

If we asked a *collective* random variable y that, from x, allows to reconstitute ω with the correct probability law, it would always be possible, as a direct consequence of lemma B.1.

8. It should be noticed that the ω_i's may not be independent (the same is true for the X_i's).

Proposition B.3 (independence implies decomposability)**.** *If the random variables* $(\omega_1, \ldots, \omega_n)$ *are independent, then* $(\Omega, \mathbb{P})$ *is* X*-decomposable.*

Proof. Simply apply lemma B.1 for each $i \in \{1, \ldots, n\}$, which defines a measure ν_i over each set $\mathcal{Y}_i$. Then, define ν as the product measure of the ν_i's. □

Proposition B.4 (a necessary condition for decomposability)**.** *If* $(\Omega, \mathbb{P})$ *is* X*-decomposable then, for each set of indexes* $V \in \mathcal{P}(\{1, \ldots, n\})$*, for each* $(A_i)_{i \in V} \in \prod_{i \in V} \tau_{\Omega_i}$*, for each* $x = (x_1, \ldots, x_n)$ *of positive probability:*

$$\mathbb{P}\big((\forall i \in V, \omega_i \in A_i) | X = x\big) = \mathbb{P}\big((\forall i \in V, \omega_i \in A_i) | \forall i \in V, X_i = x_i\big). \quad (3)$$

Proof. On one hand:

$$\begin{aligned}\mathbb{P}\big((\forall i \in V, \omega_i \in A_i) | X = x\big) &= (\mu \times \nu)\big((\forall i \in V, Y_i(x_i) \in A_i) | X = x\big) \\ &= \nu(\forall i \in V, Y_i(x_i) \in A_i).\end{aligned}$$

On the other hand, it is easy to show similarly that:

$$\mathbb{P}\big((\forall i \in V, \omega_i \in A_i) | \forall i \in V, X_i = x_i\big) = \nu(\forall i \in V, Y_i(x_i) \in A_i).$$

□

However, condition (3) does not ensure that $(\Omega, \mathbb{P})$ is X-decomposable.

As a counterexample, let us take $n = 2$. The variable ω_i (with $i = 1$ or $i = 2$) may take 4 values, noted $\omega_i^{(1)}$ to $\omega_i^{(4)}$. The variable X_i may take 2 values, $x_i^{(a)}$ and $x_i^{(b)}$. The following table gives the correspondence between the ω_i's and the x_i's, as well as measure $\mathbb{P}$.

$\mathbb{P}$	$\omega_1^{(1)} \to x_1^{(a)}$	$\omega_1^{(2)} \to x_1^{(a)}$	$\omega_1^{(3)} \to x_1^{(b)}$	$\omega_1^{(4)} \to x_1^{(b)}$
$\omega_2^{(1)} \to x_2^{(a)}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	0
$\omega_2^{(2)} \to x_2^{(a)}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{8}$
$\omega_2^{(3)} \to x_2^{(b)}$	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$\omega_2^{(4)} \to x_2^{(b)}$	0	$\frac{1}{8}$	$\frac{1}{8}$	0

It is easily checked that condition (3) is met. Indeed, for example:

$$\mathbb{P}(\omega_1^{(1)} | x_1^{(a)} \ \& \ x_2^{(a)}) = \frac{1}{2} = \mathbb{P}(\omega_1^{(1)} | x_1^{(a)}).$$

Now, let us suppose that $(\Omega, \mathbb{P})$ is X-decomposable. Let ν be a suitable measure for this decomposition. For $(\alpha, \beta, \gamma, \delta) \in \{1, 2\} \times \{3, 4\} \times \{1, 2\} \times \{3, 4\}$, let us note:

$$\nu(\alpha, \beta, \gamma, \delta) = \nu\left(y_1(x_1^{(a)}) = \omega_1^{(\alpha)} \ \& \ y_1(x_1^{(b)}) = \omega_1^{(\beta)} \ \& \ y_2(x_2^{(a)}) = \omega_2^{(\gamma)} \ \& \ y_2(x_2^{(b)}) = \omega_2^{(\delta)}\right).$$

We notice the following facts.

- $0 = \mathbb{P}(\omega_1^{(3)} \text{ \& } \omega_2^{(3)} | x_1^{(b)} \text{ \& } x_2^{(b)}) = \sum_{(\alpha,\gamma)} \nu(\alpha, 3, \gamma, 3)$. Since all terms in this sum are nonnegative, we have in particular $\nu(1, 3, 1, 3) = 0$.
- $0 = \mathbb{P}(\omega_1^{(1)} \text{ \& } \omega_2^{(4)} | x_1^{(a)} \text{ \& } x_2^{(b)}) = \sum_{(\beta,\gamma)} \nu(1, \beta, \gamma, 4)$ hence $\nu(1, 3, 1, 4) = 0$ and $\nu(1, 4, 1, 4) = 0$.
- $0 = \mathbb{P}(\omega_1^{(4)} \text{ \& } \omega_2^{(1)} | x_1^{(b)} \text{ \& } x_2^{(a)}) = \sum_{(\alpha,\delta)} \nu(\alpha, 4, 1, \delta)$ hence $\nu(1, 4, 1, 3) = 0$.

So, $\frac{1}{4} = \mathbb{P}(\omega_1^{(1)} \text{ \& } \omega_2^{(1)} | x_1^{(a)} \text{ \& } x_2^{(a)}) = \sum_{(\beta,\delta)} \nu(1, \beta, 1, \delta) = \nu(1, 3, 1, 3) + \nu(1, 3, 1, 4) + \nu(1, 4, 1, 3) + \nu(1, 4, 1, 4) = 0$: this contradiction proves that $(\Omega, \mathbb{P})$ is not decomposable.

In fact, in this counterexample, it can be shown that there exists a *signed measure* ν that meets the usual relations for decomposition. However, it is not enough for us: indeed, the proof of slicing theorem III.11 uses the growth property of integration, which is based on its positivity property.

C Attempts of generalization for the slicing theorem

Remark C.1 (failure of generalization to inclusion)**.** In slicing theorem III.11, instead of having a voting system $f' = f_y$ such that $\rho_{\mathbb{P}}(f') \leq \rho_{\mathbb{P}}(f)$ for a specific culture $\mathbb{P}$, it would be stronger to have an inclusion:

$$\{\omega \in \Omega \text{ s.t. } f' \text{ is manipulable in } \omega\} \subseteq \{\omega \in \Omega \text{ s.t. } f \text{ is manipulable in } \omega\}.$$

Indeed, this would lead to decrease the manipulability rate for any culture $\mathbb{P}$.

Since this generalization seems difficult, we will strengthen our assumptions and weaken our demands.

- We assume that f meets the Condorcet criterion.
- We still demand that f' depends only on binary relations of preference.
- We only demand that f' meets the informed majority coalition criterion (which is a weaker requirement than f' being a slice of f, given our assumption that f meets the Condorcet criterion).

Unfortunately, we will prove that such an f' does not always exist.

First, we remark that if such an f' exists, then by Condorcification theorem II.9, we can demand that f' meets the Condorcet criterion.

Let us consider $n = 3$ voters and $m = 3$ candidates A,B,C. The electoral space is defined as follows. The state of each voter is given by a strict order of preference and a bit with value 0 or 1. Here is the voting system f that we consider:

- If there is a Condorcet winner, she is elected;
- If there is none, if at least two bits are 1 then C is elected, otherwise (i.e. if at least two bits are 0) B is elected.

Now, let us consider the three following situations.

Situation	Voter			Condorcet winner
	1	2	3	
ω	A C , 0 B	B C , 1 A	C A , 1 B	C
ϕ	A B , 1 C	B A , 0 C	C A , 1 B	A
ψ	A B , 0 C	B C , 0 A	C B , 1 A	B

f is not manipulable in ω. On one hand, to manipulate in favor of B, only the second voter is interested, the others stay sincere. But then, B stays Condorcet loser so, since there are three victories in the matrix of duels, there is a Condorcet winner (who is not B). On the other hand, to manipulate in favor of A, the first voter may try to make the situation appear as without Condorcet winner. But then, there will always be two bits equal to 1, so the winner will be C and the manipulation will fail.

Similarly, it can be shown that f is manipulable neither in ϕ nor in ψ.

Now, if f' depends only on binary relations of preference and meets the Condorcet criterion, let us consider the following family of situations χ, where the bits of each voters do not matter for f'.

Situation	Voter			Condorcet winner
	1	2	3	
χ	A B C	B C A	C A B	None

If $f'(\chi) = $ A (resp. B, C), then f' is manipulable in ω (resp. ϕ, ψ) towards χ. Hence, f' must be manipulable in at least one of the three situations ω, ϕ, ψ, so f' can not meet the desired properties.

So, it seems that in general, we will not have better than $\rho_{\mathbb{P}}(f') \leq \rho_{\mathbb{P}}(f)$ for a specific culture $\mathbb{P}$.

Remark C.2 (decomposability is important)**.** That said, we may ask whether slicing theorem III.11 holds true when removing the condition of decomposability (and without another assumption to replace it). But it does not.

Indeed, consider the example of previous remark and the probability measure $\mathbb{P}(\omega) = \mathbb{P}(\phi) = \mathbb{P}(\psi) = \frac{1}{3}$. Then, the initial voting system f has a manipulability rate equal to 0 (f is manipulable in some situations, but they come with zero probability). However, any SBVS f' that depends only on binary relations of preference and that meets the informed majority coalition criterion has a manipulability rate no lower than $\frac{1}{3}$.

Remark C.3 (finding a weaker condition than decomposability)**.** At last, we may wonder whether slicing theorem III.11 holds true with an assumption that is weaker than decomposability, for example the one presented in proposition III.7. We do not know if it is the case, and that is an interesting lead for future work.

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