α-junctions of categorical mass functions

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Abstract

The set of α-junctions is the set of linear associative and commutative combination operators for belief functions. Consequently, the properties of α-junctive rules make them particularly attractive on a theoretic point of view. However, they are rarely used in practice except for the $\alpha = 1$ case which corresponds to the widely used and well understood conjunctive and disjunctive rules. The lack of success of α-junctions when $\alpha < 1$ is mainly explained by two reasons. First, they require a greater computation load due to a more complex mathematical definition. Second, the mass function obtained after combination is hard to interpret and sometimes counter-intuitive. Pichon and Denœux [4] brought a significant contribution to circumvent both of these two limitations. In this article, it is intended to pursue these efforts toward a better understanding of α-junctions. To that end, this study is focused on the behavior of α-junctions when categorical mass functions are used as entries of an α-junctive combination rule. It is shown that there exists a conjunctive and a disjunctive canonical decomposition of the mass function obtained after combination.

1 Introduction

The belief function theory (BFT) is an appealing framework for reasoning under uncertainty when imperfect data need to be aggregated through an information fusion process. Indeed, imprecise and uncertain pieces of evidence can be efficiently represented and aggregated as part of the BFT. Combination rules are well-defined mathematical operators designed for such a purpose.

In [9], Smets introduced a family of combination rules known as α-junctions. This family is the union of two sub-families: the α-conjunctive rules and the α-disjunctive rules. These rules possess interesting properties, each of them being clearly justified in an information fusion context. When the parameter α is set to 1, two classical rules are retrieved: the conjunctive and disjunctive rules. However, for other values of α, performing the combination requires an increased computation time and the results are sometimes hard to interpret.

Pichon and Denœux [4] alleviated these drawbacks. First, they explained that combination results are far better understood if α is viewed as a parameter related to the truthfulness of information sources. In addition, they provided means to fasten α-junction computations.

Besides, it is known that the BFT restricted to categorical mass functions equipped with the conjunctive and disjunctive rules boils down to Cantor's set theory. In this article, it is intended to analyze the same matter when the conjunctive and disjunctive rules are replaced with α-junctions for a given $\alpha < 1$. Pichon already briefly discussed this matter in [6]. Some additional results or analyses are given for the direct computation of combined categorical mass functions as well as for other set-functions representing combined evidence (commonality, implicability, conjunctive and disjunctive weight functions). In addition, a conjunctive and a disjunctive canonical decomposition of these mass functions are also introduced. In section 2 some mathematical notations are given and some definitions are re-called. Section 3 and 4 present the obtained results for α-conjunctive and α-disjunctive rules respectively. Section 5 concludes the paper.

2 Belief function framework: notations and definitions

In this section, mathematical notations for classical belief function concepts are given. The reader is expected to be familiar with belief function basics and consequently some definitions are not recalled. More material on belief functions basics is found for instance in [1]. A greater stress is given to a reminder on α-junctions.

2.1 Belief function basics

Suppose one has collected several bodies of evidence $\{Ev_i\}_{i=1}^{M}$. For a given body of evidence Ev_i, the corresponding **mass function** representing this piece of evidence is denoted by m_i. Mass functions are set-functions with respect to a **frame of discernment** denoted by Ω. The power set 2^Ω is the set of all subsets of Ω and it is the domain of mass functions. For any $A \in 2^\Omega$, the **cardinality** of this set is denoted by $|A|$ and $|\Omega| = n$. The cardinality of 2^Ω is denoted by $N = 2^n$. Mass functions have $[0,1]$ as codomain and they sum to one. A **focal element** of a mass function m_i is a set $A \subseteq \Omega$ such that $m_i(A) > 0$. A mass function having only one focal element A is called a **categorical mass** function and it is denoted by m_A.

Several alternatives for evidence representation are commonly used in the BFT. The **belief** and **commonality** functions bel_i and q_i are respectively the inverse Möbius and inverse co-Möbius transforms of the mass function m_i. The **plausibility** function pl_i is the conjugate of bel_i and the **implicability** function b_i is such that $\forall X \subseteq \Omega$, $b_i(X) = bel_i(X) + m_i(\emptyset)$. There is a one-to-one correspondence between a mass function m_i and any of these four functions.

If the reliability of the evidence encoded in a mass function can be evaluated through a coefficient $\alpha \in [0,1]$, then a so-called **discounting** operation on m can be performed. A discounted mass function is denoted by m^α and we have :

$$m^\alpha = (1-\alpha)m + \alpha m_\Omega. \tag{1}$$

α is called the **discounting rate**. Since m_Ω represents a state of ignorance, this categorical mass function is called the **vacuous** mass function. Consequently, setting $\alpha = 1$ turns a mass function into the neutral element of the conjunctive rule and its corresponding evidence is discarded from further processing.

Another useful concept is the **negation** $\overline{m}$ of a mass function m. The function $\overline{m}$ is such that $\forall A \subseteq \Omega$, $\overline{m}(A) = m(\overline{A})$ with $\overline{A} = \Omega \setminus A$.

2.2 Mass function combination using α-junctions

In this subsection, a brief presentation of α-junctions is proposed. A thorough presentation is provided in [4]. Suppose f is a combination operator for mass functions, *i.e.*, $m_{12} = f(m_1, m_2)$ with m_{12} a mass function depending only on two initial mass functions m_1 and m_2. Such an operator is an ***α*-junction** if it possesses the following properties [9]:

- Linearity[1]: $\forall \lambda \in [0,1]$ and for any other mass function m we have $f(m, \lambda m_1 + (1-\lambda) m_2) = \lambda f(m, m_1) + (1-\lambda) f(m, m_2)$,
- Commutativity: $f(m_1, m_2) = f(m_2, m_1)$,
- Associativity: $f(f(m_1, m_2), m_3) = f(m_1, f(m_2, m_3))$,
- Neutral element: $\exists m_e \mid \forall \mathrm{m}, f(m, m_e) = m$,
- Anonymity: for any σ extending by set-union on 2^Ω a permutation on Ω, $f(m_1 \circ \sigma, m_2 \circ \sigma) = m_{12} \circ \sigma$,
- Context preservation: $pl_1(X) = 0$ and $pl_2(X) = 0 \implies pl_{12}(X) = 0$.

The justifications behind these properties are given in [9]. In the same article, Smets also proves that the neutral element can be either $m_\emptyset$ or m_Ω. Depending on this, two sub-families arise: the α-disjunctive rules denoted by $\textcircled{\cup}^\alpha$ and the α-conjunctive rules denoted by $\textcircled{\cap}^\alpha$. For the sake of clarity, the following notations will be used: $m_{1\cup^\alpha 2} = m_1 \textcircled{\cup}^\alpha m_2$ and $m_{1\cap^\alpha 2} = m_1 \textcircled{\cap}^\alpha m_2$. Pichon and Denœux [4] provided the following computation formulae: $\forall X \subseteq \Omega, \forall \alpha \in [0,1]$

$$m_{1\cap^\alpha 2}(X) = \sum_{(A\cap B)\cup(\overline{A}\cap\overline{B}\cap C)=X} m_1(A)\, m_2(B)\, \alpha^{|\overline{C}|}\overline{\alpha}^{|C|}, \tag{2}$$

$$m_{1\cup^\alpha 2}(X) = \sum_{(A\cap\overline{B})\cup(\overline{A}\cap B)\cup(A\cap B\cap C)=X} m_1(A)\, m_2(B)\, \alpha^{|C|}\overline{\alpha}^{|\overline{C}|}, \tag{3}$$

with $\overline{\alpha} = 1 - \alpha$. Note that they also provide faster means to compute the combined mass function using matrix calculus. It is also noteworthy that, if $\alpha = 1$, the classical conjunctive and disjunctive rules are retrieved. We denote these rules by $\textcircled{\cap} = \textcircled{\cap}^1$ and by $\textcircled{\cup} = \textcircled{\cup}^1$.

Concerning the interpretation of α-junctions, Pichon and Denœux [4] state that for any $\omega \in \Omega$:

[1] The operator is linear on the vector space spanned by categorical mass functions but the output of the operator remains a mass function only in case of convex combination.

- for α-conjunctions, α is understood as the belief that at least one of the sources tells the truth, given that the event $\{\omega\}$ is true,
- for α-disjunctions, α is understood as the plausibility that both sources tell the truth, given that the event $\{\omega\}$ is true.

In [6], Pichon gives further explanations and justifications of this interpretation. He shows that α-conjunctions are understood as a particular case of a combination process introduced in [7] where meta-knowledge on the truthfulness of information sources is formalized.

3 α-conjunctive combination of categorical mass functions

In this section, several results related to the combination of categorical mass functions using an α-conjunctive rule are given. A straightforward formula for the computation of α-conjunction of categorical mass functions is evoked in [6]. We state this result in a slightly more formal way:

Proposition 1. *Let A and $B \subseteq \Omega$. $\forall X \subseteq \Omega$, one has:*

$$m_{A \cap^\alpha B}(X) = \begin{cases} \alpha^{|\overline{A \Delta B}| - |X|} \overline{\alpha}^{|X| - |A \cap B|} & \text{if } A \cap B \subseteq X \subseteq \overline{A \Delta B} \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

with Δ the set symmetric difference.

Proof. A sketch of proof is already given in [6]. We provide a few more details in here. Applying equation (2) with categorical mass functions gives:

$$m_{A \cap^\alpha B}(X) = \sum_{\substack{C \subseteq \Omega \\ (A \cap B) \cup (\overline{A} \cap \overline{B} \cap C) = X}} \alpha^{|\overline{C}|} \overline{\alpha}^{|C|}$$

Observing that no subset C can satisfy $(A \cap B) \cup (\overline{A} \cap \overline{B} \cap C) = X$ unless $A \cap B \subseteq X \subseteq \overline{A \Delta B}$ accounts for the two seperate cases in equation (4) depending on the condition $A \cap B \subseteq X \subseteq \overline{A \Delta B}$.
Suppose $A \cap B \subseteq X \subseteq \overline{A \Delta B}$ is true. Let $C_1 = C \cap (A \cup B)$ and $C_2 = C \cap (\overline{A \cup B})$. Since $A \cup B$ together with $\overline{A \cup B}$ is a partition of Ω, one has $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = \emptyset$. Observing that $(A \cap B) \cup (\overline{A} \cap \overline{B} \cap C) = (A \cap B) \cup C_2 = X \implies C_2 = X \setminus (A \cap B)$, we deduce that choosing C is tantamount to choosing C_1 which lives in $2^{A \cup B}$. This gives:

$$\begin{aligned} m_{A \cap^\alpha B}(X) &= \sum_{C_1 \subseteq A \cup B} \alpha^{|\overline{C_1 \cup X \setminus (A \cap B)}|} \overline{\alpha}^{|C_1 \cup X \setminus (A \cap B)|} \\ &= \alpha^{n - |X| + |A \cap B|} \overline{\alpha}^{|X| - |A \cap B|} \sum_{C_1 \subseteq A \cup B} \left({}^{\overline{\alpha}}/_{\alpha} \right)^{|C_1|} \\ &= \alpha^{n - |X| + |A \cap B|} \overline{\alpha}^{|X| - |A \cap B|} \left({}^{\overline{\alpha}}/_{\alpha} + 1 \right)^{|A \cup B|} \\ &= \alpha^{|\overline{A \Delta B}| - |X|} \overline{\alpha}^{|X| - |A \cap B|}. \quad \square \end{aligned}$$

□

Figure 1 illustrates the variety of potential focal sets of mass function $m_{A \cap^\alpha B}$.

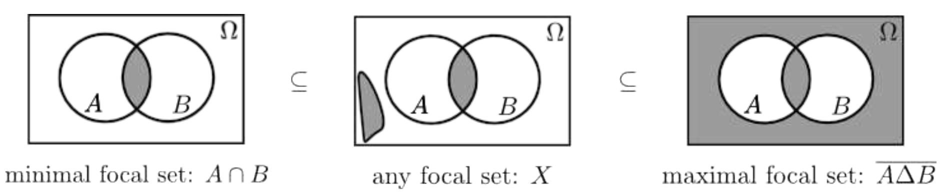

Figure 1: Example of focal sets of mass function $m_{A \cap^\alpha B}$.

First It can be noted that according to proposition 1:

$$A \cup B = \Omega \implies m_{A \cap^\alpha B} = m_{A \cap B}.$$

When $A \cup B \neq \Omega$, proposition 1 also sheds light on the fact that the α-conjunction of two deterministic sets yields a random set[2] [3]. This means that some latent uncertainty has been unveiled by the combination process and that this uncertainty is not encoded in the initial mass functions. Following the interpretation of Pichon and Denœux, the uncertainty observed in $m_{A\cap^\alpha B}$ comes from the uncertainty on the truthfulness of the sources. This uncertainty is expressed on another frame of discernment Θ and $m_{A\cap^\alpha B}$ is the marginal of a broader mass function on $\Omega \times \Theta$.

Let us now introduce some results on the commonality function $q_{A\cap^\alpha B}$ and a canonical conjunctive decomposition of $m_{A\cap^\alpha B}$.

Proposition 2. *Let A and $B \subseteq \Omega$. $\forall X \subseteq \Omega$, one has:*

$$q_{A\cap^\alpha B}(X) = \begin{cases} \overline{\alpha}^{|X\setminus(A\cap B)|} & \text{if } X \subseteq \overline{A\Delta B} \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Proof. By definition of commonality function and using proposition 1, one has:

$$\begin{aligned} q_{A\cap^\alpha B}(X) &= \sum_{\substack{Y\supseteq X \\ A\cap B\subseteq Y\subseteq \overline{A\Delta B}}} \alpha^{|\overline{A\Delta B}|-|Y|}\overline{\alpha}^{|Y|-|A\cap B|}, \\ &= \sum_{(A\cap B)\cup X\subseteq Y\subseteq \overline{A\Delta B}} \alpha^{|\overline{A\Delta B}|-|Y|}\overline{\alpha}^{|Y|-|A\cap B|}. \end{aligned}$$

Observing that no subset Y can satisfy $(A\cap B)\cup X \subseteq Y \subseteq \overline{A\Delta B}$ unless $X \subseteq \overline{A\Delta B}$ accounts for the two separate cases in equation (5) depending on the condition $X \subseteq \overline{A\Delta B}$. Now if $X \subseteq \overline{A\Delta B}$, one has:

$$\begin{aligned} q_{A\cap^\alpha B}(X) &= \alpha^{|\overline{A\Delta B}|}\overline{\alpha}^{-|A\cap B|} \sum_{W\subseteq \overline{A\Delta B}\setminus((A\cap B)\cup X)} \left({}^{\overline{\alpha}}/_{\alpha}\right)^{|W\cup((A\cap B)\cup X)|}, \\ &= \alpha^{|\overline{A\Delta B}|}\overline{\alpha}^{-|A\cap B|} \sum_{W\subseteq \overline{A\cup B}\setminus X} \left({}^{\overline{\alpha}}/_{\alpha}\right)^{|W|+|(A\cap B)\cup X|}, \\ &= \alpha^{|\overline{A\Delta B}|-|(A\cap B)\cup X|}\overline{\alpha}^{-|A\cap B|+|(A\cap B)\cup X|} \sum_{W\subseteq \overline{A\cup B}\setminus X} \left({}^{\overline{\alpha}}/_{\alpha}\right)^{|W|}, \\ &= \alpha^{|\overline{A\cup B}|-|X\setminus(A\cap B)|}\overline{\alpha}^{|X\setminus(A\cap B)|}\left({}^{\overline{\alpha}}/_{\alpha}+1\right)^{|\overline{A\cup B}\setminus X|}, \\ &= \overline{\alpha}^{|X\setminus(A\cap B)|}. \quad \square \end{aligned}$$

$\square$

Proposition 3. *Let A and $B \subseteq \Omega$. $\forall X \subseteq \Omega$, one has:*

$$m_{A\cap^\alpha B} = m_{\overline{A\Delta B}} \circledcap \left(\bigcirc\!\!\!\!\cap_{y\in\overline{A\cup B}} m^{\overline{\alpha}}_{\overline{A\Delta B}\setminus\{y\}} \right). \tag{6}$$

Proof. Proving equation (6) is equivalent to proving that $q_{A\cap^\alpha B} = g$ with g a set function such that $\forall X \subseteq \Omega$:

$$g(X) = q_{\overline{A\Delta B}}(X) \prod_{y\in\overline{A\cup B}} q_{\overline{A\Delta B}\setminus\{y\}}{}^{\overline{\alpha}}(X).$$

$q^{\overline{\alpha}}_{\overline{A\Delta B}\setminus\{y\}}$ is the commonality function corresponding to $m^{\overline{\alpha}}_{\overline{A\Delta B}\setminus\{y\}}$:

$$q^{\overline{\alpha}}_{\overline{A\Delta B}\setminus\{y\}}(X) = \begin{cases} 1 & \text{if } X \subseteq \overline{A\Delta B}\setminus\{y\} \\ \overline{\alpha} & \text{otherwise} \end{cases}.$$

$q_{\overline{A\Delta B}}$ is the commonality function corresponding to $m_{\overline{A\Delta B}}$:

$$q_{\overline{A\Delta B}}(X) = \begin{cases} 1 & \text{if } X \subseteq \overline{A\Delta B} \\ 0 & \text{otherwise} \end{cases}.$$

[2]Mass functions can also be viewed as random set distributions.

If $X \not\subseteq \overline{A\Delta B}$, then $q_{\overline{A\Delta B}}(X) = 0 \implies g(X) = 0$ and consequently, given proposition 2, $q_{A\cap^\alpha B}$ and g coincide on these sets.
All other remaining sets X in 2^Ω are such that $X \subseteq \overline{A\Delta B}$. Under this assumption and given the definition of $q^{\overline{\alpha}}_{\overline{A\Delta B}\setminus\{y\}}$, one can thus write $g(X) = \overline{\alpha}^{|C|}$ with $C = \left\{ y \in \overline{A\cup B} \mid X \not\subseteq \overline{A\Delta B} \setminus \{y\} \right\} \subset \Omega$. It can be proved that $C = X \setminus (A \cap B)$ thereby proving that $q_{A\cap^\alpha B}$ and g also coincide when $X \subseteq \overline{A\Delta B}$. $\square$ $\square$

Let us first provide a toy example to better grasp the gist of proposition 3:

Example 1. *Suppose that $\Omega = \{a, b, c\}$, $A = \{a\}$ and $B = \{a, b\}$. Consequently, we have $\overline{A\Delta B} = \{a, c\}$ and $\overline{A\cup B} = \{c\}$. The mass functions before combination, those involved in the conjunctive decomposition in equation (6) as well as the output mass function $m_{A\cap^\alpha B}$ are as follows:*

subsets:	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	Ω
$m_A = m_{\{a\}}$		*1*						
$m_B = m_{\{a,b\}}$				*1*				
$m_{\overline{A\Delta B}} = m_{\{a,c\}}$						*1*		
$m^{\overline{\alpha}}_{\overline{A\Delta B}\setminus\{c\}} = m^{\overline{\alpha}}_{\{a\}}$		α						$\overline{\alpha}$
$m_{A\cap^\alpha B}$		α				$\overline{\alpha}$		

Proposition 3 could not have been anticipated by Smets' work [8] on canonical decomposition because $m_{A\cap^\alpha B}$ is dogmatic, *i.e.* $m_{A\cap^\alpha B}(\Omega) = 0$. For the same reason, the decomposition of $m_{A\cap^\alpha B}$ is not unique in Smet's sense. Nonetheless, provided that a restriction from 2^Ω to $2^{\overline{A\Delta B}}$ is performed, then uniqueness result applies. Indeed, the restriction of $m_{A\cap^\alpha B}$ to $2^{\overline{A\Delta B}}$ is a non-dogmatic mass function on the frame $\overline{A\Delta B}$ and therefore the decomposition is unique. Since there is no restriction to a greater set than $\overline{A\Delta B}$ that remains non-dogmatic, we say that this decomposition is still canonical by abuse of language. This phenomenon is also illustrated in example 1 in which $m_{A\cap^\alpha B}$ happens to be a simple mass function if defined on $2^{\overline{A\Delta B}}$.
Following notations and definitions given in [1], we define the conjunctive weight function of an α-conjunction of two categorical mass functions $w_{A\cap^\alpha B}$ as follows:

$$\forall X \subseteq \Omega,\ w_{A\cap^\alpha B}(X) = \begin{cases} 0 & \text{if } X = \overline{A\Delta B}, \\ \overline{\alpha} & \text{if } X \subsetneq \overline{A\Delta B} \text{ and } |X| = |\overline{A\Delta B}| - 1, \\ 1 & \text{otherwise }. \end{cases}$$

Conjunctive weights are interesting in the sense that they represent the elementary pieces of evidence that lead to the current state of knowledge. These weights also induce an information content related partial order for mass functions. They can also be used to define other combination rules [1, 5].
Besides, the proposed conjunctive decomposition allows the following interpretation of α-conjunctions of categorical mass functions: given $\overline{A\Delta B}$, there are $|\overline{A\cup B}|$ sources supporting with strength α respectively that any element $y \in \overline{A\cup B}$ may be discarded and all of these sources are truthful.

4 α-disjunctive combination of categorical mass functions

In this section, the dual results of those of section 3 are given for the combination of categorical mass functions using an α-disjunctive rule. Proofs are not given because they are obtained by applying the De Morgan laws [10] to results of section 3. The De Morgan laws state that for any mass functions m_1 and m_2 on a frame Ω, one has:

$$\overline{m_1 \circledcap^\alpha m_2} = \overline{m}_1 \circledcup^\alpha \overline{m}_2, \tag{7}$$
$$\overline{m_1 \circledcup^\alpha m_2} = \overline{m}_1 \circledcap^\alpha \overline{m}_2. \tag{8}$$

Proposition 4. *Let A and $B \subseteq \Omega$. $\forall X \subseteq \Omega$, one has:*

$$m_{A\cup^\alpha B}(X) = \begin{cases} \alpha^{|X| - |A\Delta B|} \overline{\alpha}^{|A\cup B| - |X|} & \text{if } A\Delta B \subseteq X \subseteq A \cup B \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Figure 2 illustrates the variety of potential focal sets of mass function $m_{A\cup^\alpha B}$. It can be noted that according to proposition 4:

$$A \cap B = \emptyset \implies m_{A\cup^\alpha B} = m_{A\cup B}. \tag{10}$$

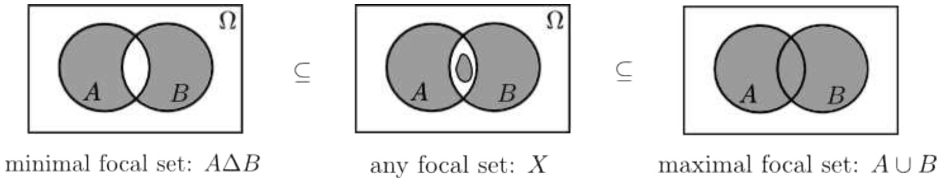

Figure 2: Example of focal sets of mass function $m_{A\cup^\alpha B}$.

Proposition 5. *Let A and $B \subseteq \Omega$. $\forall X \subseteq \Omega$, one has:*

$$b_{A\cup^\alpha B}\left(X\right)=\begin{cases}\overline{\alpha}^{|(A\cup B)\setminus X|} & \textit{if } A\Delta B\subseteq X\\ 0 & \textit{otherwise}\end{cases}. \tag{11}$$

Proposition 6. *Let A and $B \subseteq \Omega$. $\forall X \subseteq \Omega$, one has:*

$$m_{A\cup^\alpha B}=m_{A\Delta B} \circledcirc \left(\bigcircledcup_{y\in A\cap B} \underline{m}^{\overline{\alpha}}_{(A\Delta B)\cup\{y\}}\right), \tag{12}$$

with $\underline{m}^{\overline{\alpha}}_{(A\Delta B)\cup\{y\}}$ denoting a negative simple mass function which is such that $\underline{m}^{\overline{\alpha}}_{(A\Delta B)\cup\{y\}}=\overline{\alpha}m_\emptyset+\alpha m_{(A\Delta B)\cup\{y\}}$.

Example 2. *(Example 1 continued). Suppose that $\Omega=\{a,b,c\}$, $A=\{a\}$ and $B=\{a,b\}$. Consequently, we have $A\Delta B=\{b\}$ and $A\cap B=\{a\}$. The mass functions before combination, those involved in the conjunctive decomposition in equation (12) as well as the output mass function $m_{A\cup^\alpha B}$ are as follows:*

subsets:	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	Ω
$m_A=m_{\{a\}}$		*1*						
$m_B=m_{\{a,b\}}$				*1*				
$m_{A\Delta B}=m_{\{b\}}$			*1*					
$\underline{m}^{\overline{\alpha}}_{A\Delta B\cup\{a\}}=\underline{m}^{\overline{\alpha}}_{\{a,b\}}$	$\overline{\alpha}$			α				
$m_{A\cup^\alpha B}$			$\overline{\alpha}$	α				

The existence of the proposed disjunctive decomposition, like in the conjunctive case, could not have been anticipated using existing theorems. From section 3, the conjunctive decomposition of $m_{\overline{A}\cap^\alpha\overline{B}}$ is unique in some sense, therefore the disjunctive decomposition of $m_{A\cup^\alpha B}$ is unique to the same regard. We say that it is canonical by abuse of language. In compliance with [1], we define the disjunctive weight function of an α-disjunction of two categorical mass functions $v_{A\cup^\alpha B}$ as follows:

$$\forall X\subseteq\Omega,\ v_{A\cup^\alpha B}\left(X\right)=\left\{\begin{array}{ll}0 & \text{if } X=A\Delta B,\\ \overline{\alpha} & \text{if } A\Delta B\subsetneq X \text{ and } |X|=|\overline{A\Delta B}|+1,\\ 1 & \text{otherwise .}\end{array}\right.$$

Besides, the proposed disjunctive decomposition allows the following interpretation of α-disjunctions of categorical mass functions: there are $|A\cap B|$ sources supporting $A\Delta B$ with strength $\overline{\alpha}$ and $A\Delta B$ plus any element $y\in A\cap B$ with strength α and at least one of these sources is truthful.

Furthermore, it can be noted that any combination of categorical mass functions using an α-junction can be decomposed both conjunctively and disjunctively. Indeed, any mass function $m_{A\cap^\alpha B}$ can be decomposed conjunctively using proposition 3. Now let $C=\overline{A\cup B}\cup X$ and $D=\overline{A\cup B}\cup Y$ with $\{X,Y\}$ a partition of $A\cap B$. We thus have :

$$\begin{array}{rcl}C\cap D & = & \overline{A\cup B},\\ C\Delta D & = & A\cap B,\\ C\cup B & = & \overline{A\Delta B}.\end{array}$$

Using propositions 1 and 4, it is immediate that $m_{C\cup^{\overline{\alpha}}D}=m_{A\cap^\alpha B}$. By using proposition 6 on $m_{C\cup^{\overline{\alpha}}D}$, a disjunctive decomposition of $m_{A\cap^\alpha B}$ is also obtained.

5 Conclusion

In this article, α-junctions of categorical mass functions have been investigated. We provided straightforward equations for the computation of several set functions pertaining to evidence theory in both the conjunctive and disjunctive cases. In particular, a canonical conjunctive (respectively disjunctive) decomposition of the α-conjunction (respectively α-disjunction) of categorical mass functions have been obtained. In this particular situation, an α-conjunction (respectively an α-disjunction) is thus a series of purely conjunctive (respectively disjunctive) combinations. This leads to new complementary interpretations of α-junctions of deterministic pieces of information that are compliant with Pichon and Denœux's interpretation [4].

Concerning the generalization of these results for the α-junctions of any mass functions, it can only be concluded that an α-conjunction (respectively an α-disjunction) is a convex combination of series of purely conjunctive (respectively disjunctive) combinations. Pichon and Denœux actually already proposed decompositions of α-junctions of any mass functions, but these decompositions are obtained using a cross product of two frames of discernment. In future works, we hope to provide results on α-junction decompositions on a single frame.

It would be also interesting to investigate the ties between the conjunctive and disjunctive weights obtained in this article with the α-conjunctive and α-disjunctive weights introduced in chap. 7 of [5]. These other weights are defined using signed belief functions and consequently take their values in $(-\infty, +\infty) \setminus \{0\}$.

Finally, we also hope to apply α-junctions in information fusion problems involving partially truthful pieces of evidence. Truthfulness issues in information fusion arise in the presence of an unreliable or malicious information source. An unreliable source is accidentally untruthful whereas a malicious source is purposely untruthful (see [2] for an example of a such an application). α-junctions are appealing combination tools for the latter case. Indeed, if the value of α can be inferred using contextual information, an α-junction is likely to efficiently circumvent erroneous pieces of evidence.

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