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Abstract. The asymptotic behavior of the expected size of the convex hull of uniformly random points in a convex body in $\mathbb{R}^d$ is polynomial for a smooth body and polylogarithmic for a polytope. We construct a body whose expected size of the convex hull oscillates between these two behaviors when the number of points increases.

Keywords: convex hull, random point distribution, geometric probability

1 Introduction

Consider a sequence of points in a convex body in dimension d whose convex hull is dynamically maintained when the points are inserted one by one, the convex hull size may increase, decrease, or remain constant when a new point is added. Studying the expected size of the convex hull when the points are evenly distributed in the convex is a classical problem of probabilistic geometry that yields some surprising facts. For example, although it seems quite natural to think that the expected size of the convex hull is increasing with n the number of points, this fact is only formally proven for n big enough Devillers et al. (2013). The asymptotic behavior of the expected size is known to be polylogarithmic for a polytopal body and polynomial for a smooth one. If for a polytope or a smooth body, the asymptotic behavior is *somehow* "nice", for "most" convex bodies the behavior is unpredictable (Bárány, 1989, corollary 3). It is possible to construct strange convex objects that have no such nice behaviors and this note exhibits a convex body, such that the behavior of the expected size of a random polytope oscillates between the polytopal and smooth behaviors when n increases.

More formally, let K be a convex body in $\mathbb{R}^d$ and $(x_1, \ldots, x_n)$ a sample of n points chosen uniformly and independently at random in K. Let K_n be the convex hull of these points and $f_0(K_n)$ the number of vertices of K_n.
It is well known Bárány (2008); Reitzner (2010) that if P is a polytope, then

$$\mathbb{E}f_0(P_n) = c_{d,P} \log^{d-1} n + o(\log^{d-1} n) \tag{1}$$

and if K is a smooth convex body (i.e with $\mathcal{C}^2$ boundary with a positive Gaussian curvature), then

$$\mathbb{E}f_0(K_n) = c_{d,K} n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}}) \tag{2}$$

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where $c_{d,P}$ and $c_{d,K}$ are constants depending only on d and on the convex body.
These are the two extreme behaviors : every random polytope of a convex body in $\mathbb{R}^d$ has a behavior between (1) and (2) for n large enough (Bárány, 1989, corollary 3).
For general convex bodies, we cannot expect such a beautiful formula.

Theorem 1 (Bàràny-Larman, 1988) *For any function $G(n) \to_{n\to\infty} \infty$ and for most (in the Baire category sense) convex bodies K in $\mathbb{R}^d$,*

$$G(n)\log^{d-1} n > \mathbb{E}f_0(K_n)$$

for infinitely many n and

$$G(n)^{-1}n^{\frac{d-1}{d+1}} < \mathbb{E}f_0(K_n)$$

for infinitely many n.

Note that this *"most"* does not contain convex polytopes and smooth convex bodies, which are the most used in practice.

In this paper, we present an explicit example of a convex body which has this chaotic behavior.

Notations Let's introduce some notations used in this paper:

- $\mathcal{V}(K)$ will denote the $d-$dimensional volume of the convex body K;
- $K \oplus L$ will denote the Minkowski sum of K and L, defined as

$$J \oplus K = \{x + y \,|x \in J, y \in K\};$$

- $d_H(J, K)$ will denote the Hausdorff distance of the convex bodies K and L, defined as

$$d_H(J, K) = \min\{r \in \mathbb{R}^+ \,|J \subset K \oplus B_r, K \subset J \oplus B_r\}$$

  where B_r is the Euclidean ball centered in 0 with radius r in $\mathbb{R}^d$.

2 Approximations of convex bodies

In this section, we present an intermediate lemma about random polytopes of close (in terms of Hausdorff distance) convex bodies. Let K be a convex body in $\mathbb{R}^d$ and K_n be a random polytope in K. We want to show that if L is an approximation of K with small Hausdorff distance, L_n is approximating the asymptotic behavior of K_n for some value of n.
Let's assume that the expected size of K_n is in $c_{d,K}g(n, d) + o(g(n, d))$, where $c_{d,K}$ is a constant and g some function.
Then, for every close-enough compact set L containing K, L_n has an expected size as close as we want from $c_{d,K}g(n, d)$ for values of n as big as we want.
The idea of this lemma is very simple: if L is very close to K, the volume in $L \setminus K$ is small, and points chosen uniformly in L are very unlikely to be in $L \setminus K$. Then, even if asymptotically the expected size of L_n is different from the size of K_n, there exists some n where the expected size of L_n is as close as we want to $c_{d,K}g(n, d)$.

Lemma 2 *Let K be a convex body in $\mathbb{R}^d$ such that*

$$\mathbb{E}f_0(K_n) = c_{d,K}g(n,d) + o(g(n,d)). \tag{3}$$

Let $\varepsilon > 0$ and $N \in \mathbb{N}$.
Then, there exist $p > N$ and $\alpha > 0$ such that for any compact set L containing K with $d_H(K,L) < \alpha$,

$$\frac{\mathbb{E}f_0(L_p)}{c_{d,K}g(p,d)} \in [1-\varepsilon, 1+\varepsilon].$$

Proof: First, for all $n \in \mathbb{N}^*$

$$\mathbb{E}f_0(L_n) = \mathbb{P}(L_n \subset K)\mathbb{E}(f_0(L_n)|L_n \subset K) + \mathbb{P}(L_n \not\subset K)\mathbb{E}(f_0(L_N)|L_n \not\subset K).$$

As the points are uniformly distributed, $\mathbb{E}(f_0(L_n)|L_n \subset K) = \mathbb{E}(K_n)$.
Using (3), let's choose p such that

$$\frac{\mathbb{E}f_0(K_p)}{c_{d,K}g(p,d)} \in \left[1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}\right], \tag{4}$$

then

$$\frac{\mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_p)|L_p \subset K)}{c_{d,K}g(p,d)} \leq 1 + \frac{\varepsilon}{2}.$$

As

$$\mathbb{P}(L_p \not\subset K) = 1 - \mathbb{P}(L_p \subset K) = 1 - \left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p$$

we get

$$\mathbb{P}(L_p \not\subset K)\mathbb{E}(f_0(L_p)|L_p \not\subset K) \leq \left(1 - \left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p\right)p.$$

Now, as $1 \geq \frac{\mathcal{V}(K)}{\mathcal{V}(L)} \geq \frac{\mathcal{V}(K)}{\mathcal{V}(K \oplus B_\alpha)} \to_{\alpha \to 0} 1$ we can choose α such that

$$\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p \geq \max\left(1 - c_{d,K}\frac{g(p,d)}{p}\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right). \tag{5}$$

Finally,

$$\mathbb{E}(f_0(L_p)) \leq c_{d,K}g(p,d)\left(1 + \frac{\varepsilon}{2}\right) + c_{d,K}g(p,d)\frac{\varepsilon}{2} = c_{d,K}g(p,d)(1+\varepsilon). \tag{6}$$

For the lower bound, using (4) and (5) we get

$$\begin{aligned}
\mathbb{E}f_0(L_p) &\geq \mathbb{P}(L_p \subset K)\mathbb{E}(f_0(L_n)|L_n \subset K) \\
&\geq c_{d,K}g(p,d)\left[\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p (1 - \frac{\varepsilon}{2})\right] \\
&\geq c_{d,K}g(p,d)\left[\left(\frac{\mathcal{V}(K)}{\mathcal{V}(L)}\right)^p - \frac{\varepsilon}{2}\right] \\
&\geq c_{d,K}g(p,d)\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right) \\
&= c_{d,K}g(p,d)(1-\varepsilon).
\end{aligned} \tag{7}$$

Inequalities (7) and (6) prove the lemma. □

3 Construction of the convex body

Given an increasing function G, we want to construct a convex body in $\mathbb{R}^d$ where the size of a convex hull of random points has a chaotic behavior between $\log^{d-1} n$ and $n^{\frac{d-1}{d+1}}$ on some values arbitrarily big. More formally,

Theorem 3 *Let $G : \mathbb{N}^* \to \mathbb{R}_+^*$ an increasing function such that $G(n) \to_{n\to\infty} \infty$. We can construct a convex body K such that:*
For all $N \in \mathbb{N}^*$, there exist $M_1, M_2 > N$, where

$$\mathbb{E} f_0(K_{M_1}) < G(M_1) \log^{d-1} M_1$$

and

$$\mathbb{E} f_0(K_{M_2}) > G(M_2)^{-1}\, M_2^{\frac{d-1}{d+1}}.$$

Proof: The main idea of the proof is, starting from a convex body $K^{(0)}$, to iterate smooth and polytopal approximations. Lemma 2 will give us some number of points where the behavior of the random convex body will be very close to $n^{\frac{d-1}{d+1}}$ (which is the behavior for smooth convex bodies) or very close to $\log^{d-1} n$ (which is the behavior for polytopes).

Iterations We create an increasing sequence of convex bodies starting from the unit ball, made of polytopal and smooth approximations.
Let's define $K^{(0)}$ as the unit ball.
For all $n \in \mathbb{N}^*$, $K^{(n)}$ is an approximation of $K^{(n-1)}$ where $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$, with $(\beta_i)_{i\in\mathbb{N}}$ some decreasing sequence, as shown in Figure 1.

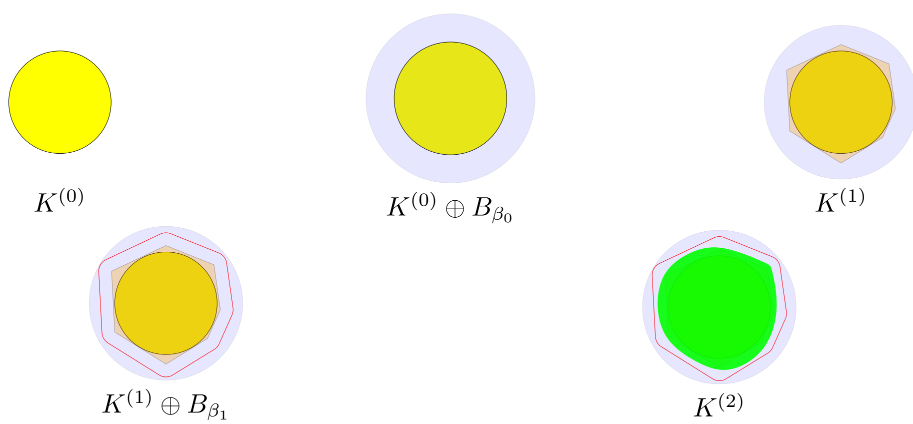

Fig. 1: Iterations are made of polygonal and smooth approximations

- If n is odd, $K^{(n-1)}$ is a smooth convex body, so $K^{(n)}$ is a convex polytope.
  Let's choose $q_n > n$, such that

$$\frac{\mathbb{E} f_0(K^{(n-1)})}{q_n^{\frac{d-1}{d+1}}} > \frac{2}{G(q_n)}.$$

We define

$$\varepsilon_n := 1 - \frac{2}{c_{d,K^{(n-1)}} G(q_n)}. \tag{8}$$

Using Lemma 2,with $\varepsilon = \varepsilon_n$, there exist α_{n-1} and $p_n > q_n$ such that for every compact set L containing $K^{(n-1)}$ with $d_H(K^{(n-1)}, L) < \alpha_{n-1}$,

$$\mathbb{E} f_0(L_{p_n}) > c_{d,K^{(n-1)}} p_n{}^{\frac{d-1}{d+1}} (1 - \varepsilon_n). \tag{9}$$

Therefore,

$$\begin{aligned} \mathbb{E} f_0(L_{p_n}) &> c_{d,K^{(n-1)}} p_n{}^{\frac{d-1}{d+1}} (1 - \varepsilon_n) \\ &> p_n{}^{\frac{d-1}{d+1}} G(q_n)^{-1} \\ &> p_n{}^{\frac{d-1}{d+1}} G(p_n)^{-1}. \end{aligned} \tag{10}$$

Now let's define $\beta_{n-1} := \min(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2})$ if $n > 1$ and $\beta_0 := \frac{\alpha_0}{2}$. We define $K^{(n)}$ as a convex polytope with $d_H(K^{(n-1)}, K^{(n)}) < \beta_{n-1}$, so (10) works for $L = K^{(n)}$.

- If n is even, $K^{(n-1)}$ is a convex polytope, so $K^{(n)}$ is a smooth approximation.

Let's choose q_n such that

$$\frac{\mathbb{E} f_0(K^{(n-1)})}{\log^{d-1} q_n} < \frac{G(q_n)}{2}$$

and define

$$\varepsilon_n := \frac{G(q_n)}{2 c_{d,K^{(n-1)}}} - 1.$$

Using Lemma 2 with $\varepsilon = \varepsilon_n$, there exist α_{n-1} and $p_n > q_n$ such that for every compact set with $d_H(K^{(n-1)}, L) < \alpha_{n-1}$,

$$\mathbb{E} f_0(L_{p_n}) < c_{d,K^{(n-1)}} \log^{d-1}(p_n)(1 + \varepsilon_n).$$

Finally,

$$\begin{aligned} \mathbb{E} f_0(L_{p_n}) &< G(q_n) \log^{d-1}(p_n) \\ &< G(p_n) \log^{d-1}(p_n). \end{aligned} \tag{11}$$

Again, we define $\beta_{n-1} = \min(\frac{\alpha_{n-1}}{2}, \frac{\beta_{n-2}}{2})$. We define $K^{(n)}$ as a smooth approximation of $K^{(n-1)}$ such that $d_H(K^{(n-1)}, K^{(n)}) < \ \beta_{n-1}$, so (11) works for $L = K^{(n)}$.

Note that for all $m > n \in \mathbb{N}$,

$$d_H(K^{(n)}, K^{(m)}) \leq \sum_{k=n}^{m-1} d_H(K^{(k)}, K^{(k+1)}) < \sum_{k=n}^{m-1} \beta_k$$
$$\leq \sum_{k=0}^{m-n-1} \frac{\beta_n}{2^k} \leq 2\beta_n \leq \alpha_n.$$

That means for all $m > n$, the property (10) or (11) (depending on the evenness of n) are also true for $K^{(m)}$.
Now, defining $K = \overline{\cup_{i=0}^{\infty} K^{(i)}}$, the property (10) and (11) are true for arbitrary $n \in \mathbb{N}$ with $L = K$, by considering $K^{(n)}$ and $K^{(n+1)}$.
As we can choose q_n as big as we want for any n (it will just decrease α_{n-1}), we can choose this sequence to be increasing. As a result, $\mathbb{E}f_0(K_n)$ will have a chaotic behavior within $n^{\frac{d-1}{d+1}}/G(n)$ and $G(n)\log^{d-1} n$, as shown in Figure 2.

□

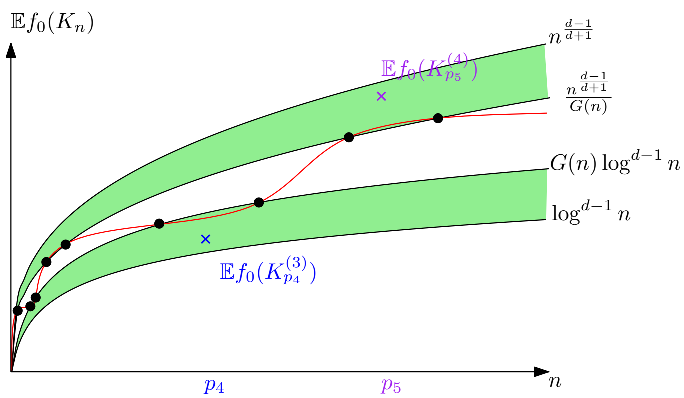

Fig. 2: The expected size of the random polytope of K.

Concluding remarks We have constructed a convex body K such that the expected size of the convex hull of a random polytope in K has a chaotic behavior. This construction is the limit of a sequence of bodies $(K^{(i)})$ that alternate polytopes and smooth shapes so it is difficult to provide an explicit description of K, in this note we just show that constructing such a sequence is possible by a repeated application of Lemma 2 but there is no obstacle, except long and painful computations, to a more constructive version with explicit description of the sequence. Notice that in such a case the complexity of $K^{(i)}$ will be increasing quite rapidly. Actually, since $K^{(i)}$ is constrained in a slab of width β_i around $K^{(i-1)}$, the size of $K^{(i)}$ can be lower bounded for polytopes, see Böröczky Jr. (2000): $|K^{(i)}| = \Omega\left(\beta_i^{-\frac{d-1}{2}}\right)$ and since $\beta_i < \frac{\alpha_0}{2^i}$ we get, at least, an exponential behavior for the size of $K^{(i)}$. Even with a constructive description of the $K^{(i)}$, the description of K as the limit of the $K^{(i)}$ will remain quite abstract, but will allow to develop a membership test, given a point p, $p \in \mathrm{int}(K)$ can be decided by computing the sequence $K^{(i)}$ up to an index where $p \in K^{(i)}$ or $p \notin K^{(i)} \oplus B_{\beta_i}$.

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