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Abstract: Highly conducting thin deposits may blind eddy current probes in non-destructive testing of steam generator tubes and thus should be identified. In this report, various asymptotic models are studied to model the axisymmetric thin conducting layers by effective transmission conditions depending on re-scaling parameter and asymptotic expansion order, so as to avoid the high computational cost in a full model due to those thin layers. We also select the most adapted models for practical configurations via numerical comparisons in a simplified case.

Key-words: axisymmetric eddy current model, asymptotic models, effective transmission conditions.

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Validité de quelques modèles asymptotiques pour l'inspection par courant de Foucault de dépôts fins fortement conducteurs

Résumé : Des couches minces hautement conductrices peuvent masquer des défauts problématiques lors d'un contrôle non-destructif des tubes dans un générateur de vapeur via des sondes courant de Foucault. Ainsi il est essentiel de pouvoir en tenir compte. Dans ce rapport, on étudie des modèles asymptotiques avec différentes conditions de transmission effectives ayant pour objectif de modéliser une couche mince axis-symétrique afin de réduire le coût de calcul numérique. Parmi des conditions de transmission qui dépendent d'un paramètre de redimensionnement et de l'ordre du développement asymptotique, on en sélectionne les plus pertinentes pour les applications pratiques via des tests numériques dans des configurations simplifiées.

Mots-clés : modèle de courant de Foucault axis-symétrique, modèles asymptotiques, conditions de transmission effectives.

1 Introduction

Non-destructive eddy current testing of the steam generator tubes is an essential task for the safety and failure-free operating of nuclear power plants. This testing generally aims to detect harmful defects such as cracks of tubes and clogging deposits in cooling circuit between tubes and supporting plates (see for example [1–4, 8, 9, 13–15]). These problematic faults can nevertheless be masked by some thin layers of copper covering the shell side of the tube due to its high conductivity. This is why it is important to be able to evaluate their influence on eddy current testing.

The copper layers are characterized by a very high conductivity (as compared to steam generator tubes) and a very small thickness, see Table 1. A major numerical challenge to deal with this problem with the full eddy current model is the expensive computational cost resulting from the fact that the domain discretization should use a very fine mesh of the same scale to the thickness of the thin layer. To reduce the numerical cost, we replace the thin layer by some effective transmission conditions using the asymptotic expansion of the solution with respect to the thickness of the deposit, which yields the so-called asymptotic models. A rich literature on asymptotic models has been developed and we may cite among others Tordeux [12], Claeys [5], Delourme [6], Poignard [10] and the references therein for different approaches and various applications.

	tube wall	copper layer
conductivity (in $S \cdot m^{-1}$)	$\sigma_t = 0.97 \times 10^6$	$\sigma_c = 58.0 \times 10^6$
thickness (in mm)	$r_{t_2} - r_{t_1} = 1.27$	$0.005 \sim 0.1$

Table 1: Conductivity and scale differences between tube wall and copper layer.

In this report, we consider the case where a copper layer of constant thickness covers axisymmetrically the shell side of the tube. According to the choice of a rescaling parameter m for the conductivity and the asymptotic expansion order n, we can obtain a family of effective transmission conditions $\mathcal{Z}_{m,n}$ linking up the solutions at the two sides of the thin layer. We aim to choose the appropriate effective transmission conditions, i.e. the parameters (m, n), with which the direct asymptotic model not only gives a good approximation of the full model in real configuration, but also allows to establish quick inversion methods.

Although mainly considering here the case of thin layers with constant thickness, we shall introduce the asymptotic method for general thin deposit layers, i.e. with variable thickness (Section 2). With that method, some effective transmission conditions with different parameters (m, n) are calculated for thin layers with constant thickness (Section 3). It is worth mentioning that similar studies have been developed in [11] for thin conducting sheet with constant thickness with a different geometrical setting. Finally in Section 4 we give some 1-D numerical examples which allow to verify and compare the asymptotic models with these different transmission conditions, and then discuss the most pertinent model in view of direct and inverse simulations.

2 Asymptotic approximation of axisymmetric eddy current model

This section concentrates on the construction of asymptotic models for eddy current problems with the presence of highly conducting thin layers. The objective is to get the effective transmission conditions on the interface between the thin layer and the tube with which the variational asymptotic model has no longer the volume integral on the thin layer domain.

Let us briefly introduce the axisymmetric eddy current problem. For more details readers may refer to [7]. In the cylindrical coordinates, a vector field $\boldsymbol{a}$ can be decomposed into the meridian part $\boldsymbol{a}_m = a_r\boldsymbol{e}_r + a_z\boldsymbol{e}_z$ and the azimuthal part $\boldsymbol{a}_\theta = a_\theta\boldsymbol{e}_\theta$. $\boldsymbol{a}$ is axisymmetric if $\partial_\theta\boldsymbol{a}$ vanishes. Under the assumption of axisymmetry and the high conductivity / low frequency regime ($\omega\epsilon \ll \sigma$), the 3-D time-harmonic Maxwell equations for the electric and magnetic fields $(\boldsymbol{E},\boldsymbol{H})$

$$\begin{cases} \operatorname{curl}\boldsymbol{H} + (\mathrm{i}\omega\epsilon - \sigma)\boldsymbol{E} = \boldsymbol{J} & \text{in } \mathbb{R}^3, \\ \operatorname{curl}\boldsymbol{E} - \mathrm{i}\omega\mu\boldsymbol{H} = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

with a divergence-free axisymmetric applied source $\boldsymbol{J}$ ($\operatorname{div}\boldsymbol{J} = 0$) can be reduced to a second order equation on a 2-D domain $\mathbb{R}^2_+ := \{(r,z) : r \geq 0, z \in \mathbb{R}\}$ for the azimuthal part of the electric field E_θ that we denote in the sequel by $u = E_\theta$:

$$-\operatorname{div}\left(\frac{1}{\mu r}\nabla(ru)\right) - \mathrm{i}\omega\sigma u = \mathrm{i}\omega J_\theta = \mathrm{i}\omega J \quad \text{in } \mathbb{R}^2_+, \tag{2}$$

where $\nabla := (\partial_r, \partial_z)^t$ and $\operatorname{div} := \nabla\cdot$ are gradient and divergence operators in 2-D Cartesian coordinates. Assume that $J \in L^2(\mathbb{R}^2_+)$ has compact support, and that μ and σ are in $L^\infty(\mathbb{R}^2_+)$ such that $\mu \geq \mu_v > 0$, $\sigma \geq 0$ and that $\mu = \mu_v$, $\sigma = 0$ for $r \geq r_0$ sufficiently large. Then problem (2) with a decay condition ($u \to 0$ as $r^2 + z^2 \to 0$) has a unique solution in $H(\mathbb{R}^2_+)$ where for any $\Omega \subset \mathbb{R}^2_+$ we denote

$$H(\Omega) := \left\{ v : r^{1/2}(1+r^2)^{-\lambda/2} v \in L^2(\Omega),\, r^{-1/2}\nabla(rv) \in L^2(\Omega) \right\}$$

with λ any real > 1 (see [7]). Let us indicate that if Ω is bounded in the r-direction then $H(\Omega)$ is equivalent to the following space for which we shall use the same notation

$$H(\Omega) := \left\{ v : r^{1/2} v \in L^2(\Omega),\, r^{-1/2}\nabla(rv) \in L^2(\Omega) \right\}.$$

Hence, the eddy current equation (2) writes in the variational form

$$a(u,v) := \int_\Omega \left(\frac{1}{\mu r}\nabla(ru)\cdot\nabla(r\bar{v}) - \mathrm{i}\omega\sigma r w\bar{v}\right)\mathrm{d}r\,\mathrm{d}z = \int_\Omega \mathrm{i}\omega J\bar{v}r\,\mathrm{d}r\,\mathrm{d}z \qquad \forall v \in H(\Omega). \tag{3}$$

For numerical reasons, the computational domain will be truncated in radial direction at $r = r_*$ where $r_* > r_0$ is sufficiently large and impose a Neumann boundary condition on $r = r_*$. Then the solution for the truncated problem would satisfy (2) on $\Omega = B_{r_*} := \{(r,z) \in \mathbb{R}^2 : 0 \leq r \leq r_*\}$. This is why we shall use in the sequel the generic notation for the variational space $H(\Omega)$ with Ω denoting $\mathbb{R}^2_+$ or B_{r_*}. We also recall that the problem on $\Omega = B_{r_*}$ can be equivalently truncated to a bounded domain $B_{r_*,z_*} = \{(r,z) \in \mathbb{R}^2 : 0 \leq r \leq r_*, |z| < z_*\}$ by introducing appropriate Dirichlet-to-Neumann operators on $z = \pm z_*$. This would be convenient for accelerating numerical evaluation of the solution (see [7]). As a corollary of the well-posedness of the problem for u we can state:

Corollary 2.1. *Assume that the source $J \in L^2(\Omega)$ with compact support. Then the variational formulation* (3) *has a unique solution u in $H(\Omega)$.*

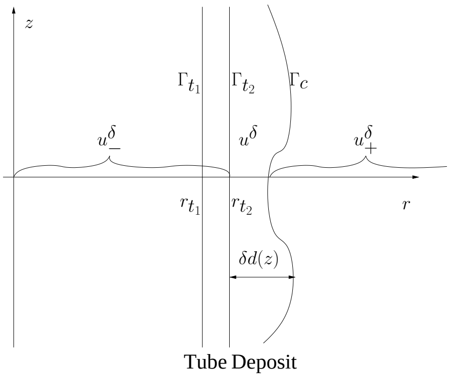


Figure 1: Representation of a thin layer deposit.

2.1 Rescaled in-layer eddy current equation

We consider a thin layer of deposit with high conductivity (in our case, a layer of copper) covering axisymmetrically (a part of) the shell side of the tube (see Figure 1 for a radial cut of the setting in the cylindrical coordinate system). The first step is to rewrite the in-layer eddy current equation by rescaling the coordinate in the transverse direction and the conductivity with respect to the layer thickness. The analytical solution of this rescaled equation allows to get the relationship between the boundary values of both Dirichlet and Neumann type on the two longitudinal boundaries of the thin layer.

On the domain of problem Ω, we set

$$\Omega_\pm := \{(r,z) \in \Omega : r \gtrless r_{t_2}\}$$

The thin layer is depicted by the domain $\Omega_c^\delta \subset \Omega_+$. We denote by $u_\pm^\delta$ the fields outside the deposit layer, with u_-^δ in Ω_- and u_+^δ in $\Omega_+ \setminus \Omega_c^\delta$ (at the shell side of the deposit layer), and by u^δ the in-layer field, i.e. in Ω_c^δ (see Figure 1). Assume that the thickness $f_\delta(z)$ at the vertical position z is of the order δ

$$f_\delta(z) = \delta d(z),$$

where δ is a small parameter and $d(z)$ is independent of δ. Assuming that the deposit conductivity writes

$$\sigma_c = \frac{\sigma_m}{\delta^m}, \tag{4}$$

where σ_m is an appropriately re-scaled conductivity and m the re-scaling parameter. We will particularly interest in the cases where $m = 0, 1, 2$. So in the deposit layer the eddy current equation (2) writes

$$-\text{div}\left(\frac{1}{r}\nabla(ru^\delta)\right) - \frac{k_m^2}{\delta^m}u^\delta = 0 \qquad \text{on } \Omega_c^\delta, \tag{5}$$

where $k_m^2 := \mathrm{i}\omega\mu_c\sigma_m$. We consider the variable substitution

$$\rho = \frac{r - r_{t_2}}{\delta}, \quad \rho \in [0, d(z)],$$

and we denote by $\tilde{u} = \tilde{u}(\rho, z) := u^\delta(r_{t_2} + \delta\rho, z)$ the re-scaled in-layer solution. From (5), one gets

$$\frac{r^2}{\delta^2}\partial_\rho^2 \tilde{u} + \frac{r}{\delta}\partial_\rho \tilde{u} - \tilde{u} + r^2\left(\partial_z^2 \tilde{u} + \frac{k_m^2}{\delta^m}\tilde{u}\right) = 0.$$

By substituting r with $r_{t_2} + \rho\delta$, we get for $m = 0, 1$,

$$\partial_\rho^2 \tilde{u} = -\delta \mathcal{B}_m^1 \tilde{u} - \delta^2 \mathcal{B}_m^2 \tilde{u} - \delta^3 \mathcal{B}_m^3 \tilde{u} - \delta^4 \mathcal{B}_m^4 \tilde{u}, \tag{6}$$

with

$$\begin{cases} \mathcal{B}_0^1 = \dfrac{2\rho}{r_{t_2}}\partial_\rho^2 + \dfrac{1}{r_{t_2}}\partial_\rho, \\ \mathcal{B}_0^2 = \dfrac{\rho^2}{r_{t_2}^2}\partial_\rho^2 + \dfrac{\rho}{r_{t_2}^2}\partial_\rho - \dfrac{1}{r_{t_2}^2} + \partial_z^2 + k_0^2, \\ \mathcal{B}_0^3 = \dfrac{2\rho}{r_{t_2}}\left(\partial_z^2 + k_0^2\right), \\ \mathcal{B}_0^4 = \dfrac{\rho^2}{r_{t_2}^2}\left(\partial_z^2 + k_0^2\right), \end{cases} \qquad \begin{cases} \mathcal{B}_1^1 = \dfrac{2\rho}{r_{t_2}}\partial_\rho^2 + \dfrac{1}{r_{t_2}}\partial_\rho + k_1^2, \\ \mathcal{B}_1^2 = \dfrac{\rho^2}{r_{t_2}^2}\partial_\rho^2 + \dfrac{\rho}{r_{t_2}^2}\partial_\rho - \dfrac{1}{r_{t_2}^2} + \partial_z^2 + \dfrac{2\rho}{r_{t_2}}k_1^2, \\ \mathcal{B}_1^3 = \dfrac{2\rho}{r_{t_2}}\partial_z^2 + \dfrac{\rho^2}{r_{t_2}^2}k_1^2, \\ \mathcal{B}_1^4 = \dfrac{\rho^2}{r_{t_2}^2}\partial_z^2. \end{cases} \tag{7}$$

For $m = 2$, we consider a weighted in-layer field

$$w^\delta(r, z) := \sqrt{r}u^\delta(r, z) \quad r \in [r_{t_2}, r_{t_2} + f_\delta(z)],$$

and after re-scaling one has

$$\tilde{w}(\rho, z) := w^\delta(r_{t_2} + \delta\rho, z)$$

So the eddy current equation for the in-layer field u^δ (5) becomes here

$$\frac{1}{\delta^2}\partial_\rho^2 \tilde{w} - \frac{3}{4r^2}\tilde{w} + \frac{k_2^2}{\delta^2}\tilde{w} + \partial_z^2 \tilde{w} = 0.$$

By substituting r with $r_{t_2} + \rho\delta$, we obtain

$$\left(\partial_\rho^2 + k_2^2\right)\tilde{w} = -\delta \mathcal{B}_2^1 \tilde{w} - \delta^2 \mathcal{B}_2^2 \tilde{w} - \delta^3 \mathcal{B}_2^3 \tilde{w} - \delta^4 \mathcal{B}_2^4 \tilde{w}, \tag{8}$$

with

$$\begin{cases} \mathcal{B}_2^1 = \dfrac{2\rho}{r_{t_2}}\left(\partial_\rho^2 + k_2^2\right), \\ \mathcal{B}_2^2 = \dfrac{\rho^2}{r_{t_2}^2}\left(\partial_\rho^2 + k_2^2\right) - \dfrac{3}{4r_{t_2}^2} + \partial_z^2, \\ \mathcal{B}_2^3 = \dfrac{2\rho}{r_{t_2}}\partial_z^2, \\ \mathcal{B}_2^4 = \dfrac{\rho^2}{r_{t_2}^2}\partial_z^2. \end{cases} \tag{9}$$

2.2 Taylor developments for u_+^δ

We would like to extend the solution outside the deposit layer u_+^δ through the layer domain till the interface Γ_{t_2}, i.e. from $\Omega_+ \setminus \Omega_c^\delta$ to Ω_+, such that the transmission conditions on Γ_c between u and u_+^δ could be expressed in terms of u_+^δ on Γ_{t_2}. As u_+^δ satisfies the eddy current equation with coefficients $\mu = \mu_v$ and $\sigma = \sigma_v = 0$ in $\Omega_+ \setminus \Omega_c^\delta$, it is natural to assume that its extension on Ω_c^δ satisfies the same equation

$$-\mathrm{div}\left(\frac{1}{\mu_v r}\nabla(ru_+^\delta)\right) = 0 \qquad \text{in } \Omega_+.$$

Using the variable substitution $\nu = r - r_{t_2}$, one rewrites the above equation in the following form

$$\sum_{j=0}^{4} \nu^j \mathcal{A}_j\left(\nu\partial_\nu, \partial_z\right) u_+^\delta = 0, \tag{10}$$

where

$$\begin{aligned}
\mathcal{A}_0\left(\nu\partial_\nu, \partial_z\right) &= \left(\nu\partial_\nu\right)^2 - \nu\partial_\nu, \\
\mathcal{A}_1\left(\nu\partial_\nu, \partial_z\right) &= \frac{2}{r_{t_2}}\left(\nu\partial_\nu\right)^2 - \frac{1}{r_{t_2}}\nu\partial_\nu, \\
\mathcal{A}_2\left(\nu\partial_\nu, \partial_z\right) &= \frac{1}{r_{t_2}^2}\left(\nu\partial_\nu\right)^2 - \frac{1}{r_{t_2}^2} + \partial_z^2, \\
\mathcal{A}_3\left(\nu\partial_\nu, \partial_z\right) &= \frac{2}{r_{t_2}}\partial_z^2, \\
\mathcal{A}_4\left(\nu\partial_\nu, \partial_z\right) &= \frac{1}{r_{t_2}^2}\partial_z^2.
\end{aligned}$$

If the asymptotic expansion of u_+^δ with respect to δ writes

$$u_+^\delta(r,z) = \sum_{n=0}^{\infty} \delta^n u_+^n(r,z),$$

then each term $u_+^n(r,z)$ verifies the same equation (10). With Taylor series expansion, one has

$$u_+^n(r_{t_2}+\nu, z) = \sum_{k=0}^{\infty} \nu^k u_+^{n,k}(z) \quad \text{where} \quad u_+^{n,k}(z) = \frac{1}{k!}\left(\partial_\nu^k u_+^n\right)(r_{t_2}, z).$$

Since

$$\nu\partial_\nu\left(\nu^k u_+^{n,k}(z)\right) = k\left(\nu^k u_+^{n,k}(z)\right),$$

we can indeed write $\mathcal{A}_i\left(\nu\partial_\nu, \partial_z\right)$ as $\mathcal{A}_i\left(k, \partial_z\right)$ while it is applied to $\left(\nu^k u_+^{n,k}(z)\right)$. Thus, from (10)

$$\sum_{j=0}^{4}\sum_{k=0}^{\infty} \mathcal{A}_j(k,\partial_z)(\nu^{k+j} u_+^{n,k}) = 0.$$

The equality at order $\mathcal{O}(\nu^k)$ yields

$$\mathcal{A}_0\left(k,\partial_z\right)u_+^{n,k} = -\sum_{j=1}^{4}\mathcal{A}_j\left(k-j,\partial_z\right)u_+^{n,k-j},$$

with $u_+^{n,-1} = u_+^{n,-2} = u_+^{n,-3} = u_+^{n,-4} = 0$. Now we consider $\mathcal{A}_0\left(k,\partial_z\right) = k^2 - k$. For $k \geq 2$, $\mathcal{A}_0\left(k,\partial_z\right) \neq 0$, thus invertible with its inverse $\mathcal{A}_0^{-1}\left(k,\partial_z\right) = \frac{1}{k^2-k}$. So we have

$$u_+^{n,k} = -\mathcal{A}_0^{-1}\left(k,\partial_z\right)\left(\sum_{j=1}^{4}\mathcal{A}_j\left(k-j,\partial_z\right)u_+^{n,k-j}\right), \quad k \geq 2. \tag{11}$$

Now we define recurrently two families of operators $\{\mathcal{S}_k^0\left(\partial_z\right), \mathcal{S}_k^1\left(\partial_z\right)\}$:

$$\mathcal{S}_0^0 := \mathrm{Id}, \quad \mathcal{S}_0^1 := 0, \quad \mathcal{S}_1^0 := 0, \quad \mathcal{S}_1^1 := \mathrm{Id},$$

$$k \geq 2 \quad \begin{cases} \mathcal{S}_k^0 := -\mathcal{A}_0^{-1}\left(k,\partial_z\right)\left(\displaystyle\sum_{j=1}^{4}\mathcal{A}_j\left(k-j,\partial_z\right)\mathcal{S}_{k-j}^0\left(\partial_z\right)\right), \\ \mathcal{S}_k^1 := -\mathcal{A}_0^{-1}\left(k,\partial_z\right)\left(\displaystyle\sum_{j=1}^{4}\mathcal{A}_j\left(k-j,\partial_z\right)\mathcal{S}_{k-j}^1\left(\partial_z\right)\right). \end{cases} \tag{12}$$

From the recurrent relation (11), one observes

$$u_+^{n,k}(z) = \mathcal{S}_k^0\left(\partial_z\right)u_+^n(r_{t_2},z) + \mathcal{S}_k^1\left(\partial_z\right)\partial_r u_+^n(r_{t_2},z).$$

Therefore we have the following developments

$$\begin{cases} u_+^n(r_{t_2}+\nu,z) = \displaystyle\sum_{k=0}^{\infty}\nu^k\left(\mathcal{S}_k^0\left(\partial_z\right)u_+^n + \mathcal{S}_k^1\left(\partial_z\right)\partial_r u_+^n\right)(r_{t_2},z), \\ \partial_r u_+^n(r_{t_2}+\nu,z) = \displaystyle\sum_{k=0}^{\infty}\nu^k(k+1)\left(\mathcal{S}_{k+1}^0\left(\partial_z\right)u_+^n + \mathcal{S}_{k+1}^1\left(\partial_z\right)\partial_r u_+^n\right)(r_{t_2},z). \end{cases}$$

We also define the operators

$$\widetilde{\mathcal{S}}_k^0 := \mathcal{S}_k^0 - \frac{1}{r_{t_2}}\mathcal{S}_k^1, \qquad \widetilde{\mathcal{S}}_k^1 := \frac{1}{r_{t_2}}\mathcal{S}_k^1. \tag{13}$$

Then the Taylor series expansions write

$$\begin{cases} u_+^n(r_{t_2}+\nu,z) = \displaystyle\sum_{k=0}^{\infty}\nu^k\left(\widetilde{\mathcal{S}}_k^0\left(\partial_z\right)u_+^n + \widetilde{\mathcal{S}}_k^1\partial_r(ru_+^n)\right)(r_{t_2},z), \\ \partial_r(ru_+^n)(r_{t_2}+\nu,z) = \displaystyle\sum_{k=0}^{\infty}\nu^k(k+1)\left((r_{t_2}\widetilde{\mathcal{S}}_{k+1}^0 + \widetilde{\mathcal{S}}_k^0)u_+^n + (r_{t_2}\widetilde{\mathcal{S}}_{k+1}^1 + \widetilde{\mathcal{S}}_k^1)\partial_r(ru_+^n)\right)(r_{t_2},z). \end{cases} \tag{14}$$

2.3 Transmission conditions between the in-layer field u^δ and $u^\delta_\pm$

We consider the transmission conditions on the two boundaries Γ_{t_2} and Γ_c of the thin layer. The transmission conditions on Γ_{t_2} linking up u^δ_- and u^δ write

$$\begin{cases} u^\delta_-|_{r_{t_2}} = u^\delta|_{r_{t_2}}, & (15a)\\ \dfrac{1}{\mu_t}\partial_r(ru^\delta_-)\Big|_{r_{t_2}} = \dfrac{1}{\mu_c}\partial_r(ru^\delta)\Big|_{r_{t_2}}, & (15b)\end{cases}$$

and the transmission conditions between u^δ_+ and u^δ on Γ_c write

$$\begin{cases} u^\delta_+|_{\Gamma_c} = u^\delta|_{\Gamma_c}, & (16a)\\ \dfrac{1}{\mu_v}\partial_n(ru^\delta_+)\Big|_{\Gamma_c} = \dfrac{1}{\mu_c}\partial_n(ru^\delta)\Big|_{\Gamma_c}. & (16b)\end{cases}$$

The unit normal and tangential vectors on Γ_c at the point $(r_{t_2} + \delta d(z), z)$ are respectively

$$n = \frac{(1, -\delta d'(z))}{\sqrt{1 + (\delta d'(z))^2}}, \qquad \tau = \frac{(\delta d'(z), 1)}{\sqrt{1 + (\delta d'(z))^2}}.$$

(16a) implies the continuity of the tangential derivatives of the field, that is

$$\tau \cdot \nabla u^\delta_+|_{r_{t_2}+\delta d} = \tau \cdot \nabla u^\delta|_{r_{t_2}+\delta d}$$

$$\Big(\delta d'\partial_r(ru^\delta_+) + \partial_z(ru^\delta_+)\Big)\Big|_{r_{t_2}+\delta d} = \Big(\delta d'\partial_r(ru^\delta) + \partial_z(ru^\delta)\Big)\Big|_{r_{t_2}+\delta d}. \tag{17}$$

On the other hand, (16b) yields

$$\frac{1}{\mu_v}\Big(\partial_r(ru^\delta_+) - \delta d'\partial_z(ru^\delta_+)\Big)\Big|_{r_{t_2}+\delta d} = \frac{1}{\mu_c}\Big(\partial_r(ru^\delta) - \delta d'\partial_z(ru^\delta)\Big)\Big|_{r_{t_2}+\delta d}. \tag{18}$$

From (17) and (18), we get the transmission conditions on Γ_c as

$$\begin{cases} u^\delta|_{r_{t_2}+\delta d} = u^\delta_+|_{r_{t_2}+\delta d}, & (19a)\\ \partial_r(ru^\delta)\big|_{r_{t_2}+\delta d} = \left(\dfrac{\frac{\mu_c}{\mu_v} + (\delta d')^2}{1 + (\delta d')^2}\partial_r(ru^\delta_+) + (1 - \dfrac{\mu_c}{\mu_v})\dfrac{\delta d'}{1 + (\delta d')^2}\partial_z(ru^\delta_+)\right)\Bigg|_{r_{t_2}+\delta d}. & (19b)\end{cases}$$

2.4 Procedure for obtaining effective transmission conditions between $u^\delta_\pm$

Given a re-scaling parameter $m \in \mathbb{N}$ in (4), we write the re-scaled in-layer eddy current problem (5) as a Cauchy problem for the re-scaled in-layer solution $\tilde{u}$ with initial values given by the transmission conditions (15) between u^δ and u^δ_- on Γ_{t_2}. The boundary values of $\tilde{u}$ on Γ_c should, after going back to the initial scale, match the transmission conditions (19) between u^δ and u^δ_+ on Γ_c. This yields the effective transmission conditions between u^δ_- and u^δ_+ on Γ_{t_2} by considering the Taylor series expansion (14) which allows to extend u^δ_+ to the interface Γ_{t_2} (19).

In asymptotic expansions, we develop $u^\delta_\pm$ and $\tilde{u}$ with respect to δ:

$$u^\delta_\pm = \sum_{n=0}^{\infty}\delta^n u^n_\pm, \qquad \tilde{u} = \sum_{n=0}^{\infty}\delta^n u^n.$$

We denote by $\mathcal{Z}_{m,n}$ the approximate transmission conditions between $u_\pm^\delta$ on Γ_{t_2} with re-scaling parameter m at order n in the asymptotic expansion $(\mathcal{O}(\delta^n))$. Therefore, we can obtain a family of asymptotic models with different approximate transmission conditions $\mathcal{Z}_{m,n}$ according to the choice of (m,n).

3 Asymptotic models for deposits with constant thickness

To determine (m,n) with which the asymptotic model using $\mathcal{Z}_{m,n}$ is both a good approximation and easy to deduce inverse methods, we study a simplified case where the deposit layer on the shell side of the tube has constant thickness δ.

3.1 Transmission conditions between $\tilde{u}$ (or $\tilde{w}$) and $u_\pm^\delta$

1. $m = 0, 1$.

The transmission conditions at Γ_{t_2} (15) yield

$$\begin{cases} u_-^\delta|_{r=r_{t_2}} = \tilde{u}|_{\rho=0}, \\ \dfrac{1}{\mu_T}\,\partial_r(ru_-^\delta)\Big|_{r=r_{t_2}} = \dfrac{1}{\mu_c}\left(\dfrac{r_{t_2}+\delta\rho}{\delta}\partial_\rho\tilde{u}+\tilde{u}\right)\Big|_{\rho=0}, \end{cases}$$

which imply

$$\begin{cases} u^n|_{\rho=0} = u_-^n|_{r=r_{t_2}}, & \text{(20a)} \\ \partial_\rho u^n\big|_{\rho=0} = -\dfrac{1}{r_{t_2}}u_-^{n-1}|_{r=r_{t_2}} + \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\,\partial_r(ru_-^{n-1})\big|_{r=r_{t_2}}. & \text{(20b)} \end{cases}$$

From the transmission conditions at Γ_c (19) we have

$$\begin{cases} u_+^\delta|_{r=r_{t_2}+\delta} = \tilde{u}|_{\rho=1}, \\ \dfrac{1}{\mu_v}\,\partial_r(ru_+^\delta)\Big|_{r=r_{t_2}+\delta} = \dfrac{1}{\mu_c}\left(\dfrac{r_{t_2}+\delta\rho}{\delta}\partial_\rho\tilde{u}+\tilde{u}\right)\Big|_{\rho=1}. \end{cases}$$

Combined with the Taylor expansions (14), the above conditions yield

$$\begin{cases} u^n|_{\rho=1} = \displaystyle\sum_{k=0}^{n}\left(\widetilde{\mathcal{S}}_k^0 u_+^{n-k} + \widetilde{\mathcal{S}}_k^1\partial_r(ru_+^{n-k})\right)\Big|_{r=r_{t_2}}, & \text{(21a)} \\ \partial_\rho u^n\big|_{\rho=1} = \displaystyle\sum_{l=0}^{n-1}\sum_{k=0}^{n-l-1}\frac{(-1)^l}{r_{t_2}^{l+1}}\Big\{-\left(\widetilde{\mathcal{S}}_k^0 u_+^{n-l-k-1} + \widetilde{\mathcal{S}}_k^1\partial_r(ru_+^{n-l-k-1})\right) \\ \quad +\dfrac{\mu_c}{\mu_v}(k+1)\left((r_{t_2}\widetilde{\mathcal{S}}_{k+1}^0+\widetilde{\mathcal{S}}_k^0)u_+^{n-l-k-1} + (r_{t_2}\widetilde{\mathcal{S}}_{k+1}^1+\widetilde{\mathcal{S}}_k^1)\partial_r(ru_+^{n-l-k-1})\right)\Big\}\Big|_{r_{t_2}}. & \text{(21b)} \end{cases}$$

2. $m = 2$.

From the definition of w, we have

$$\begin{cases} \tilde{w} = \sqrt{r_{t_2}+\rho\delta}\,\tilde{u}, \\ \partial_\rho\tilde{w} = \dfrac{\delta}{2\sqrt{r_{t_2}+\rho\delta}}\tilde{u} + \sqrt{r_{t_2}+\rho\delta}\,\partial_\rho\tilde{u}. \end{cases}$$

Then after some calculates, the transmission conditions at Γ_{t_2} (15) is transformed as

$$\begin{cases} \tilde{w}|_{\rho=0} = \sqrt{r_{t_2}}\tilde{u}|_{\rho=0} = \sqrt{r_{t_2}}u_-^\delta|_{r=r_{t_2}}, \\ \partial_\rho \tilde{w}\big|_{\rho=0} = \dfrac{\delta}{2\sqrt{r_{t_2}}}\tilde{u}|_{\rho=0} + \sqrt{r_{t_2}}\,\partial_\rho \tilde{u}\big|_{\rho=0} = -\dfrac{\delta}{2\sqrt{r_{t_2}}}u_-^\delta|_{r=r_{t_2}} + \dfrac{\mu_c}{\mu_t}\dfrac{\delta}{\sqrt{r_{t_2}}}\,\partial_r(ru_-^\delta)\big|_{r=r_{t_2}}, \end{cases}$$

which yields

$$\begin{cases} w^n|_{\rho=0} = \sqrt{r_{t_2}}u_-^n|_{r=r_{t_2}}, & \text{(22a)} \\ \partial_\rho w^n\big|_{\rho=0} = \dfrac{1}{\sqrt{r_{t_2}}}\left(-\dfrac{1}{2}u_-^{n-1} + \dfrac{\mu_c}{\mu_t}\partial_r(ru_-^{n-1})\right)\Big|_{r=r_{t_2}}. & \text{(22b)} \end{cases}$$

Similarly, the transmission conditions at Γ_c (19) become

$$\begin{cases} \tilde{w}|_{\rho=1} = \sqrt{r_{t_2}+\delta}\,u_+^\delta|_{r=r_{t_2}+\delta} \\ \qquad = \left(\sqrt{r_{t_2}} + \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\frac{(2k-3)!!}{2^k}r_{t_2}^{-\frac{2k-1}{2}}\delta^k\right)u_+^\delta|_{r=r_{t_2}+\delta} \\ \partial_\rho \tilde{w}\big|_{\rho=1} = \dfrac{\delta}{2\sqrt{r_{t_2}+\delta}}u|_{\rho=1} + \sqrt{r_{t_2}+\delta}\,\partial_\rho u\big|_{\rho=1} \\ \qquad = \delta\left(\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{(2k-1)!!}{2^k}r_{t_2}^{-\frac{2k+1}{2}}\delta^k\right)\left(-\frac{1}{2}u_+^\delta + \frac{\mu_c}{\mu_v}\partial_r(ru_+^\delta)\right)\Big|_{r=r_{t_2}+\delta}. \end{cases}$$

Together with the Taylor developments (14), the above transmission conditions yield

$$\begin{cases} w^n|_{\rho=1} = \displaystyle\sum_{k=0}^{n}a_k\sum_{l=0}^{n-k}\left(\widetilde{\mathcal{S}}_l^0(\partial_z)u_+^{n-k-l} + \widetilde{\mathcal{S}}_l^1(\partial_z)\partial_r(ru_+^{n-k-l})\right)\Big|_{r=r_{t_2}}, & \text{(23a)} \\ \partial_\rho w^n\big|_{\rho=1} = \displaystyle\sum_{k=0}^{n-1}b_k\sum_{l=0}^{n-k-1}\Bigg\{\left(-\frac{1}{2}\widetilde{\mathcal{S}}_l^0 + \frac{\mu_c}{\mu_v}(l+1)(r_{t_2}\widetilde{\mathcal{S}}_{l+1}^0 + \widetilde{\mathcal{S}}_l^0)\right)u_+^{n-k-l-1} & \\ \qquad + \left(-\dfrac{1}{2}\widetilde{\mathcal{S}}_l^1 + \dfrac{\mu_c}{\mu_v}(l+1)(r_{t_2}\widetilde{\mathcal{S}}_{l+1}^1 + \widetilde{\mathcal{S}}_l^1)\right)\partial_r(ru_+^{n-k-l-1})\Bigg\}\Big|_{r=r_{t_2}}. & \text{(23b)} \end{cases}$$

where

$$a_0 = \sqrt{r_{t_2}}, \qquad a_k|_{k\geq 1} = \frac{(-1)^{k-1}}{k!}\frac{(2k-3)!!}{2^k}r_{t_2}^{-\frac{2k-1}{2}},$$
$$b_k = \frac{(-1)^k}{k!}\frac{(2k-1)!!}{2^k}r_{t_2}^{-\frac{2k+1}{2}}.$$

3.2 Computing algorithm for the re-scaled in-layer field $\tilde{u}$ (or $\tilde{w}$)

In this section, we follow the procedure in 2.4 and give the detailed computing steps. Given $m = 0, 1$ or 2, we resolve the corresponding problem (6) or (8) in the thin deposit layer to obtain the transmission conditions between u_-^n and u_+^n (or between w_-^n and w_+^n) on Γ_{t_2} from the transmission conditions (20) - (21) between u^n and $u_\pm^n$ (or the transmission conditions (22) - (23) between w^n and $w_\pm^n$).

1. $m = 0, 1$.

We consider a general Cauchy problem with an arbitrary second member f for the same differential equation as in (6):

$$\partial_\rho^2 \check{u} = f \qquad \rho \in [0,1].$$

With the initial values at $\rho = 0$, the solution $\check{u}$ writes

$$\check{u}(\rho) = \check{u}|_{\rho=0} + \partial_\rho \check{u}\big|_{\rho=0}\,\rho + \int_0^\rho \int_0^s f(t)\,\mathrm{d}t\,\mathrm{d}s.$$

And at $\rho = 1$,

$$\begin{cases} \check{u}|_{\rho=1} = \check{u}|_{\rho=0} + \partial_\rho \check{u}\big|_{\rho=0} + \displaystyle\int_0^1 \int_0^s f(t)\,\mathrm{d}t\,\mathrm{d}s, \\ \partial_\rho \check{u}\big|_{\rho=1} = \partial_\rho \check{u}\big|_{\rho=0} + \displaystyle\int_0^1 f(t)\,\mathrm{d}t. \end{cases} \tag{24}$$

From the above resolvent and the re-scaled eddy current equation (6), it follows that the asymptotic expansions $\{u^n\}_n$ of $\tilde{u}$ can be obtained recurrently via the following Cauchy problems

$$\partial_\rho^2 u^n = -\sum_{j=1}^{4} \mathcal{B}_m^j u^{n-j}, \quad u^{-1} = u^{-2} = u^{-3} = u^{-4} = 0,$$

with initial values given by the transmission conditions (20) at $\rho = 0$. Then the boundary values of u^n at $\rho = 1$ given by (24) should coincident with those given by the transmission conditions (21) on Γ_c. These equalities give recurrently the transmission conditions linking up $u^n_\pm$ on Γ_{t_2}.

2. $m = 2$.

We consider the Cauchy problem with the same operator as in problem (8) and an arbitrary second member f

$$\left(\partial_\rho^2 + k_2^2\right)\check{w} = f \qquad \rho \in [0,1],$$

with initial values at $\rho = 0$. Its solution $\check{w}$ writes

$$\check{w} = (\check{w}|_{\rho=0} - (v\star f)|_{\rho=0})\cos(k_2\rho) + \frac{1}{k_2}\left(\partial_\rho \check{w}\big|_{\rho=0} - \partial_\rho (v\star f)\big|_{\rho=0}\right)\sin(k_2\rho) + v\star f, \tag{25}$$

where $v = \frac{1}{2\mathrm{i}k_2} e^{\mathrm{i}k_2|\rho|}$ is the fundamental solution, i.e. the solution of the problem with a Dirac distribution as second member:

$$\left(\partial_\rho^2 + k_2^2\right)\check{w} = \delta_0.$$

One computes

$$\begin{aligned} v\star f(\rho) &= \int_0^\rho \frac{1}{2\mathrm{i}k_2} e^{\mathrm{i}k_2(\rho-\xi)} f(\xi)\,\mathrm{d}\xi + \int_\rho^1 \frac{1}{2\mathrm{i}k_2} e^{\mathrm{i}k_2(\xi-\rho)} f(\xi)\,\mathrm{d}\xi, \\ \partial_\rho (v\star f)(\rho) &= \int_0^\rho \frac{1}{2} e^{\mathrm{i}k_2(\rho-\xi)} f(\xi)\,\mathrm{d}\xi - \int_\rho^1 \frac{1}{2} e^{\mathrm{i}k_2(\xi-\rho)} f(\xi)\,\mathrm{d}\xi. \\ v\star f(0) &= \int_0^1 \frac{1}{2\mathrm{i}k_2} e^{\mathrm{i}k_2\xi} f(\xi)\,\mathrm{d}\xi = -\frac{1}{\mathrm{i}k_2}\partial_\rho (v\star f)(0), \\ v\star f(1) &= \int_0^1 \frac{1}{2\mathrm{i}k_2} e^{\mathrm{i}k_2(1-\xi)} f(\xi)\,\mathrm{d}\xi = \frac{1}{\mathrm{i}k_2}\partial_\rho (v\star f)(1). \end{aligned}$$

By substituting the above terms in (25), we obtain

$$\begin{cases} \breve{w}|_{\rho=1} = \cos(k_2)\,(\breve{w} - v\star f)\,|_{\rho=0} + \dfrac{\sin(k_2)}{k_2}\,(\partial_\rho \breve{w} + \mathrm{i}k_2 v\star f)|_{\rho=0} + (v\star f)|_{\rho=1}, \\ \partial_\rho \breve{w}\big|_{\rho=1} = -k_2\sin(k_2)\,(\breve{w} - v\star f)\,|_{\rho=0} + \cos(k_2)\,(\partial_\rho \breve{w} + \mathrm{i}k_2 v\star f)\big|_{\rho=0} + \mathrm{i}k_2(v\star f)|_{\rho=1}. \end{cases} \tag{26}$$

Therefore, from the above resolvent procedure and the problem (8), the asymptotic expansions $\{w^n\}_n$ of $\tilde{w}$ verify recurrently the following Cauchy problems

$$\left(\partial_\rho^2 + k_2^2\right) w^0 = -\sum_{j=1}^{4} \mathcal{B}_2^j w^{n-j}, \qquad w^{-1} = w^{-2} = w^{-3} = w^{-4} = 0,$$

with initial values given by the transmission conditions (22) at $\rho = 0$ (on Γ_{t_2}). Their solutions give the boundary values (26) that should coincide with the transmission conditions (23) at $\rho = 1$ (on Γ_{t_2}), which implies recurrently the transmission conditions connecting $u_\pm^n$ on Γ_{t_2}.

3.3 Computation of some approximate transmission conditions $\mathcal{Z}_{m,n}$

In this section, we follow the computing algorithm described in the previous section 3.2 and give the transmission conditions $\mathcal{Z}_{m,n}$ on Γ_{t_2} for $m = 0, 1, 2$ and $n = 0, 1, 2$. To simplify the computations, we suppose $\mu_c = \mu_v$ which is the case for copper. We will use the first $\widetilde{\mathcal{S}}_k^i(\partial_z)$ operators in the Taylor developments (14) with their explicit expressions

$$\begin{aligned} &\widetilde{\mathcal{S}}_0^0 = \mathrm{Id}, && \widetilde{\mathcal{S}}_0^1 = 0,\\ &\widetilde{\mathcal{S}}_1^0 = -\frac{1}{r_{t_2}}, && \widetilde{\mathcal{S}}_1^1 = \frac{1}{r_{t_2}},\\ &\widetilde{\mathcal{S}}_2^0 = \frac{1}{r_{t_2}^2} - \frac{1}{2}\partial_z^2, && \widetilde{\mathcal{S}}_2^1 = -\frac{1}{2r_{t_2}^2}. \end{aligned}$$

We denote by $u_\pm^\delta$ the approximated fields of $u_\pm^\delta$ up to the asymptotic developments order, that is

$$\begin{aligned} &u_\pm^\delta = u_\pm^0 && \text{order } 0,\\ &u_\pm^\delta = u_\pm^0 + \delta u_\pm^1 && \text{order } 1,\\ &u_\pm^\delta = u_\pm^0 + \delta u_\pm^1 + \delta^2 u_\pm^2 && \text{order } 2. \end{aligned}$$

We also introduce the following notations

$$\begin{aligned} &\gamma_0^\pm u^\delta := u_\pm^\delta|_{r_{t_2}}, && \gamma_1^\pm u^\delta := \partial_r(r u_\pm^\delta)|_{r_{t_2}},\\ &[\gamma_0 u^\delta] := \gamma_0^+ u^\delta - \gamma_0^- u^\delta, && [\mu^{-1}\gamma_1 u^\delta] := \mu_v^{-1}\gamma_1^+ u^\delta - \mu_t^{-1}\gamma_1^- u^\delta,\\ &\langle \gamma_0 u^\delta\rangle := \frac{1}{2}(\gamma_0^+ u^\delta + \gamma_0^- u^\delta), && \langle \mu^{-1}\gamma_1 u^\delta\rangle := \frac{1}{2}(\mu_v^{-1}\gamma_1^+ u^\delta + \mu_t^{-1}\gamma_1^- u^\delta), \end{aligned}$$

Readers may skip the fastidious computational details and refer to the following expressions for the corresponding approximate transmission conditions.

$\mathcal{Z}_{0,0}$	(33)	or	(34)	$\mathcal{Z}_{0,1}$	(41)	or	(42)	$\mathcal{Z}_{0,2}$	(47)	or	(48)
$\mathcal{Z}_{1,0}$	(54)	or	(55)	$\mathcal{Z}_{1,1}$	(60)	or	(61)	$\mathcal{Z}_{1,2}$	(65)	or	(66)
$\mathcal{Z}_{2,0}$	(70)			$\mathcal{Z}_{2,1}$	(74)			$\mathcal{Z}_{2,2}$	(78)		

3.3.1 Rescaling parameter $m = 0$

1. Order $n = 0$.

From the asymptotic development (6) and the transmission conditions (20) on Γ_{t_2} for u^0, we have the differential equation for u^0

$$\begin{cases} \partial_\rho^2 u^0 = 0 \qquad \rho \in [0,1], \\ u^0|_{\rho=0} = u_-^0|_{r_{t_2}}, \\ \partial_\rho u^0\big|_{\rho=0} = 0, \end{cases}$$

which yields

$$u^0(\rho) = u_-^0|_{r_{t_2}} \quad \rho \in [0,1].$$

Thus, with the first transmission condition (21a) on Γ_c for u^0, which is

$$u^0|_{\rho=1} = \widetilde{\mathcal{S}}_0^0 u_+^0|_{r_{t_2}} + \widetilde{\mathcal{S}}_0^1 \partial_r(r u_+^0)|_{r_{t_2}} = u_+^0|_{r_{t_2}}, \tag{27}$$

we have

$$u_-^0|_{r_{t_2}} = u_+^0|_{r_{t_2}}. \tag{28}$$

Similarly, considering (6) and the transmission conditions (20) on Γ_{t_2} for u^1, we write the differential problem for u^1 as

$$\begin{cases} \partial_\rho^2 u^1 = -\mathcal{B}_0^1 u^0 = 0 \qquad \rho \in [0,1], \\ u^1|_{\rho=0} = u_-^1|_{r_{t_2}}, \\ \partial_\rho u^1\big|_{\rho=0} = -\dfrac{1}{r_{t_2}} u_-^0|_{r_{t_2}} + \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}}, \end{cases}$$

which implies

$$\partial_\rho u^1 = -\frac{1}{r_{t_2}}\left(u_-^0|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}}\right), \tag{29}$$

$$u^1 = u_-^1|_{r_{t_2}} - \frac{1}{r_{t_2}}\left(u_-^0|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}}\right)\rho. \tag{30}$$

The second transmission condition (21b) on Γ_c for u^1 writes

$$\partial_\rho u^1|_{\rho=1} = -\frac{1}{r_{t_2}}\left(u_+^0|_{r_{t_2}} - \frac{\mu_c}{\mu_v}\partial_r(r u_+^0)|_{r_{t_2}}\right). \tag{31}$$

Hence, the equalities (29) and (31) yield

$$\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}} = \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(r u_+^0)|_{r_{t_2}}. \tag{32}$$

(28) and (32) imply the approximate transmission conditions at order 0 on Γ_{t_2}

$$\mathcal{Z}_{0,0} \quad \begin{cases} u_-^\delta = u_+^\delta, & \text{(33a)} \\ \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(r u_-^\delta) = \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_v}\partial_r(r u_+^\delta), & \text{(33b)} \end{cases}$$

which write also

$$\mathcal{Z}_{0,0} \quad \begin{cases} [\gamma_0 u^\delta] = 0, & (34a)\\ [\mu^{-1}\gamma_1 u^\delta] = 0. & (34b)\end{cases}$$

We remark that $\mathcal{Z}_{0,0}$ given by (33) are simply the transmission conditions between the tube wall and the vacuum, as if the deposit layer does not exist.

2. Order $n = 1$.

The first transmission condition (21a) for u^1 on Γ_c

$$\begin{aligned} u^1|_{\rho=1} &= \widetilde{\mathcal{S}}_0^0 u_+^1|_{r_{t_2}} + \widetilde{\mathcal{S}}_0^1 \partial_r(r u_+^1)|_{r_{t_2}} + \widetilde{\mathcal{S}}_1^0 u_+^0|_{r_{t_2}} + \widetilde{\mathcal{S}}_1^1 \partial_r(r u_+^0)|_{r_{t_2}} \\ &= u_+^1|_{r_{t_2}} - \frac{1}{r_{t_2}}\left(u_+^0|_{r_{t_2}} - \partial_r(r u_+^0)|_{r_{t_2}}\right) \end{aligned} \tag{35}$$

together with the equality (30) imply

$$u_-^1|_{r_{t_2}} + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}} = u_+^1|_{r_{t_2}} + \frac{1}{r_{t_2}}\partial_r(r u_-^0)|_{r_{t_2}}. \tag{36}$$

One get the differential problem for u^2 from (6) using the previous expansions u^0, u^1 and the transmission conditions (20) on Γ_{t_2} for u^2

$$\begin{cases} \partial_\rho^2 u^2 = -\mathcal{B}_0^1 u^1 - \mathcal{B}_0^2 u^0 \\ \quad = \left(\frac{2}{r_{t_2}^2} - (\partial_z^2 + k_0^2)\right) u_-^0|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}} \qquad \rho \in [0,1], \\ u^2|_{\rho=0} = u_-^2|_{r_{t_2}}, \\ \partial_\rho u^2\big|_{\rho=0} = -\frac{1}{r_{t_2}} u_-^1|_{r_{t_2}} + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(r u_-^1)|_{r_{t_2}}. \end{cases}$$

Thus

$$\begin{aligned} \partial_\rho u^2 = &-\frac{1}{r_{t_2}}\left(u_-^1|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(r u_-^1)|_{r_{t_2}}\right) \\ &+ \left(\left(\frac{2}{r_{t_2}^2} - (\partial_z^2 + k_0^2)\right) u_-^0|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}}\right)\rho, \end{aligned} \tag{37}$$

$$\begin{aligned} u^2 = &u_-^2|_{r_{t_2}} - \frac{1}{r_{t_2}}\left(u_-^1|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(r u_-^1)|_{r_{t_2}}\right)\rho \\ &+ \left(\left(\frac{2}{r_{t_2}^2} - (\partial_z^2 + k_0^2)\right) u_-^0|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(r u_-^0)|_{r_{t_2}}\right)\frac{\rho^2}{2}. \end{aligned} \tag{38}$$

The explicit expression of the second transmission condition (21b) on Γ_c for u^2 is

$$\partial_\rho u^2|_{\rho=1} = -\frac{1}{r_{t_2}}\left(u_+^1|_{r_{t_2}} - \frac{\mu_c}{\mu_v}\partial_r(r u_+^1)|_{r_{t_2}}\right) + \left(\frac{2}{r_{t_2}^2} - \frac{\mu_c}{\mu_v}\partial_z^2\right) u_+^0|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\partial_r(r u_+^0)|_{r_{t_2}}. \tag{39}$$

Thus, (37) and (39) imply

$$\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(r u_-^1) - \left(k_0^2 + \partial_z^2\right) u_-^0 = \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(r u_+^1) - \frac{\mu_c}{\mu_v}\partial_z^2 u_+^0. \tag{40}$$

(36) and (40) lead to the first order approximate transmission conditions

$$\mathcal{Z}_{0,1}\quad \begin{cases} u_-^\delta + \dfrac{\delta}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^\delta) = u_+^\delta + \dfrac{\delta}{r_{t_2}}\partial_r(ru_+^\delta), & \text{(41a)}\\ \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^\delta) - \delta\left(k_0^2+\partial_z^2\right)u_-^\delta = \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_v}\partial_r(ru_+^\delta) - \delta\dfrac{\mu_c}{\mu_v}\partial_z^2 u_+^\delta. & \text{(41b)}\end{cases}$$

Considering (34) we have the equivalent expressions

$$\mathcal{Z}_{0,1}\quad \begin{cases} [\gamma_0 u^\delta] = 0, & \text{(42a)}\\ [\mu^{-1}\gamma_1 u^\delta] = -\delta\dfrac{r_{t_2}k_0^2}{\mu_c}\langle\gamma_0 u^\delta\rangle. & \text{(42b)}\end{cases}$$

3. Order $n = 2$.

From (38) and the first transmission condition (21a) for u^2 on Γ_c, which is

$$u^2|_{\rho=1} = u_+^2|_{r_{t_2}} - \frac{1}{r_{t_2}}\left(u_+^1|_{r_{t_2}} - \partial_r(ru_+^1)|_{r_{t_2}}\right) \tag{43}$$

one gets

$$\begin{aligned} u_-^2|_{r_{t_2}} + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}} + \frac{1}{2r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}} - \frac{1}{2}k_0^2u_-^0|_{r_{t_2}}\\ = u_+^2|_{r_{t_2}} + \frac{1}{r_{t_2}}\partial_r(ru_+^1)|_{r_{t_2}} + \frac{1}{2r_{t_2}^2}\partial_r(ru_+^0)|_{r_{t_2}}. \end{aligned} \tag{44}$$

To get the second transmission conditions connecting $u_\pm^2$ one has to consider the Cauchy problem for u^3 derived from (6) and the transmission conditions (20) on Γ_{t_2} for u^3

$$\begin{cases} \partial_\rho^2 u^3 = -\mathcal{B}_0^1u^2 - \mathcal{B}_0^2u^1 - \mathcal{B}_0^3u^0\\ \qquad = \left(\dfrac{2}{r_{t_2}^2} - (\partial_z^2+k_0^2)\right)u_-^1|_{r_{t_2}} - \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}\\ \qquad + \rho\left(\dfrac{3}{r_{t_2}^3} - \dfrac{1}{r_{t_2}}(\partial_z^2+k_0^2)\right)\left(-2u_-^0|_{r_{t_2}} + \dfrac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) \qquad \rho\in[0,1],\\ u^3|_{\rho=0} = u_-^3|_{r_{t_2}},\quad \partial_\rho u^3\big|_{\rho=0} = -\dfrac{1}{r_{t_2}}u_-^2|_{r_{t_2}} + \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^2)|_{r_{t_2}}. \end{cases}$$

On one hand, we get from the above Cauchy problem

$$\begin{aligned} \partial_\rho u^3|_{\rho=1} = &-\frac{1}{r_{t_2}}u_-^2|_{r_{t_2}} + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^2)|_{r_{t_2}} + \left(\frac{2}{r_{t_2}^2} - \frac{\mu_c}{\mu_v}\partial_z^2\right)u_-^1|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}\\ &+ \frac{1}{2}\left(\frac{3}{r_{t_2}^3} - \frac{1}{r_{t_2}}(\partial_z^2+k_0^2)\right)\left(-2u_-^0|_{r_{t_2}} + \frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right). \end{aligned}$$

One the other hand, the second transmission condition (21b) on Γ_c for u^3 writes explicitly

$$\begin{aligned} \partial_\rho u^3|_{\rho=1} = &-\frac{1}{r_{t_2}}\left(u_+^2|_{r_{t_2}} - \frac{\mu_c}{\mu_v}\partial_r(ru_+^2)|_{r_{t_2}}\right) + \left(\frac{2}{r_{t_2}^2} - \partial_z^2\right)u_+^1|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\partial_r(ru_+^1)|_{r_{t_2}}\\ &+ \left(-\frac{3}{r_{t_2}^3} + \frac{1}{2r_{t_2}}(1+\frac{\mu_c}{\mu_t})\partial_z^2\right)u_+^0|_{r_{t_2}} + \left(\frac{3}{2r_{t_2}^3} - \frac{1}{2r_{t_2}}\frac{\mu_c}{\mu_t}\partial_z^2\right)\partial_r(ru_+^0)|_{r_{t_2}}. \end{aligned} \tag{45}$$

The above two equalities result in

$$\begin{aligned}&\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru^2_-) - \left(k_0^2+\partial_z^2\right)u^1_- + \frac{1}{2r_{t_2}}\left(k_0^2+\partial_z^2\right)u^0_- - \frac{1}{2r_{t_2}}\frac{\mu_c}{\mu_t}k_0^2\partial_r(ru^0_-)\\ =&\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(ru^2_+) - \frac{\mu_c}{\mu_v}\partial_z^2 u^1_+ + \frac{1}{2r_{t_2}}\frac{\mu_c}{\mu_v}\partial_z^2 u^0_+. \end{aligned} \tag{46}$$

We derive from the previous transmission conditions (44) and (46)

$$\mathcal{Z}_{0,2}\quad\begin{cases}\left(1-\dfrac{\delta^2}{2}k_0^2\right)u^\delta_- + \left(\dfrac{\delta}{r_{t_2}}+\dfrac{\delta^2}{2r_{t_2}^2}\right)\dfrac{\mu_c}{\mu_t}\partial_r(ru^\delta_-) = u^\delta_+ + \left(\dfrac{\delta}{r_{t_2}}+\dfrac{\delta^2}{2r_{t_2}^2}\right)\partial_r(ru^\delta_+), & \text{(47a)}\\ \left(\dfrac{1}{r_{t_2}}-\dfrac{\delta^2}{2r_{t_2}}k_0^2\right)\dfrac{\mu_c}{\mu_t}\partial_r(ru^\delta_-) + \left(-\delta+\dfrac{\delta^2}{2r_{t_2}}\right)\left(k_0^2+\partial_z^2\right)u^\delta_- & \\ \qquad = \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_v}\partial_r(ru^\delta_+) + \left(-\delta+\dfrac{\delta^2}{2r_{t_2}}\right)\dfrac{\mu_c}{\mu_v}\partial_z^2 u^\delta_+. & \text{(47b)}\end{cases}$$

From (34), (42), the above conditions write also

$$\mathcal{Z}_{0,2}\quad\begin{cases}[\gamma_0 u^\delta] = \delta^2\dfrac{k_0^2}{2}\langle\gamma_0 u^\delta\rangle, & \text{(48a)}\\ [\mu^{-1}\gamma_1 u^\delta] = \left(-\delta\dfrac{r_{t_2}k_0^2}{\mu_c}+\delta^2\dfrac{k_0^2}{2\mu_c}\right)\langle\gamma_0 u^\delta\rangle - \delta^2\dfrac{k_0^2}{2}\langle\mu^{-1}\gamma_1 u^\delta\rangle. & \text{(48b)}\end{cases}$$

3.3.2 Rescaling parameter $m = 1$

1. Order $n = 0$.

The asymptotic development (6) and the transmission conditions (20) on Γ_{t_2} for u^0 lead to the Cauchy problem for u^0 with initial values at $\rho = 0$

$$\begin{cases}\partial_\rho^2 u^0 = 0 \qquad \rho\in[0,1],\\ u^0|_{\rho=0} = u^0_-|_{r_{t_2}},\\ \partial_\rho u^0\big|_{\rho=0} = 0,\end{cases}$$

which yields

$$u^0(\rho) = u^0_-|_{r_{t_2}} \quad \rho\in[0,1]. \tag{49}$$

Taking $\rho = 1$ in (49) and considering the transmission condition on Γ_c for u^0 (27), one gets

$$u^0_-|_{r_{t_2}} = u^0_+|_{r_{t_2}}. \tag{50}$$

Then we consider the Cauchy problem for u^1 given by the asymptotic development (6) and by the transmission conditions (20) on Γ_{t_2} for u^1

$$\begin{cases}\partial_\rho^2 u^1 = -\mathcal{B}_1^1 u^0 = -k_1^2 u^0_-|_{r_{t_2}} \qquad \rho\in[0,1],\\ u^1|_{\rho=0} = u^1_-|_{r_{t_2}},\\ \partial_\rho u^1\big|_{\rho=0} = -\dfrac{1}{r_{t_2}}u^0_-|_{r_{t_2}} + \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru^0_-)|_{r_{t_2}},\end{cases}$$

which implies

$$\partial_\rho u^1 = -\frac{1}{r_{t_2}}\left(u_-^0|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) - \rho k_1^2 u_-^0|_{r_{t_2}}, \tag{51}$$

$$u^1 = u_-^1|_{r_{t_2}} - \frac{\rho^2}{r_{t_2}}\left(u_-^0|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right)\rho - \frac{1}{2}k_1^2 u_-^0|_{r_{t_2}}. \tag{52}$$

(51) and the transmission condition on Γ_c for u^1 (31) imply

$$\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}} - k_1^2 u_-^0|_{r_{t_2}} = \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(ru_+^0)|_{r_{t_2}}. \tag{53}$$

(50) and (53) give the approximate transmission conditions at order 0 on Γ_{t_2} for $u_\pm^\delta$

$$\mathcal{Z}_{1,0} \quad \begin{cases} u_-^\delta = u_+^\delta, & \text{(54a)} \\ -k_1^2 u_-^\delta + \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^\delta) = \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_v}\partial_r(ru_+^\delta). & \text{(54b)} \end{cases}$$

That is

$$\mathcal{Z}_{1,0} \quad \begin{cases} [\gamma_0 u^\delta] = 0, & \text{(55a)} \\ [\mu^{-1}\gamma_1 u^\delta] = -\dfrac{r_{t_2}k_1^2}{\mu_c}\langle\gamma_0 u^\delta\rangle. & \text{(55b)} \end{cases}$$

We remark that the transmission conditions $\mathcal{Z}_{1,0}$ given by (54) are indeed the classical boundary impedance conditions which take into account the deposit layer.

2. Order $n = 1$.

From (52) and the transmission condition on Γ_c for u^1 (35) we have

$$u_-^1|_{r_{t_2}} + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}} - \frac{1}{2}k_1^2 u_-^0|_{r_{t_2}} = u_+^1|_{r_{t_2}} + \frac{1}{r_{t_2}}\partial_r(ru_+^0)|_{r_{t_2}}. \tag{56}$$

Using the asymptotic development (6) and the transmission conditions (20) on Γ_{t_2} for u^2, we get

$$\begin{cases} \partial_\rho^2 u^2 = -\mathcal{B}_1^1 u^1 - \mathcal{B}_1^2 u^0 \\ \qquad = -k_1^2 u_-^1|_{r_{t_2}} + \left(\dfrac{2}{r_{t_2}^2} - \partial_z^2\right) u_-^0|_{r_{t_2}} - \dfrac{1}{r_{t_2}^2}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}} \\ \qquad + \rho k_1^2\left(\dfrac{2}{r_{t_2}}u_-^0|_{r_{t_2}} - \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) + \rho^2\dfrac{k_1^4}{2}u_-^0|_{r_{t_2}} \qquad \rho \in [0,1], \\ u^2|_{\rho=0} = u_-^2|_{r_{t_2}}, \\ \partial_\rho u^2\big|_{\rho=0} = -\dfrac{1}{r_{t_2}}u_-^1|_{r_{t_2}} + \dfrac{1}{r_{t_2}}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}. \end{cases}$$

So the above Cauchy problem is solved by

$$
\begin{aligned}
\partial_\rho u^2 = & -\frac{1}{r_{t_2}}\left(u_-^1|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}\right) \\
& + \rho\left(-k_1^2 u_-^1|_{r_{t_2}} + \left(\frac{2}{r_{t_2}^2} - \partial_z^2\right) u_-^0|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) \\
& + \rho^2\frac{k_1^2}{2}\left(\frac{2}{r_{t_2}}u_-^0|_{r_{t_2}} - \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) + \rho^3\frac{k_1^4}{6}u_-^0|_{r_{t_2}}, \qquad (57)\\
u^2 = & \, u_-^2|_{r_{t_2}} - \frac{\rho}{r_{t_2}}\left(u_-^1|_{r_{t_2}} - \frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}\right) \\
& + \rho^2\frac{1}{2}\left(-k_1^2 u_-^1|_{r_{t_2}} + \left(\frac{2}{r_{t_2}^2} - \partial_z^2\right) u_-^0|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) \\
& + \rho^3\frac{k_1^2}{6}\left(\frac{2}{r_{t_2}}u_-^0|_{r_{t_2}} - \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) + \rho^4\frac{k_1^4}{24}u_-^0|_{r_{t_2}}. \qquad (58)
\end{aligned}
$$

Using (57) at $\rho = 1$ and the explicit expression of the transmission condition for u^2 on Γ_{t_2} (39), one has on Γ_{t_2} the equality

$$
-k_1^2 u_-^1 + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1) - \left(\partial_z^2 - \frac{k_1^2}{2r_{t_2}} - \frac{k_1^4}{6}\right) u_-^0 - \frac{k_1^2}{2r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0) = \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(ru_+^1) - \frac{\mu_c}{\mu_v}\partial_z^2 u_+^0. \qquad (59)
$$

(56) and (59) yield the approximate transmission conditions at order 1 between $u_\pm^\delta$ on Γ_{t_2}

$$
\mathcal{Z}_{1,1} \quad \left\{
\begin{aligned}
& \left(1 - \frac{\delta}{2}k_1^2\right) u_-^\delta + \frac{\delta}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^\delta) = u_+^\delta + \frac{\delta}{r_{t_2}}\partial_r(ru_+^\delta), && (60a)\\
& \left(-k_1^2 - \delta\left(\partial_z^2 - \frac{k_1^2}{2r_{t_2}} - \frac{k_1^4}{6}\right)\right) u_-^\delta + \left(\frac{1}{r_{t_2}} - \delta\frac{k_1^2}{2r_{t_2}}\right)\frac{\mu_c}{\mu_t}\partial_r(ru_-^\delta) \\
& \qquad = -\delta\frac{\mu_c}{\mu_v}\partial_z^2 u_+^\delta + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(ru_+^\delta). && (60b)
\end{aligned}
\right.
$$

With (61), the above conditions yield the equivalent expressions

$$
\mathcal{Z}_{1,1} \quad \left\{
\begin{aligned}
& [\gamma_0 u^\delta] = \delta\frac{k_1^2}{2}\langle\gamma_0 u^\delta\rangle, && (61a)\\
& [\mu^{-1}\gamma_1 u^\delta] = \frac{r_{t_2}}{\mu_c}\left(-k_1^2 + \delta(\frac{k_1^2}{2r_{t_2}} + \frac{k_1^4}{6})\right)\langle\gamma_0 u^\delta\rangle - \delta\frac{k_1^2}{2}\langle\mu^{-1}\gamma_1 u^\delta\rangle. && (61b)
\end{aligned}
\right.
$$

3. Order $n = 2$.

We derive from (58) and the transmission condition on Γ_c for u^1 (43)

$$
\begin{aligned}
& u_-^2 - \frac{1}{2}k_1^2 u_-^1 + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1) - \left(\frac{k_1^2}{6r_{t_2}} - \frac{k_1^4}{24}\right) u_-^0 + \left(\frac{1}{2r_{t_2}^2} - \frac{k_1^2}{6r_{t_2}}\right)\frac{\mu_c}{\mu_t}\partial_r(ru_-^0) \\
& \quad = u_+^2 + \frac{1}{r_{t_2}}\partial_r(ru_+^1) + \frac{1}{2r_{t_2}^2}\partial_r(ru_+^0). \qquad (62)
\end{aligned}
$$

Now we consider the Cauchy problem for u^3 with initial values at $\rho = 0$. From the asymptotic development (6) and the transmission conditions (20) on Γ_{t_2} for u^3, the Cauchy problem writes

$$\begin{cases} \partial_\rho^2 u^3 = -\mathcal{B}_1^1 u^2 - \mathcal{B}_1^2 u^1 - \mathcal{B}_1^3 u^0 \\ \quad = -k_1^2 u^2|_{r_{t_2}} + \left(\frac{2}{r_{t_2}^2} - \partial_z^2\right) u_-^1|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}} \\ \quad + \rho\Bigg(k_1^2(\frac{2}{r_{t_2}}u_-^1|_{r_{t_2}} - \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}) \\ \qquad + (-\frac{6}{r_{t_2}^3} + \frac{2}{r_{t_2}}\partial_z^2)u_-^0|_{r_{t_2}} + (\frac{3}{r_{t_2}^3} - \frac{1}{r_{t_2}}\partial_z^2)\frac{\mu_c}{\mu t}\partial_r(ru_-^0)|_{r_{t_2}}\Bigg) \\ \quad + \rho^2\left(\frac{k_1^4}{2}u_-^1|_{r_{t_2}} + k_1^2(-\frac{9}{2r_{t_2}^2} + \partial_z^2)u_-^0|_{r_{t_2}} + \frac{2k_1^2}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) \\ \quad + \rho^3 k_1^4\left(-\frac{1}{2r_{t_2}}u_-^0|_{r_{t_2}} + \frac{1}{6r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) - \rho^4\frac{k_1^6}{24}u_-^0|_{r_{t_2}} \qquad \rho\in[0,1], \\ u^3_{\rho=0} = u_-^3|_{r_{t_2}}, \\ \partial_\rho u^3\big|_{\rho=0} = -\frac{1}{r_{t_2}}u_-^2|_{r_{t_2}} + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^2)|_{r_{t_2}}. \end{cases}$$

Then we obtain

$$\begin{aligned} \partial_\rho u^3 &= -\frac{1}{r_{t_2}}u_-^2|_{r_{t_2}} + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^2)|_{r_{t_2}} \\ &+ \rho\left(-k_1^2u^2|_{r_{t_2}} + \left(\frac{2}{r_{t_2}^2} - \partial_z^2\right)u_-^1|_{r_{t_2}} - \frac{1}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}\right) \\ &+ \frac{\rho^2}{2}\Bigg(k_1^2(\frac{2}{r_{t_2}}u_-^1|_{r_{t_2}} - \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1)|_{r_{t_2}}) \\ &\qquad + (-\frac{6}{r_{t_2}^3} + \frac{2}{r_{t_2}}\partial_z^2)u_-^0|_{r_{t_2}} + (\frac{3}{r_{t_2}^3} - \frac{1}{r_{t_2}}\partial_z^2)\frac{\mu_c}{\mu t}\partial_r(ru_-^0)|_{r_{t_2}}\Bigg) \\ &+ \frac{\rho^3}{3}\left(\frac{k_1^4}{2}u_-^1|_{r_{t_2}} + k_1^2(-\frac{9}{2r_{t_2}^2} + \partial_z^2)u_-^0|_{r_{t_2}} + \frac{2k_1^2}{r_{t_2}^2}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) \\ &+ \frac{\rho^4}{4}k_1^4\left(-\frac{1}{2r_{t_2}}u_-^0|_{r_{t_2}} + \frac{1}{6r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^0)|_{r_{t_2}}\right) - \rho^5\frac{k_1^6}{120}u_-^0|_{r_{t_2}} \end{aligned} \tag{63}$$

Taking $\rho = 1$ in (63) and considering the transmission condition for u^3 on Γ_{t_2} (45), we have

$$\begin{aligned} &-k_1^2u_-^2 + \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^2) - \left(\partial_z^2 - \frac{k_1^2}{2r_{t_2}} - \frac{k_1^4}{6}\right)u_-^1 - \frac{k_1^2}{2r_{t_2}}\frac{\mu_c}{\mu_t}\partial_r(ru_-^1) \\ &+ \left((\frac{1}{2r_{t_2}} + \frac{k_1^2}{3})\partial_z^2 - (\frac{2k_1^2}{3r_{t_2}^2} + \frac{k_1^4}{12r_{t_2}} + \frac{k_1^6}{120})\right)u_-^0 + \left(\frac{k_1^2}{2r_{t_2}^2} + \frac{k_1^4}{24r_{t_2}}\right)\frac{\mu_c}{\mu_t}\partial_r(ru_-^0) \\ &= \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(ru_+^2) - \partial_z^2u_+^1 + \frac{1}{2r_{t_2}}\frac{\mu_c}{\mu_v}\partial_z^2u_+^0. \end{aligned} \tag{64}$$

From (62) and (64), we conclude the second order approximate transmission conditions for

$u_\pm^\delta$ on Γ_{t_2}

$$
\mathcal{Z}_{1,2}\quad \begin{cases}
\left(1-\frac{\delta}{2}k_1^2-\delta^2\big(\frac{k_1^2}{6r_{t_2}}-\frac{k_1^4}{24}\big)\right)u_-^\delta+\left(\frac{\delta}{r_{t_2}}+\delta^2\big(\frac{1}{2r_{t_2}^2}-\frac{k_1^2}{6r_{t_2}}\big)\right)\frac{\mu_c}{\mu_t}\partial_r(ru_-^\delta)\\
\quad =u_+^\delta+\left(\frac{\delta}{r_{t_2}}+\frac{\delta^2}{2r_{t_2}^2}\right)\partial_r(ru_+^\delta), & (65a)\\
\left(-k_1^2-\delta\big(\partial_z^2-\frac{k_1^2}{2r_{t_2}}-\frac{k_1^4}{6}\big)+\delta^2\left(\big(\frac{1}{2r_{t_2}}+\frac{k_1^2}{3}\big)\partial_z^2-\frac{2k_1^2}{3r_{t_2}^2}-\frac{k_1^4}{12r_{t_2}}-\frac{k_1^6}{120}\right)\right)u_-^\delta\\
\quad +\left(\frac{1}{r_{t_2}}-\delta\frac{k_1^2}{2r_{t_2}}+\delta^2\big(\frac{k_1^2}{2r_{t_2}^2}+\frac{k_1^4}{24r_{t_2}}\big)\right)\frac{\mu_c}{\mu_t}\partial_r(ru_-^\delta)\\
\quad =\left(-\delta+\frac{\delta^2}{2r_{t_2}}\frac{\mu_c}{\mu_v}\right)\partial_z^2u_+^\delta+\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}\partial_r(ru_+^\delta). & (65b)
\end{cases}
$$

Considering (55), (61), the above conditions also write

$$
\mathcal{Z}_{1,2}\quad \begin{cases}
[\gamma_0u^\delta]=\left(\delta\frac{k_1^2}{2}-\big(\frac{k_1^2}{6r_{t_2}}+\delta^2\frac{k_1^4}{12}\big)\right)\langle\gamma_0u^\delta\rangle+\delta^2\frac{\mu_ck_1^2}{3r_{t_2}}\langle\mu^{-1}\gamma_1u^\delta\rangle, & (66a)\\
[\mu^{-1}\gamma_1u^\delta]=\frac{r_{t_2}}{\mu_c}\left(-k_1^2+\delta\big(\frac{k_1^2}{2r_{t_2}}+\frac{k_1^4}{6}\big)+\delta^2\big(\frac{5k_1^2}{6}\partial_z^2-\frac{2k_1^2}{3r_{t_2}^2}-\frac{5k_1^4}{12r_{t_2}}-\frac{17k_1^6}{240}\big)\right)\langle\gamma_0u^\delta\rangle\\
\qquad -\left(\delta\frac{k_1^2}{2}+\delta^2\big(\frac{k_1^2}{2r_{t_2}}+\frac{k_1^4}{6}\big)\right)\langle\mu^{-1}\gamma_1u^\delta\rangle. & (66b)
\end{cases}
$$

3.3.3 Rescaling parameter $m = 2$

1. Order $n = 0$.

The asymptotic development (8) and the transmission conditions (22) on Γ_{t_2} for w^0 lead to the Cauchy problem for w^0 with initial values at $\rho = 0$

$$
\begin{cases}
(\partial_\rho^2+k_2^2)w^0=0 \qquad \rho\in[0,1],\\
w^0|_{\rho=0}=\sqrt{r_{t_2}}u_-^0|_{r_{t_2}},\\
\partial_\rho|_{\rho=0}=0.
\end{cases}
$$

The solution and its derivative writes

$$
\begin{cases}
w^0=\sqrt{r_{t_2}}u_-^0|_{r_{t_2}}\cos(k_2\rho), & (67a)\\
\partial_\rho w^0=-\sqrt{r_{t_2}}u_-^0|_{r_{t_2}}k_2\sin(k_2\rho). & (67b)
\end{cases}
$$

We consider the transmission conditions (22) on Γ_c for w^0, which writes explicitly

$$
\begin{cases}
w^0|_{\rho=1}=\sqrt{r_{t_2}}u_+^0|_{r_{t_2}}, & (68a)\\
\partial_\rho w^0|_{\rho=1}=0. & (68b)
\end{cases}
$$

By taking $\rho = 1$ in (67) and by comparing them with (68), we get

$$
u_-^0|_{r_{t_2}}=u_+^0|_{r_{t_2}}=0. \qquad (69)
$$

Therefore, at order 0, the approximate transmission conditions $\mathcal{Z}_{2,0}$ for $u^\delta_\pm$ on Γ_{t_2} becomes Dirichlet boundary conditions

$$\mathcal{Z}_{2,0} \qquad u^\delta_- = u^\delta_+ = 0. \tag{70}$$

These conditions model the conductive deposit as a perfect conductor.

2. Order $n = 1$.

The Cauchy problem for w^1 writes

$$\begin{cases} (\partial^2_\rho + k_2^2)w^1 = -\mathcal{B}^1_2 w^0 = 0 \qquad \rho \in [0,1], \\ w^1|_{\rho=0} = \sqrt{r_{t_2}}\, u^1_-|_{r_{t_2}}, \\ \partial_\rho w^1|_{\rho=0} = \dfrac{1}{\sqrt{r_{t_2}}}\left(-\dfrac{1}{2}u^0_-|_{r_{t_2}} + \dfrac{\mu_c}{\mu_t}\partial_r(ru^0_-)|_{r_{t_2}}\right), \end{cases}$$

where the initial values are just the transmission conditions (22) on Γ_{t_2}. Thus we have

$$\begin{cases} w^1 = \sqrt{r_{t_2}}\, u^1_-|_{r_{t_2}} \cos(k_2\rho) + \dfrac{1}{\sqrt{r_{t_2}}}\left(-\dfrac{1}{2}u^0_- + \dfrac{\mu_c}{\mu_t}\partial_r(ru^0_-)\right)\Big|_{r_{t_2}} \dfrac{\sin(k_2\rho)}{k_2}, & \text{(71a)} \\ \partial_\rho w^1 = -\sqrt{r_{t_2}}\, u^1_-|_{r_{t_2}} k_2 \sin(k_2\rho) + \dfrac{1}{\sqrt{r_{t_2}}}\left(-\dfrac{1}{2}u^0_- + \dfrac{\mu_c}{\mu_t}\partial_r(ru^0_-)\right)\Big|_{r_{t_2}} \cos(k_2\rho). & \text{(71b)} \end{cases}$$

Otherwise, the transmission conditions (22) on Γ_c for w^1 write

$$\begin{cases} w^1|_{\rho=1} = \sqrt{r_{t_2}}\, u^1_+|_{r_{t_2}} - \dfrac{1}{2\sqrt{r_{t_2}}}u^0_+|_{r_{t_2}} + \dfrac{1}{\sqrt{r_{t_2}}}\partial_r(ru^0_+)|_{r_{t_2}} & \text{(72a)} \\ \partial_\rho w^1|_{\rho=1} = -\dfrac{1}{2\sqrt{r_{t_2}}}u^0_+|_{r_{t_2}} + \dfrac{1}{\sqrt{r_{t_2}}}\dfrac{\mu_c}{\mu_v}\partial_r(ru^0_+)|_{r_{t_2}} & \text{(72b)} \end{cases}$$

Taking $\rho = 1$ in (71) and considering (72) imply

$$\begin{cases} \cos(k_2)u^1_- + \dfrac{1}{r_{t_2}}\dfrac{\sin(k_2)}{k_2}\dfrac{\mu_c}{\mu_t}\partial_r(ru^0_-) = u^1_+ + \dfrac{1}{r_{t_2}}\partial_r(ru^0_+), & \text{(73a)} \\ -\sin(k_2)u^1_- + \dfrac{1}{r_{t_2}}\dfrac{\cos(k_2)}{k_2}\dfrac{\mu_c}{\mu_t}\partial_r(ru^0_-) = \dfrac{1}{r_{t_2}}\dfrac{1}{k_2}\dfrac{\mu_c}{\mu_v}\partial_r(ru^0_+). & \text{(73b)} \end{cases}$$

So from (73) we obtain the first order approximate transmission conditions between $u^\delta_\pm$ on Γ_{t_2}

$$\mathcal{Z}_{2,1} \quad \begin{cases} \cos(k_2)u^\delta_- + \dfrac{\delta}{r_{t_2}}\dfrac{\sin(k_2)}{k_2}\dfrac{\mu_c}{\mu_t}\partial_r(ru^\delta_-) = u^\delta_+ + \dfrac{\delta}{r_{t_2}}\partial_r(ru^\delta_+), & \text{(74a)} \\ -\sin(k_2)u^\delta_- + \dfrac{\delta}{r_{t_2}}\dfrac{\cos(k_2)}{k_2}\dfrac{\mu_c}{\mu_t}\partial_r(ru^\delta_-) = \dfrac{\delta}{r_{t_2}}\dfrac{1}{k_2}\dfrac{\mu_c}{\mu_v}\partial_r(ru^\delta_+). & \text{(74b)} \end{cases}$$

3. Order $n = 2$.

With the transmission conditions (22) for w^2 on Γ_{t_2} as initial values, the Cauchy problem for w^2 writes

$$\begin{cases} (\partial^2_\rho + k_2^2)w^2 = -\mathcal{B}^1_2 w^1 - \mathcal{B}^2_2 w^0 = \sqrt{r_{t_2}}\left(\dfrac{3}{4r^2_{t_2}} - \partial_z^2\right)u^0_-|_{r_{t_2}}\cos(k_2\rho) \qquad \rho \in [0,1], \\ w^2|_{\rho=0} = \sqrt{r_{t_2}}\, u^2_-|_{r_{t_2}}, \\ \partial_\rho w^2|_{\rho=0} = \dfrac{1}{\sqrt{r_{t_2}}}\left(-\dfrac{1}{2}u^1_-|_{r_{t_2}} + \dfrac{\mu_c}{\mu_t}\partial_r(ru^1_-)|_{r_{t_2}}\right), \end{cases}$$

Using the computing algorithm for $m = 2$ described in Section 3.2, in particular the formula (26), one obtains

$$
\begin{cases}
w^2|_{\rho=1} = \left(\sqrt{r_{t_2}}u_-^2 + \dfrac{1-e^{i2k_2}-i2k_2}{8k_2^2}\sqrt{r_{t_2}}(\partial_z^2 - \dfrac{3}{4r_2^2})u_-^0\right)\Big|_{r_{t_2}} \cos(k_2) \\
\quad + \left\{\dfrac{1}{\sqrt{r_{t_2}}}\left(-\dfrac{1}{2}u_-^1 + \dfrac{\mu_c}{\mu_t}\partial_r(ru_-^1)\right) - \dfrac{i(1-e^{i2k_2}-i2k_2)}{8k_2}\sqrt{r_{t_2}}(\partial_z^2 - \dfrac{3}{4r_2^2})u_-^0\right\}\Big|_{r_{t_2}} \dfrac{\sin(k_2)}{k_2} \\
\quad + \left(\dfrac{i\sin k_2}{4k_2^2} + \dfrac{ie^{ik_2}}{4k_2}\right)\sqrt{r_{t_2}}(\partial_z^2 - \dfrac{3}{4r_2^2})u_-^0|_{r_{t_2}}, \qquad (75a) \\
\partial_\rho w^1 = -\left(\sqrt{r_{t_2}}u_-^2 + \dfrac{1-e^{i2k_2}-i2k_2}{8k_2^2}\sqrt{r_{t_2}}(\partial_z^2 - \dfrac{3}{4r_2^2})u_-^0\right)\Big|_{r_{t_2}} k_2\sin(k_2) \\
\quad + \left\{\dfrac{1}{\sqrt{r_{t_2}}}\left(-\dfrac{1}{2}u_-^1 + \dfrac{\mu_c}{\mu_t}\partial_r(ru_-^1)\right) - \dfrac{i(1-e^{i2k_2}-i2k_2)}{8k_2}\sqrt{r_{t_2}}(\partial_z^2 - \dfrac{3}{4r_2^2})u_-^0\right\}\Big|_{r_{t_2}} \cos(k_2) \\
\quad - \left(\dfrac{\sin k_2}{4k_2} + \dfrac{e^{ik_2}}{4}\right)\sqrt{r_{t_2}}(\partial_z^2 - \dfrac{3}{4r_2^2})u_-^0|_{r_{t_2}}. \qquad (75b)
\end{cases}
$$

Otherwise, the transmission conditions (23) for w^2 on Γ_c write

$$
\begin{cases}
w^2|_{\rho=1} = \sqrt{r_{t_2}}u_+^2|_{r_{t_2}} - \dfrac{1}{2\sqrt{r_{t_2}}}u_+^1|_{r_{t_2}} + \dfrac{1}{\sqrt{r_{t_2}}}\partial_r(ru_+^1)|_{r_{t_2}} \\
\qquad -\dfrac{\sqrt{r_{t_2}}}{2}\left(\partial_z^2 - \dfrac{3}{4r_{t_2}^2}\right)u_+^0|_{r_{t_2}}, \qquad (76a) \\
\partial_\rho w^2|_{\rho=1} = -\dfrac{1}{2\sqrt{r_{t_2}}}u_+^1|_{r_{t_2}} + \dfrac{1}{\sqrt{r_{t_2}}}\dfrac{\mu_c}{\mu_v}\partial_r(ru_+^1)|_{r_{t_2}} \\
\qquad -\sqrt{r_{t_2}}\left(\dfrac{\mu_c}{\mu_v}\partial_z^2 - \dfrac{3}{4r_{t_2}^2}\right)u_+^0|_{r_{t_2}} + \dfrac{1}{2r_{t_2}\sqrt{r_{t_2}}}\left(\dfrac{\mu_c}{\mu_v} - 1\right)\partial_r(ru_+^0)|_{r_{t_2}}. \qquad (76b)
\end{cases}
$$

From (75) and (76) we have

$$
\begin{cases}
\cos(k_2)u_-^2 + \dfrac{1}{r_{t_2}}\dfrac{\sin(k_2)}{k_2}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^1) + \dfrac{1}{2r_{t_2}^2}\left(\dfrac{\csc(k_2)}{k_2} - \dfrac{\cos(k_2)}{k_2^2}\right)\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^0) \\
\qquad = u_+^2 + \dfrac{1}{r_{t_2}}\partial_r(ru_+^1) - \dfrac{1}{2r_{t_2}^2}\left[\left(\dfrac{1}{k_2^2} - \dfrac{\cot(k_2)}{k_2}\right)\dfrac{\mu_c}{\mu_v} - 1\right]\partial_r(ru_+^0), \qquad (77a) \\
-\sin(k_2)u_-^2 + \dfrac{1}{r_{t_2}}\dfrac{\cos(k_2)}{k_2}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^1) + \dfrac{1}{2r_{t_2}^2}\dfrac{\sin(k_2)}{k_2^2}\dfrac{\mu_c}{\mu_t}\partial_r(ru_-^0) \\
\qquad = \dfrac{1}{r_{t_2}}\dfrac{1}{k_2}\dfrac{\mu_c}{\mu_v}\partial_r(ru_+^1) + \dfrac{1}{2r_{t_2}^2}\dfrac{1}{k_2}\partial_r(ru_+^0). \qquad (77b)
\end{cases}
$$

Therefore, we conclude that the second order approximate transmission conditions $\mathcal{Z}_{2,2}$ between $u_\pm^\delta$ on Γ_{t_2} write

$$\mathcal{Z}_{2,2}\quad\begin{cases}\cos(k_2)u_-^\delta+\left(\frac{\delta}{r_{t_2}}\frac{\sin(k_2)}{k_2}+\frac{\delta^2}{2r_{t_2}^2}\left(\frac{\csc(k_2)}{k_2}-\frac{\cos(k_2)}{k_2^2}\right)\right)\frac{\mu_c}{\mu_t}\partial_r(ru_-^\delta)\\ \qquad =u_+^\delta+\left[\frac{\delta}{r_{t_2}}-\frac{\delta^2}{2r_{t_2}^2}\left(\left(\frac{1}{k_2^2}-\frac{\cot(k_2)}{k_2}\right)\frac{\mu_c}{\mu_v}-1\right)\right]\partial_r(ru_+^\delta), & \text{(78a)}\\ -\sin(k_2)u_-^\delta+\left(\frac{\delta}{r_{t_2}}\frac{\cos(k_2)}{k_2}+\frac{\delta^2}{2r_{t_2}^2}\frac{\sin(k_2)}{k_2^2}\right)\frac{\mu_c}{\mu_t}\partial_r(ru_-^\delta)\\ \qquad =\left(\frac{\delta}{r_{t_2}}\frac{1}{k_2}\frac{\mu_c}{\mu_v}+\frac{\delta^2}{2r_{t_2}^2}\frac{1}{k_2}\right)\partial_r(ru_+^\delta). & \text{(78b)}\end{cases}$$

4 Numerical tests for 1-D models

To choose the (m,n) with which the asymptotic model has a best approximation, we test numerically the asymptotic models by implementing the transmission conditions $\mathcal{Z}_{m,n}$ for $m=0,1,2$ and $n=0,1,2$ in the 1-D case, i.e. $\partial_z=0$. The advantage of using 1-D eddy current models is that they have analytic solutions, which allows us to estimate modeling errors. We write the transmission conditions $\mathcal{Z}_{m,2}$ for $m=0,1,2$ at order $n=2$ in a general form

$$\begin{cases}\alpha_1^m u_-^\delta+\beta_1^m\partial_r(ru_-^\delta)=\gamma_1^m u_+^\delta+\eta_1^m\partial_r(ru_+^\delta),\\ \alpha_2^m u_-^\delta+\beta_2^m\partial_r(ru_-^\delta)=\gamma_2^m u_+^\delta+\eta_2^m\partial_r(ru_+^\delta).\end{cases}\tag{79}$$

The transmission conditions $\mathcal{Z}_{m,n}$ at order $n=0,1$ can be derived from $\mathcal{Z}_{m,2}$ by neglecting the high order terms. We give the coefficients α_j^m, β_j^m, γ_j^m and η_j^m, $m=0,1,2$, $j=1,2$ as below.

1. $m=0$.

From $\mathcal{Z}_{0,2}$ given by (47), we have

$$\alpha_1^0=1-\delta^2\frac{k_0^2}{2},\qquad \alpha_2^0=-\delta k_0^2+\delta^2\frac{k_0^2}{2r_{t_2}},$$

$$\beta_1^0=\delta\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}+\delta^2\frac{1}{2r_{t_2}^2}\frac{\mu_c}{\mu_t},\qquad \beta_2^0=\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_t}-\delta^2\frac{k_0^2}{2r_{t_2}}\frac{\mu_c}{\mu_t},$$

$$\gamma_1^0=1,\qquad \gamma_2^0=0,$$

$$\eta_1^0=\delta\frac{1}{r_{t_2}}+\delta^2\frac{1}{2r_{t_2}^2},\qquad \eta_2^0=\frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}.$$

One gets easily the coefficients corresponding to $\mathcal{Z}_{0,0}$ (see (33)) by considering only the terms on order $\mathcal{O}(1)$ of δ, and those corresponding to $\mathcal{Z}_{0,1}$ (see (41)) by neglecting the terms on order $\mathcal{O}(\delta^2)$.

2. $m=1$.

The transmission conditions $\mathcal{Z}_{1,2}$ given by (65) yields

$$\alpha_1^1 = 1 - \delta\frac{k_1^2}{2} - \delta^2\left(\frac{k_1^2}{6r_{t_2}} - \frac{k_1^4}{24}\right), \qquad \alpha_2^1 = -k_1^2 + \delta\left(\frac{k_1^2}{2r_{t_2}} + \frac{k_1^4}{6}\right) - \delta^2\left(\frac{2k_1^2}{3r_{t_2}^2} + \frac{k_1^4}{12r_{t_2}} + \frac{k_1^6}{120}\right),$$

$$\beta_1^1 = \left(\frac{\delta}{r_{t_2}} + \delta^2(\frac{1}{2r_{t_2}^2} - \frac{k_1^2}{6r_{t_2}})\right)\frac{\mu_c}{\mu_t}, \qquad \beta_2^1 = \left(\frac{1}{r_{t_2}} - \delta\frac{k_1^2}{2r_{t_2}} + \delta^2(\frac{k_1^2}{2r_{t_2}^2} + \frac{k_1^4}{24r_{t_2}})\right)\frac{\mu_c}{\mu_t},$$

$$\gamma_1^1 = 1, \qquad \gamma_2^1 = 0,$$

$$\eta_1^1 = \delta\frac{1}{r_{t_2}} + \delta^2\frac{1}{2r_{t_2}^2}, \qquad \eta_2^1 = \frac{1}{r_{t_2}}\frac{\mu_c}{\mu_v}.$$

For $\mathcal{Z}_{1,0}$ given by (54), one needs only to take the terms on order $\mathcal{O}(1)$ of δ in the above coefficients. For $\mathcal{Z}_{1,1}$ (see (60)), we neglect the terms on order $\mathcal{O}(\delta^2)$.

3. $m = 2$.

The transmission conditions $\mathcal{Z}_{2,2}$ (78) yield

$$\alpha_1^2 = \cos(k_2), \qquad \alpha_2^2 = -\sin(k_2),$$

$$\beta_1^2 = \left(\frac{\delta}{r_{t_2}}\frac{\sin k_2}{k_2} + \frac{\delta^2}{2r_{t_2}^2}(\frac{\csc(k_2)}{k_2} - \frac{\cos(k_2)}{k_2^2})\right)\frac{\mu_c}{\mu_t}, \qquad \beta_2^2 = \left(\frac{\delta}{r_{t_2}}\frac{\cos(k_2)}{k_2} + \frac{\delta^2}{2r_{t_2}^2}\frac{\sin(k_2)}{k_2^2}\right)\frac{\mu_c}{\mu_t},$$

$$\gamma_1^2 = 1, \qquad \gamma_2^2 = 0,$$

$$\eta_1^2 = \delta\frac{1}{r_{t_2}} - \delta^2\frac{1}{2r_{t_2}^2}\left((\frac{1}{k_2^2} - \frac{\cot(k_2)}{k_2})\frac{\mu_c}{\mu_v} - 1\right), \qquad \eta_2^2 = \delta\frac{1}{r_{t_2}}\frac{1}{k_2}\frac{\mu_c}{\mu_v} + \delta^2\frac{1}{2r_{t_2}^2}\frac{1}{k_2}.$$

We observe that terms on order $\mathcal{O}(1)$ of δ in the above coefficients gives the transmission conditions $\mathcal{Z}_{2,0}$ (see (70)). If we consider additionally the terms on $\mathcal{O}(\delta)$, then we obtains the transmission conditions $\mathcal{Z}_{2,1}$ (see (74)).

With these approximate transmission conditions $\mathcal{Z}_{m,n}$, we build the 1-D asymptotic models by supposing that there is no variation in the axial (z) direction. We may introduce a Dirac distribution like applied electric current $J\delta_{r_s}$ at $r = r_s$, which yields the transmission conditions at $r = r_s$,

$$[u]_{\Gamma_s} = 0,$$
$$\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right]_{\Gamma_s} = -\mathrm{i}\omega\mu J.$$

The analytic solution of the full 1-D model writes

$$u(r) = \begin{cases} c_1 r & 0 < r < r_s, \\ c_2 r + c_3\dfrac{1}{r} & r_s < r <_{t_1}, \\ c_4 J_1(k_t r) + c_5 Y_1(k_t r) & r_{t_1} < r < r_{t_2}, \\ c_6 J_1(k_c r) + c_7 Y_1(k_c r) & r_{t_2} < r < r_{t_2} + \delta, \\ c_8\dfrac{1}{r} & r > r_{t_2} + \delta, \end{cases}$$

with $k_t^2 = \mathrm{i}\omega\mu_t\sigma_t$ and $k_c^2 = \mathrm{i}\omega\mu_c\sigma_c$.

With the transmission conditions $[u] = [\mu^{-1}\partial_r(ru)] = 0$ at $r = r_{t_1}$, r_{t_2} and $r_{t_2}+\delta$, the coefficients $\boldsymbol{c} = (c_1, \ldots, c_8)^T$ can be obtained by resolving a linear system

$$A\boldsymbol{c} = (0, -\mathrm{i}\omega\mu J, 0, \ldots, 0)^T$$

and

$$A = \begin{pmatrix} r_s & -r_s & -\frac{1}{r_s} & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_{t_1} & \frac{1}{r_{t_1}} & -J_1(k_t r_{t_1}) & -Y_1(k_t r_{t_1}) & 0 & 0 & 0 \\ 0 & \frac{2}{\mu_v} & 0 & -\frac{k_t J_0(k_t r_{t_1})}{\mu_t} & -\frac{k_t Y_0(k_t r_{t_1})}{\mu_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_1(k_t r_{t_2}) & Y_1(k_t r_{t_2}) & -J_1(k_c r_{t_2}) & -Y_1(k_c r_{t_2}) & 0 \\ 0 & 0 & 0 & \frac{k_3 J_0(k_3 r_{t_2})}{\mu_t} & \frac{k_3 Y_0(k_3 r_{t_2})}{\mu_t} & -\frac{k_c J_0(k_c r_{t_2})}{\mu_c} & -\frac{k_c Y_0(k_c r_{t_2})}{\mu_c} & 0 \\ 0 & 0 & 0 & 0 & 0 & J_1(k_c(r_{t_2}+\delta)) & Y_1(k_c(r_{t_2}+\delta)) & -\frac{1}{(r_{t_2}+\delta)} \\ 0 & 0 & 0 & 0 & 0 & \frac{k_c J_0(k_c(r_{t_2}+\delta))}{\mu_c} & \frac{k_c Y_0(k_c(r_{t_2}+\delta))}{\mu_c} & 0 \end{pmatrix}.$$

The analytic solution of the asymptotic models is in the form

$$u^\delta(r) = \begin{cases} u_-^\delta = \begin{cases} c_1^\delta r & 0 < r < r_s, \\ c_2^\delta r + c_3^\delta \dfrac{1}{r} & r_s < r <_{t_1}, \\ c_4^\delta J_1(k_t r) + c_5^\delta Y_1(k_t r) & r_{t_1} < r < r_{t_2}, \end{cases} \\ u_+^\delta = c_6^\delta \dfrac{1}{r} & r > r_{t_2}. \end{cases}$$

With the transmission conditions $[u] = [\mu^{-1}\partial_r(ru)] = 0$ at $r = r_{t_1}$ and the approximate transmission conditions $\mathcal{Z}_{m,n}$ (79) at $r = r_{t_2}$ we obtain a linear system for the coefficients $\boldsymbol{c}^\delta = (c_1^\delta, \ldots, c_6^\delta)^T$

$$A^\delta \boldsymbol{c}^\delta = (0, -\mathrm{i}\omega\mu J, 0, \ldots, 0)^T,$$

where

$$A^\delta = \begin{pmatrix} r_s & -r_s & -\frac{1}{r_s} & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & r_{t_1} & \frac{1}{r_{t_1}} & -J_1(k_t r_{t_1}) & -Y_1(k_t r_{t_1}) & 0 \\ 0 & \frac{2}{\mu_v} & 0 & -\frac{k_t J_0(k_t r_{t_1})}{\mu_t} & -\frac{k_t Y_0(k_t r_{t_1})}{\mu_t} & 0 \\ 0 & 0 & 0 & \alpha_1^m J_1(k_t r_{t_2}) + \beta_1^m k_t J_0(k_t r_{t_2}) & \alpha_1^m Y_1(k_t r_{t_2}) + \beta_1^m k_t Y_0(k_t r_{t_2}) & -\gamma_1^m \frac{1}{r_{t_2}} \\ 0 & 0 & 0 & \alpha_2^m J_1(k_t r_{t_2}) + \beta_2^m k_t J_0(k_t r_{t_2}) & \alpha_2^m Y_1(k_t r_{t_2}) + \beta_2^m k_t Y_0(k_t r_{t_2}) & -\gamma_2^m \frac{1}{r_{t_2}} \end{pmatrix}.$$

4.1 Tests with fixed re-scaled conductivities

We first fix the re-scaled conductivities of the thin layer deposits σ_m, $m = 0, 1, 2$. Ignoring the physical unities, we take in our tests

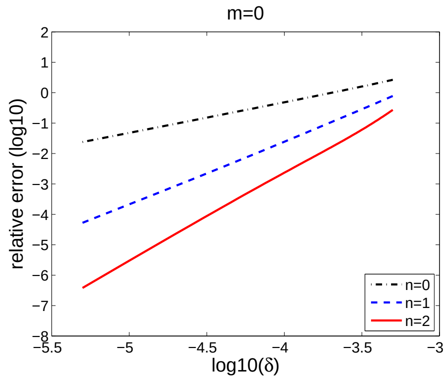
$$\sigma_0 = 5 \times 10^6,$$
$$\sigma_1 = 1 \times 10^3,$$
$$\sigma_2 = 5 \times 10^{-1}.$$

Then we evaluate the relative errors of the asymptotic models using $\mathcal{Z}_{m,n}$ $(n = 0, 1, 2)$ approximate transmission conditions with respect to the full model. We remark that here the deposit conductivity in the full model is variable according to the layer thickness δ:

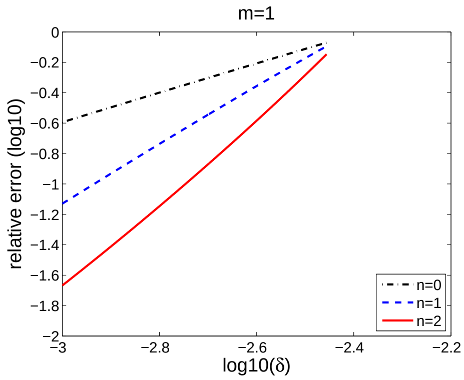
$$\sigma_c = \frac{\sigma_m}{\delta^m}.$$

We also recall that the permeability of the deposit is $\mu_c = \mu_v$, the conductivity of tube $\sigma_t = 9.7 \times 10^5 S/m$ and the permeability of tube $\mu_t = 1.01\mu_v$.

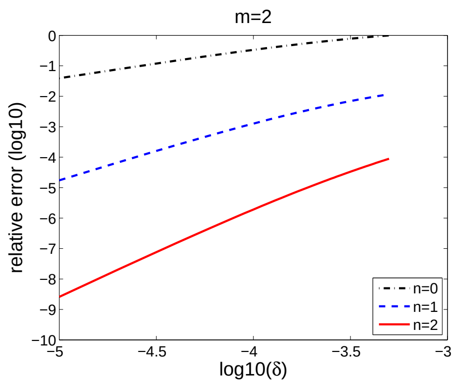
Figure 2 shows the relative errors in $L^2_{1/2}$-norm of solutions for the field outside the tube u^δ_- of the asymptotic models with respect to the full model for fixed re-scaled conductivities σ_m, $m = 0, 1, 2$. One observes that for a given re-scaling parameter m, the asymptotic models approximate better the full model as the asymptotic expansion order n increases. The slopes given in the figure validate numerically the above asymptotic models using approximate transmission conditions $\mathcal{Z}_{m,n}$ with the corresponding orders of approximation.



(a) $\sigma_0 = 5 \times 10^6$



(b) $\sigma_1 = 1 \times 10^3$

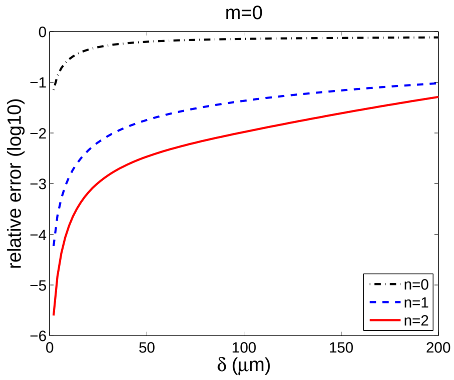


(c) $\sigma_2 = 5 \times 10^{-1}$

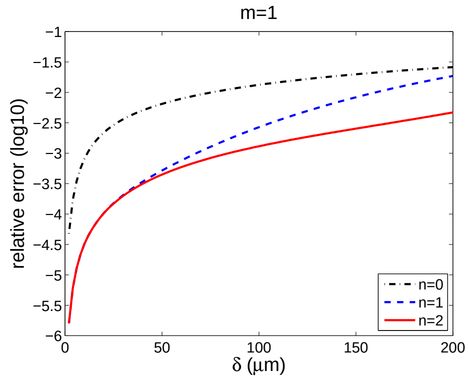
Figure 2: $L^2_{1/2}$-norm relative errors of asymptotic models with $\mathcal{Z}_{m,n}$ transmission conditions with fixed re-scaled conductivities.

4.2 Tests with real deposit conductivity

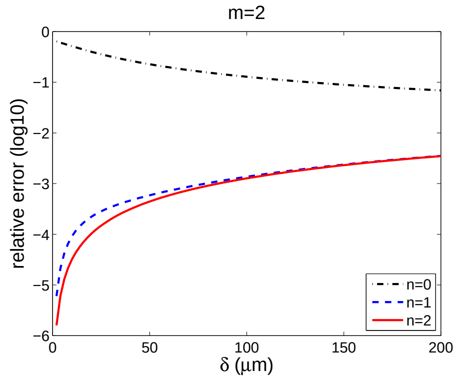
We consider a thin layer of copper covering the tube with constant thickness. The conductivity of copper is $\sigma_c = 5.8 \times 10^7 S/m$ and its permeability is $\mu_c = \mu_v$. σ_t and μ_t are the same as in the previous tests. We vary the thickness δ from $5\mu m$ to $200\mu m$ and evaluate the differences between the solutions u^δ of the asymptotic models with the solution u of the full model.



(a)



(b)



(c)

Figure 3: $L^2_{1/2}$-norm relative errors of asymptotic models with $\mathcal{Z}_{m,n}$ transmission conditions. Comparison between different expansion order n, re-scaling parameter m fixed.

Figure 3 shows that with a given re-scaling parameter $m = 0, 1$,or 2, the approximation gets better as the asymptotic expansion order n increases. One observes in Figure 3c that for a layer thickness under $200\mu m$, the asymptotic model using the transmission conditions $\mathcal{Z}_{2,0}$ is not a good approximation. This is because the $\mathcal{Z}_{2,0}$ conditions model the thin layer as perfect conductor, which is not true for copper through which the electrical field can still penetrate.

From the comparisons shown in Figure 4, we can conclude that for the asymptotic development order $n = 2$, the asymptotic model using the approximate transmission conditions $\mathcal{Z}_{2,2}$ is the best approximation of the full model among the three choices of the re-scaling parameter $m = 0, 1, 2$. However, we remark that in the corresponding coefficients α_j^2, β_j^2, γ_j^2 and η_j^2, $j = 1, 2$, the layer thickness δ, which we would like to reconstruct in the inverse problem, appears not only as polynomial factors but also implicitly in the trigonometric terms $\sin(k_2)$ and $\cos(k_2)$, now that $k_2^2 = \mathrm{i}\omega\mu_c\sigma_2 = \mathrm{i}\omega\mu_c\sigma_c\delta^2$. Hence it will be difficult to deduce the inverse problems from direct asymptotic models using $\mathcal{Z}_{2,2}$.

Meanwhile, one observes that the asymptotic models using $\mathcal{Z}_{1,n}$ are good approximations of the full model. For instance, if we choose a threshold of 1% relative error to judge whether an asymptotic model is accurate, then one observes in Figure 3b that even the asymptotic model

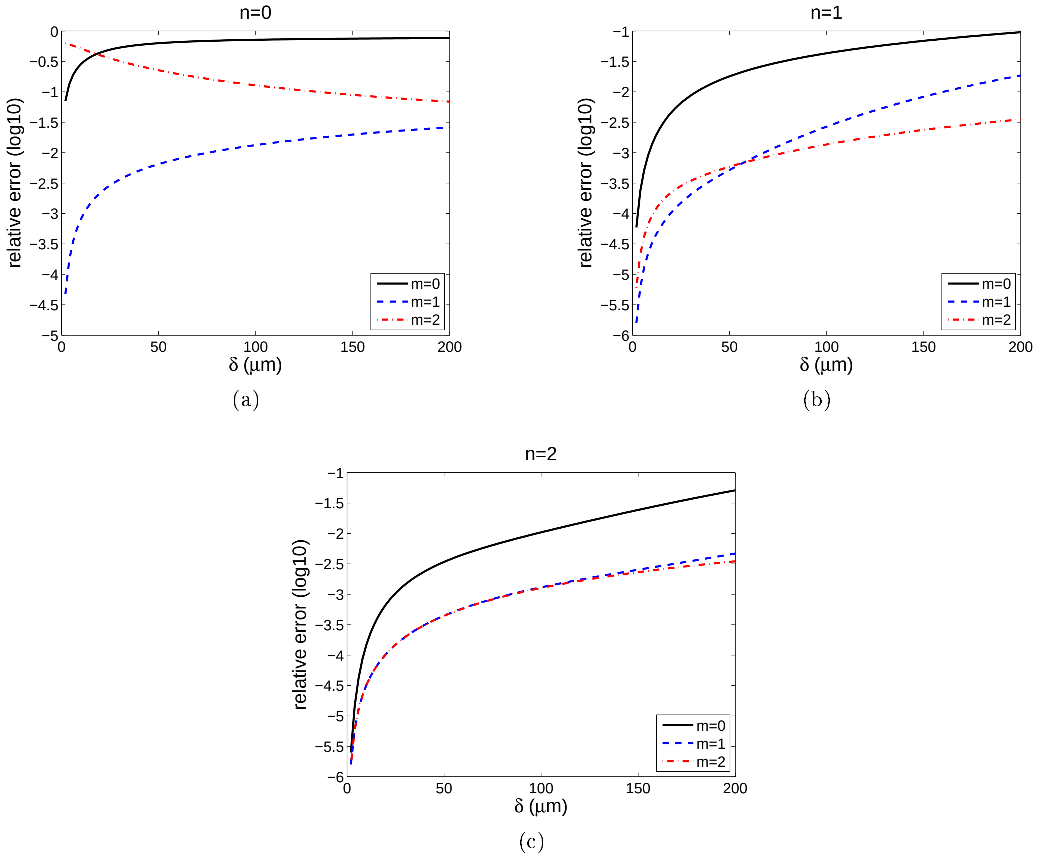


Figure 4: $L^2_{1/2}$-norm relative errors of asymptotic models with $\mathcal{Z}_{m,n}$ transmission conditions. Comparison between different re-scaling parameters m, expansion order n fixed

using $\mathcal{Z}_{1,0}$ gives a good approximation for thickness δ under $50\mu m$, which covers already a large range of interested thickness in industrial practice (see Table 1). The asymptotic model using $\mathcal{Z}_{1,1}$ ameliorates the precision for the full range of interested thickness (say, $\delta < 150\mu m$). With $m = 1$, the layer thickness δ appears only as polynomial factors in the coefficients α^1_j, β^1_j, γ^1_j and η^1_j, $j = 1, 2$, which facilitate the deduction of inverse method for the reconstruction of thickness.

Therefore, we will focus on the asymptotic models using $\mathcal{Z}_{1,n}$ with $n = 0, 1$ in the following discussion.

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