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PRECISION OF IMAGING ALGORITHMS BASED ON TOF ESTIMATION OF GUIDED WAVES

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ABSTRACT

This article addresses the problem of damage localization in an isotropic plate using guided waves. This localization is based on Time of Flight measurements thanks to an array of three sensors arbitrarily distributed on the structure. First, an exact solution of the Time of Flight problem is proposed. Then, the exact analytical Cramer-Rao Bounds, expressed as a function of the actuator and the sensors locations and the Signal to Noise Ratio of the measured signals, are presented. Statistical performance of the method is illustrated by means of Monte-Carlo simulations and compared to the Cramer-Rao Bounds. Time delays are estimated both with an Hilbert Transform method and a Maximum Likelihood Estimator. Results show that the performance of the algorithms is in agreement with the theoretical variance if Signal to Noise Ratio is low enough. These Cramer-Rao Bounds provide a way to determine the optimal position of a sensor array for a given a-priori accuracy.

KEYWORDS : *Structural health monitoring, Cramer Rao bound, damage localization, Maximum Likelihood Estimation, guided waves.*

1. INTRODUCTION

The monitoring of defects in a plate requires not only the identification of the nature of the flaw but an accurate estimation of its location. In particular, in the context of Structural Health Monitoring (SHM), some imaging techniques for damage detection are based on an array of piezoceramic sensors and actuators and Time of Flight (ToF) measurement [1]. But, the accuracy of results achieved using this kind of technique may depend on the distribution of the transducers and the signal to noise ratio (SNR) of the acquired signals.

This work addresses the problem of damage localization accuracy in an isotropic plate using guided waves. This localization is based on ToF measurements thanks to an array of three transducers arbitrarily distributed on the structure. First, the authors propose an exact solution of the ToF problem by solving a nonlinear system of coupled equations using the coordinates, supposed to be known, of the actuator and two sensors. Then, the damage localization consists in seeking the intersection of two ellipses established by two actuator-sensors pairs obtained from arrival times. A prior estimation of these arrival times is performed by a Maximum Likelihood Estimator (MLE).

However, estimations accuracy depends on the geometrical position of transducers. The exact analytical Cramer-Rao Bounds (CRBs), which give the lower bounds on the estimations variance, are presented. The calculation of these bounds is inspired by a previous work [2] whose aim was to estimate the variance of the position of Acoustic Emission (AE) events. These bounds are expressed as functions of the transducers positions and depend on the variance of ToF estimations. So, the CRBs for ToF estimations, which depend on the SNR of the data acquisition process, are also proposed. These expressions can be very useful in order to justify the *a priori* accuracy of the technique and for guiding the choice of the transducers array.

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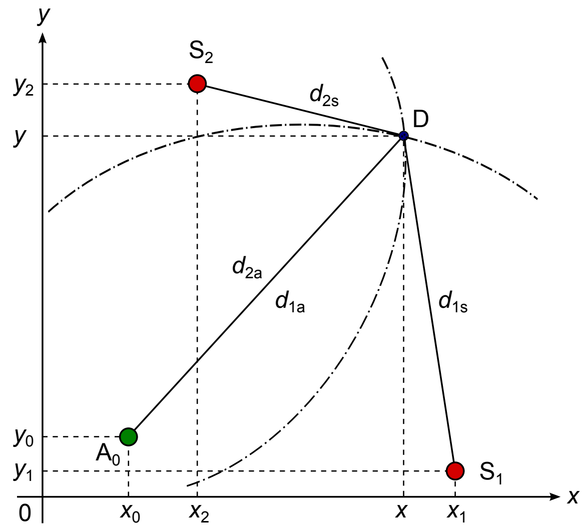

Figure 1: Coordinate system for actuator A_0, sensors S_1 and S_2, and damage D.

2. LOCALIZATION ESTIMATION

2.1 Principle of localization estimation

The basic principle for the localization calculation is the time-distance relationship implied by the acoustic wave velocity and consists in seeking a common intersection of two ellipses established by two actuator-sensors pairs [3, 4]. The arrival time t after propagation and scattering by a possible damage located at the point $D(x,y)$, combined with the sound velocity c, yields to the distance d between the acoustic source A_0 and the sensors: $d = c \cdot t$. In the case of non-dispersive propagation, each arrival time measured by sensors S_1 and S_2 yields to the equations $d_1 = c \cdot t_1 = d_{1a} + d_{1s}$ and $d_2 = c \cdot t_2 = d_{2a} + d_{2s}$, where:

$$d_{1a} = d_{2a} = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \tag{1}$$

is the Euclidean distance between the source A_0 and the damage D,

$$d_{1s} = \sqrt{(x-x_1)^2 + (y-y_1)^2}, \tag{2}$$

and,

$$d_{2s} = \sqrt{(x-x_2)^2 + (y-y_2)^2}, \tag{3}$$

are the distances between damage D and sensors S_1 and S_2 respectively as shown in Figure 1. In Equations (1), (2), and (3), (x,y) are the unknown damage coordinates, and (x_0,y_0), (x_1,y_1), and (x_2,y_2) the known coordinates of the actuator A_0 and the two sensors S_1, S_2 respectively. The goal of signal processing is to estimate the damage localization (x,y) from arrival times t_1 and t_2. However, the precision of the damage localization estimation depends not only on the actuator localization, the sensors, and the damage, but also on the accuracy of the ToF measurements from each actuator-sensors path. So, the *a priori* accuracy of the localization algorithm can be estimated thanks to the CRB expressed as a function of the parameters, x, y, x_0, y_0, x_1, y_1, x_2, y_2, and σ_τ^2, the time delay estimations variance.

2.2 Cramer-Rao bounds

The exact analytic forms of the CRBs indicate the lower limits of the estimation variance for an unknown damage coordinates (x, y). They are given by:

$$\mathrm{CRB}(x) = c^2\sigma_\tau^2 \cdot \frac{K_{21}^2 + K_{22}^2}{(K_{11}K_{22} - K_{12}K_{21})^2} \tag{4}$$

and

$$\mathrm{CRB}(y) = c^2\sigma_\tau^2 \cdot \frac{K_{11}^2 + K_{12}^2}{(K_{11}K_{22} - K_{12}K_{21})^2} \tag{5}$$

with

$$K_{11} = \frac{x - x_0}{d_{1a}} + \frac{x - x_1}{d_{1s}}, \quad K_{12} = \frac{x - x_0}{d_{2a}} + \frac{x - x_2}{d_{2s}}, \quad K_{21} = \frac{y - y_0}{d_{1a}} + \frac{y - y_1}{d_{1s}}, \quad \text{and} \quad K_{22} = \frac{y - y_0}{d_{2a}} + \frac{y - y_2}{d_{2s}},$$

and c the group velocity associated with the mode and center frequency of interest (Figure 3b). Mathematical proof of Equations (4) and (5) is given in Appendix A.

2.3 Practical localization calculation

The solution, corresponding to the intersection of two ellipses, as shown in Figure 1, can be solved and leads to the following algorithm:

1. Record of the actuator and sensors spatial coordinates, $x_0, y_0, x_1, y_1, x_2, y_2$;
2. Calculate the distance from time measurements: $d_1 = c \cdot t_1$ and $d_2 = c \cdot t_2$ where t_1 and t_2 are the times of arrival between actuator A_0 and sensors S_1 and S_2 respectively;
3. Calculate $\alpha_1 = (d_1^2 + x_0^2 + y_0^2 - x_1^2 - y_1^2)/2$ and $\alpha_2 = (d_2^2 + x_0^2 + y_0^2 - x_2^2 - y_2^2)/2$;
4. Calculate $A = -[(x_0 - x_1)d_2 - (x_0 - x_2)d_1] / [(y_0 - y_1)d_2 - (y_0 - y_2)d_1]$ and $B = (\alpha_1 d_2 - \alpha_2 d_1) / [(y_0 - y_1)d_2 - (y_0 - y_2)d_1]$;
5. Calculate:

$$\begin{aligned}
K_1 &= [(x_0 - x_1) + A(y_0 - y_1)]^2 - d_1^2(1 + A^2), \\
K_2 &= 2\{[B(y_0 - y_1) - \alpha_1][(x_0 - x_1) + A(y_0 - y_1)] + d_1^2[x_0 - A(B - y_0)]\}, \\
K_3 &= [\alpha_1 - B(y_0 - y_1)]^2 - d_1^2[x_0^2 + (B - y_0)^2]
\end{aligned}$$

6. Calculate the damage coordinates:

$$\hat{x} = \frac{-K_2 \pm \sqrt{K_2^2 - 4K_1K_3}}{2K_1} \quad \text{and} \quad \hat{y} = a\hat{x} + b. \tag{6}$$

3. Time delay estimation

3.1 Signal model

The damage localization procedure assumes that the time delays t_1 and t_2 are known. This can be achieved by measuring the group delay t_0 between the excitation signal and the measured ones, acquired after interaction with the damage. In a non-dispersive medium, a received signal is supposed to be a time-shift (t_0) version of the excitation. In this work, the received signal is modeled by the F_p carrier frequency Hanning windowed sine-wave $y(t) = s(t) + v(t) = a(t) \cdot \cos\phi(t) + v(t)$ with $a(t) = A/2\{1 + \cos[2\pi(t - t_0)/T]\} \cdot \mathrm{rect}[(t - t_0)/T]$ the envelope, $\phi(t) = 2\pi F_p(t - t_0)$ the instantaneous phase, and $v(t) = \mathcal{N}(0, \sigma_v^2)$ a zero-mean gaussian noise with variance σ_v^2. A is the maximum amplitude, t_0 the time when the envelope $a(t)$ is maximum, T the Hanning window width, and $\mathrm{rect}(\cdot)$

represents a rectangular window with a magnitude of 1 over its length T. After sampling , the discrete time measured signal is given by $y(kT_s)$, where T_s is the sampling period (throughout this document it will be noted that $y(kT_s) = y(k)$):

$$y(k) = a(k) \cdot \cos\phi(k) + v(k). \tag{7}$$

With this procedure, t_0 can be estimated from the observed signal $y(k)$. It is important to point out that t_0 can be considered as the group delay if the reference time, $t_0 = 0$, is taken on the peak value of the excitation signal envelope and assuming that the medium is non-dispersive. Under these conditions, the received signals may be considered as an attenuated amplitude time-shift versions of the excitation signal.

3.2 Cramer-Rao Bounds on time delays

The accuracy of the group delay estimation depends on the data acquisition process and especially on the SNR. The SNR is defined as the ratio between E_s, the energy of the noiseless signal $s(t)$, and E_v, the noise level over a time period of Δ. It can be shown that $E_s = 2A^2\Delta/16$ and $E_v = \sigma_v^2\Delta$. This yields to:

$$\text{SNR} = \frac{E_s}{E_v} = \frac{3}{16} \cdot \frac{A^2}{\sigma_v^2}. \tag{8}$$

It is now possible to study the impact of the SNR on the accuracy of the estimation of the group delay by calculating the CRB which gives the theoretical variance of the group delay estimation. The CRB for time delay estimation is given by:

$$\text{CRB}(t_0) = \frac{3}{(2\pi)^2} \cdot \frac{N_p}{1+3N_p^3} \cdot \frac{1}{F_s F_p} \cdot \frac{1}{\text{SNR}}. \tag{9}$$

Mathematical proof is given in Appendix B. Thanks to Equations (4), (5), and (9) it is now possible to link the *a priori* accuracy of the localization estimation to the measured signals SNR.

3.3 Maximum Likelihood Estimation

In this section, the MLE of the damage localization is derived and its implementation is described. The MLE is defined as the value of t_0 that maximizes $\wedge(\mathbf{y};t_0)$, the likelihood function is given in Appendix B by Equation (24). This estimator has been chosen to assess the CRB, because of its optimal behavior, i.e. it is unbiased, achieves the CRB and presents gaussian Probality Density Function (PDF) [5]. The MLE is found by maximizing $\wedge(\mathbf{y};t_0)$ over parameter t_0 and is given by:

$$\hat{t}_0 = \arg\min_{t_0} \{J(t_0)\} \tag{10}$$

where $J(t_0) = ||\mathbf{y} - \mathbf{s}||^2$ represents the mean square error (MSE) between signal $y(k)$ and its model $s(k)$. Since $J(t_0)$ is a nonlinear function of t_0, the optimization problem has to be solved numerically. In a first step, an initial estimate of $t_{0\text{init}}$ is obtained with a suboptimal estimator based on a Hilbert Transform (HT). This suboptimal estimator consist in calculating the group delay by identifying the maximum value of the signal envelope. The envelope $\hat{a}(k)$ is obtained implementing an HT of $y(k)$, $\hat{a}(k) = \mathscr{H}[y(k)]$. The decision is made when the estimated envelope $\hat{a}(k)$ reaches its maximum value:

$$t_{0_{\text{init}}} = \arg\max_{t_0} \{\hat{a}(k,t_0)\}. \tag{11}$$

In a second step, this initial estimate is refined using an iterative maximization procedure of $J(t_0)$ based on a Levenberg-Marquardt algorithm.

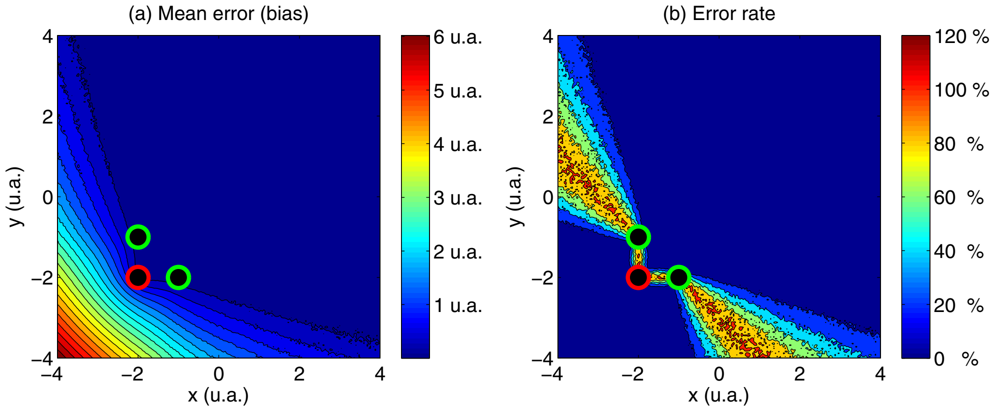

Figure 2: (a) Bias of damage localization estimation obtained with Monte-Carlo simulation - (b) Error rate; A_0 position $(x_0, y_0) = (-2, -2)$ (red dot), S_1 position $(x_1, y_1) = (-2, -1)$, and S_2 position $(x_2, y_2) = (-1, -2)$ (green dots) (arbitrary unit - a.u.) - Variances and CRBs are homogeneous to the square of a distance, $c = 1$ (u.a.) and $\sigma_\tau = 1$ (u.a.).

However, one of the main difficulties of this approach lies in the presence local minima of $J(t_0)$ which could impair the minimization procedure. This leads to an estimation significantly distant from the global minimum, which is particularly noticeable at low SNR. To overcome this issue, iterations stop when a preset threshold on the relative precision between two successive iterations reaches a preset threshold ε according to:

$$|J_{n+1}(\hat{t}_0) - J_n(\hat{t}_0)| = \varepsilon \cdot J_n(\hat{t}_0) \tag{12}$$

where $J_n(\hat{t}_0)$ and $J_{n+1}(\hat{t}_0)$ represent the values of $J(t_0)$ for the iterations n, $n+1$. In this work, this value is set to $\varepsilon = 10^{-7}$. In addition, in order to avoid convergence towards a local minimum, in situations where the initializing procedure is not close enough to the global optimum, the search is stopped after 50 iterations and the first suboptimal estimation is stored.

4. Results and conclusions

The statistical performance of the localization procedure is illustrated by means of Monte-Carlo simulations compared to the CRBs. 300 trials were run on simulated data and for several positions of the three sensors. First, damage localization procedure uses the nonlinear system of coupled equations mentioned in subsection 2.3. Then, time delays are estimated both with a HT method and a MLE. Arbitrary units (a.u.) has been chosen to describe each physical quantities. The problem parameters are $c = 1$, $\sigma_\tau = 1$, $(x_0, y_0) = (-2, -2)$ (actuator A_0), $(x_1, y_1) = (-2, -1)$ (sensors S_1), and $(x_2, y_2) = (-1, -2)$ (sensors S_2).

Figure 2(a) shows the bias of localization estimation for a particular position of the three sensors and gives the plate area where the estimations can be made and for which CRBs are valid, i.e. when bias is low. Moreover, Figure 2(b) gives the error rate, i.e. the percentage of instances where an estimation of the defect position is no longer possible (the two ellipses do not intersect). Figure 3 indicates that the performance (a) of the algorithms follows the theoretical variance (b) of time delay estimations in the valid area. Finally, Figure 4(b) shows that the MLE variances are close to the CRB when SNR is high enough (SNR $\geq$ 25 dB) as predicted by the theoretical analysis. Moreover, the initial estimates given by the HT seem to provide a good initial starting point when SNR $\geq$ 5 dB.

This study provides a simple way to determine the optimal position of sensor on an array for a given a-priori accuracy. An experimental set-up will be developed to assess this method.

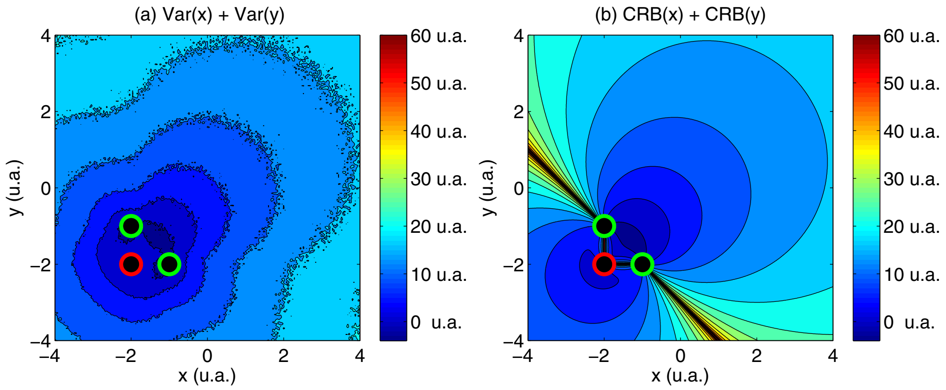

Figure 3: (a) Variance of damage localization estimation obtained with Monte-Carlo simulation - (b) Theoretical CRB; A_0 position $(x_0, y_0) = (-2, -2)$, S_1 position $(x_1, y_1) = (-2, -1)$, and S_2 position $(x_2, y_2) = (-1, -2)$ (arbitrary unit - a.u.) - Variances and CRBs are homogeneous to the square of a distance, $c = 1$ (u.a.) and $\sigma_\tau = 1$ (u.a.).

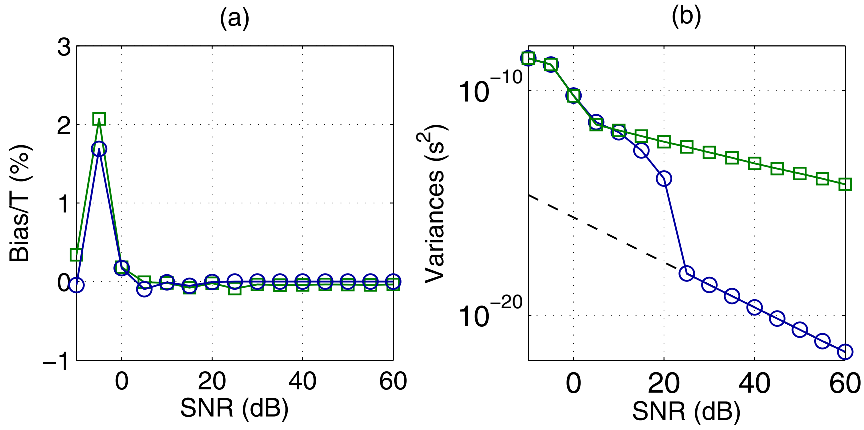

Figure 4: Bias (a) and variances (b) of the estimation of the time delay versus SNR - (□) Hilbert transform method, (○) MLE, (− − −) theoretical CRB - $N_p = 5.5$, $F_p = 390$ kHz.

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A CRAMER-RAO BOUNDS FOR THE ESTIMATION FOR THE DAMAGE COORDINATES

Let the observed data be $\tau_1 = t_1 + n_1$ and $\tau_2 = t_2 + n_2$ where the observation noises, n_1 and n_1, are zero-mean independent Gaussian processes with the same variance σ_τ^2, i.e. $n_1 \sim N(0, \sigma_\tau^2)$ and $n_2 \sim N(0, \sigma_\tau^2)$. Since τ_1 and τ_2 are non-correlated, the probability density functions (PDF) of the observations are given by:

$$p(\tau;\boldsymbol{\theta}) = p(\tau_1;\boldsymbol{\theta}) \times p(\tau_2;\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma_\tau^2}} \exp\left\{ -\frac{\left[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2\right]}{2\sigma_\tau^2} \right\} \tag{13}$$

with $m = 1,2$, $\boldsymbol{\theta} = [\theta_1, \theta_2]^T = [x,y]^T$ the unknown coordinates of the damage, and $\tau = [\tau_1, \tau_2]^T$. The log-likelihood function is then given by:

$$\wedge(\tau;\boldsymbol{\theta}) = \ln\left[p(\tau;\boldsymbol{\theta})\right] = -\frac{\ln(2\pi\sigma^2)}{2} - \frac{\left[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2\right]}{2\sigma^2}. \tag{14}$$

The CRB is the inverse of the Fisher Information Matrice (FIM) [5] whose elements are:

$$[\mathbf{I}(\boldsymbol{\theta})]_{i,j} = -E\left[\frac{\partial^2 \wedge(\tau;\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j}\right] \tag{15}$$

with $i = 1,2$ and $j = 1,2$. In order to evaluate the FIM, derivatives of $\wedge(\tau;\boldsymbol{\theta})$ need to be computed. It can be shown that the derivative of $\wedge(\tau;\boldsymbol{\theta})$ with respect to θ_i is given by:

$$\frac{\partial \wedge(\tau;\boldsymbol{\theta})}{\partial\theta_i} = -\frac{1}{2\sigma^2}\frac{\partial}{\partial\theta_i}\left[(\tau_1 - t_1)^2 + (\tau_2 - t_2)^2\right]. \tag{16}$$

The second order derivatives of $\wedge(\tau;\boldsymbol{\theta})$ are given by:

$$\begin{aligned}
\frac{\partial}{\partial\theta_j}\left[\frac{\partial \wedge(\tau;\boldsymbol{\theta})}{\partial\theta_i}\right] = &- \frac{1}{c\sigma_\tau^2}\left\{ +\frac{1}{c}\left(\frac{\theta_j - \theta_{j_0}}{d_{1a}} + \frac{\theta_j - \theta_{j_1}}{d_{1s}}\right)\left(\frac{\theta_i - \theta_{i_0}}{d_{1a}} + \frac{\theta_i - \theta_{i_1}}{d_{1s}}\right)\right. \\
&- \frac{1}{c}\left(\frac{\theta_j - \theta_{j_0}}{d_{2a}} + \frac{\theta_j - \theta_{j_2}}{d_{2s}}\right)\left(\frac{\theta_i - \theta_{i_0}}{d_{2a}} + \frac{\theta_i - \theta_{i_2}}{d_{2s}}\right) \\
&+ (\tau_1 - t_1)\left[\frac{(\theta_i - \theta_{i_0})(\theta_j - \theta_{j_0})}{d_{1a}^3} + \frac{(\theta_i - \theta_{i_1})(\theta_j - \theta_{j_1})}{d_{1s}^3}\right] \\
&\left. + (\tau_2 - t_2)\left[\frac{(\theta_i - \theta_{i_0})(\theta_j - \theta_{j_0})}{d_{2a}^3} + \frac{(\theta_i - \theta_{i_2})(\theta_j - \theta_{j_2})}{d_{2s}^3}\right]\right\}.
\end{aligned} \tag{17}$$

where $\theta_{1_0} = x_0$, $\theta_{2_0} = y_0$. By noting that $K_{j1} = (\theta_j - \theta_{j_0})/d_{1a} + (\theta_j - \theta_{j_1})/d_{1s}$ it can be shown that the expected values are:

$$-E\left[\frac{\partial^2 \wedge(\tau;\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j}\right] = \frac{1}{c^2\sigma_\tau^2}\left(K_{j1} \cdot K_{i1} + K_{j2} \cdot K_{i2}\right) \quad (i \neq j) \tag{18}$$

and

$$-E\left[\frac{\partial^2 \wedge(\tau;\boldsymbol{\theta})}{\partial\theta_i^2}\right] = \frac{1}{c^2\sigma_\tau^2}\left(K_{i1}^2 + K_{i2}^2\right) \quad (i = j). \tag{19}$$

The FIM is given by:

$$\mathbf{I}(\boldsymbol{\theta}) = \frac{1}{c^2\sigma_\tau^2}\begin{bmatrix} K_{11}^2 + K_{12}^2 & K_{11} \cdot K_{21} + K_{12} \cdot K_{22} \\ K_{11} \cdot K_{21} + K_{12} \cdot K_{22} & K_{21}^2 + K_{22}^2 \end{bmatrix} \tag{20}$$

Inverting matrix Equation (20) gives the CRB for the parameters $\theta_1 = x$ and $\theta_2 = y$:

$$\mathrm{CRB}(\theta_1) = \mathrm{CRB}(x) = c^2\sigma_\tau^2 \cdot \frac{K_{21}^2 + K_{22}^2}{(K_{11}K_{22} - K_{12}K_{21})^2}, \tag{21}$$

and

$$\mathrm{CRB}(\theta_2) = \mathrm{CRB}(y) = c^2\sigma_\tau^2 \cdot \frac{K_{11}^2 + K_{12}^2}{(K_{11}K_{22} - K_{12}K_{21})^2} \tag{22}$$

with

$$K_{11} = \frac{x - x_0}{d_{1a}} + \frac{x - x_1}{d_{1s}}, \quad K_{12} = \frac{x - x_0}{d_{2a}} + \frac{x - x_2}{d_{2s}}, \quad K_{21} = \frac{y - y_0}{d_{1a}} + \frac{y - y_1}{d_{1s}}, \quad \text{and} \quad K_{22} = \frac{y - y_0}{d_{2a}} + \frac{y - y_2}{d_{2s}}.$$

B Cramer-Rao bounds for time delay estimation

Let observed data, signal, noise be $\mathbf{y} = [y(k_1), \cdots, y(k_1 + N - 1)]^{\mathrm{T}}$, $\mathbf{s} = [s(k_1), \cdots, s(k_1 + N - 1)]^{\mathrm{T}}$, and $\mathbf{v} = [v(k_1), \cdots, v(k_1 + N - 1)]^{\mathrm{T}}$ where k_1 is the first sample and N, the number of acquired samples. So, $y(k)$ can be written in the compact form $\mathbf{y} = \mathbf{s} + \mathbf{v}$. The unknown parameters are collected in a single vector $\boldsymbol{\theta} = [x, y]^{\mathrm{T}} = [\theta_1, \theta_2]^{\mathrm{T}}$. The PDF of the observation is given by:

$$p(\mathbf{y}; t_0) = \prod_{k=k_1}^{k=k_2} \frac{1}{\sqrt{2\pi\sigma_v^2}} \exp\left\{-\frac{1}{2\sigma_v^2}\left[y(k) - s(k)\right]^2\right\} = \frac{1}{(2\pi\sigma_v^2)^{N/2}} \exp\left\{-\frac{1}{2\sigma_v^2}\|\mathbf{y} - \mathbf{s}\|^2\right\}, \tag{23}$$

with $k_2 = k_1 + N - 1$. In Equation (23) it is supposed that the N samples cover the whole duration where $\mathbf{y}$ is non-zero and $\|\cdot\|$ is the Euclidean norm. The log-likelihood function is given by:

$$\wedge(\mathbf{y}; t_0) = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|\mathbf{y} - \mathbf{s}\| \tag{24}$$

The CRB is then,

$$\mathrm{CRB}(t_0) = 1/\left\{-E\left[\frac{\partial^2 \wedge(\mathbf{y}; t_0)}{\partial t_0^2}\right]\right\}. \tag{25}$$

It can be shown that the derivative of $\wedge(\mathbf{y}; t_0)$ with respect to t_0 is:

$$\frac{\partial \wedge(\mathbf{y}; t_0)}{\partial t_0} = \frac{1}{\sigma^2}\sum_{k=k_1}^{k=k_2}\left[v(k) \cdot s'_{t_0}(k)\right] \tag{26}$$

with $v(k) = y(k) - s(k)$ and $s'_{t_0}(k) = \partial s(k)/\partial t_0$. The second order derivative is given by:

$$\frac{\partial^2 \wedge(\mathbf{y}; t_0)}{\partial t_0^2} = \frac{1}{\sigma^2}\sum_{k=k_1}^{k=k_2}\left\{-\left[s'_{t_0}(k)\right]^2 + v(k) \cdot s''_{t_0}(k)\right\} \tag{27}$$

with $s''_{t_0}(k) = \partial^2 s(k)/\partial t_0^2$. The expected value is then:

$$-E\left[\frac{\partial^2 \wedge(\mathbf{y}; t_0)}{\partial t_0^2}\right] = \frac{1}{\sigma^2}\sum_{k=k_1}^{k=k_2}\left[s'_{t_0}(k)\right]^2. \tag{28}$$

If $N = k_2 - k_1 + 1$ is large enough, it can be shown that:

$$-E\left[\frac{\partial^2 \wedge(\mathbf{y}; t_0)}{\partial t_0^2}\right] = \frac{1}{\sigma^2}N\left(\frac{A\pi}{2}\right)^2 \cdot \left(\frac{1}{T^2} + 3F_p^2\right). \tag{29}$$

Then, according to Equation (25):

$$\mathrm{CRB}(t_0) = \sigma^2\frac{N_p}{F_sF_p} \cdot \frac{1}{1 + 3 \cdot N_p^2} \cdot \left(\frac{2}{A\pi}\right)^2 = \frac{3}{(2\pi)^2} \cdot \frac{N_p}{1 + 3N_p^3} \cdot \frac{1}{F_sF_p} \cdot \frac{1}{\mathrm{SNR}}. \tag{30}$$