# Finite Wavelet Domain Method for Efficient Modeling of Lamb Wave Based Structural Health Monitoring

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# FINITE WAVELET DOMAIN METHOD FOR EFFICIENT MODELING OF LAMB WAVE BASED STRUCTURAL HEALTH MONITORING

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**ABSTRACT**

This document describes the development of an innovative and computationally efficient modeling approach for the prediction of the transient response in elastic rods and strips. The introduced beam element exploits the advantages of wavelets for the spatial discretization of the displacement field. Results are validated against with confirmed models and are found to be in agreement. The model appears to be very accurate and computationally more efficient than other popular numerical methods.

**KEYWORDS :** *Wavelet-based Finite Element, Transient Response.*

## 1 INTRODUCTION

The dynamic transient response and wave propagation in materials and structural components has attracted much interest in the development of innovative Structural Health Monitoring (SHM) methods and systems. Of particular importance is the design of active SHM systems based on linear and nonlinear guided waves. The design of the later requires the availability of modeling tools which can efficiently and robustly simulate ultrasonic Lamb wave propagation in undamaged and damaged structures. But the modeling of ultrasonic wave propagation in light-weight structures is greatly hindered by the numerical shortcomings and the high computational costs faced by Finite Element Analysis (FEA) and Finite Difference simulation methods. Presently available computational models require very fine spatial and temporal discretization, mandated by the small wavelengths and acousto-ultrasonic frequencies, and the small and local damage. Any effort to increase the accuracy of the spatial approximation using the h- or the p-method requires the introduction of intermediate nodes which render the time-simulation of Lamb waves impractical in terms of required computation time. Computational efficiency is further deteriorated in damage detection and localization, where the majority of detection methods have been developed independently, without sharing common features and synergies with modeling tools. In order to overcome this problem, preliminary work is presented towards the development of an innovative modeling method for the simulation of wave propagation in elastic isotropic beam structures. The new approach, termed thereafter as Finite Wavelet Domain (FWD) method, implements wavelet scaling functions for the spatial in-plane approximation of the displacement fields and their application to wave propagation problems in elastic strips. The FWD method will provide unique advantages compared to FE methods: ultra-high accuracy and convergence rate, capability to improve the order of approximation without increasing the size of the model and remeshing, and diagonal consistent mass matrices which will dramatically speed up explicit time integration.
The concept of wavelet-based FEA is not new. Patton and Marks [1] [2] have demonstrated the superiority of wavelet-based FEA vs. traditional approaches using a rod element. Later works also supported the fact that wavelets can be exploited as basis functions for the solution of partial

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differential equations and boundary value problems [3] [4] [5]. Further investigations have led to the development of more advanced wavelet-based FEs like rods [6], beams and plates [7] [8]. Wavelet-based time-integration schemes have also been presented [9] [10]. All these works prove that the wavelet-based simulation of elastic waves in media is both feasible and computationally more efficient than other successful numerical methods.
The present paper demonstrates a formulation of the FWD method based on Daubechies (DB) wavelet scaling functions for the wave simulation on rods and beams using first order shear theory. The following sections present an overview of the Daubechies wavelets family and their advantages as approximation functions of compact support, as well as the formulation of the FWD method. Numerical results demonstrate the capabilities of the FWD method regarding the simulation of longitudinal waves in rods, and antisymmetric Lamb waves in beams. In order to validate its accuracy, results of the FWD method are compared to traditional beam finite elements (FE).

## 2 THEORETICAL BACKGROUND

### 2.1 DAUBECHIES WAVELETS

The selection of the wavelet scaling functions, as interpolation functions, is an issue of much discussion [12]. Numerous wavelet families exist; each one with special characteristics. A review of the theoretical background of wavelets has been provided by Polyzou et al. [13]. The Daubechies (DB) wavelets family [14] seems to be more appropriate for the construction of a wavelet-based FEs. A major drawback of the DB wavelets is the lack of explicit mathematical formulation. Instead, DBs can be calculated at special points, termed as dyadic points, using the two-scale function:

$$\varphi(x)=\sum_{k=0}^{L-1} h_k \cdot \varphi(2x-k), \quad x\in[0,L-1], k\in\mathbb{Z} \tag{1}$$

where $\varphi$ is the scaling function, or father wavelet, $L$ is the order of the wavelet and $h_k$ is a set of filter coefficients uniquely defined for each wavelet. Their superiority as compared to other wavelet comes from the fact that, for a given support, DB wavelets have the maximum number of vanishing moments as compared to other wavelet families; this means that the DB wavelets can provide better polynomial approximation than any other wavelet of the same order.
The wavelet related calculations involved in finite elements are can be found in the open literature; analytical solutions have been provided for the calculation of scaling functions and wavelets, their derivatives, integrals and quadratures in bounded and infinite intervals [14] [16] [17] [18] [19]. These solutions diminish to singular eigenvalue problems whose normalization is a computationally demanding procedure, due to the recursive algorithms employed. However, these calculations are problem-independent and need to be performed only once.
The introduced FWD method is founded on the unique properties of Daubechies wavelet scaling functions, which are exploited towards the accurate and computationally efficient approximation of the displacement fields. Daubechies scaling functions are orthogonal and compactly supported, which ensures continuous approximation within a finite interval and the formulation of symmetric and banded stiffness matrices; the later significantly reduces the computational cost and improves the accuracy of explicit time integration schemes. The advantages of the FWD can be summarized as: i) high accuracy and convergence rate compared to FE methods; ii) capability to improve the order of approximation without increasing the size of the discrete model; iii) capability for local refinement of model fidelity and resolution without remeshing; and iv) diagonal consistent mass matrices, which dramatically speeds up explicit time integration.
The main concept of the introduced FWD method is that displacement fields can be expanded as series of the DB scaling function $\varphi$. The development of a Timoshenko beam and a simplified rod element are presented herein.

### 2.2 COMPUTATIONAL ISSUES

In the equations that will be derived in following sections, the concept of "connection coefficients" has been employed:

$$\Gamma_{kl}^{mn} = \int_0^1 \varphi^{(m)}\left(\xi - k\right) \cdot \varphi^{(n)}\left(\xi - l\right) d\xi \tag{2}$$

where $\varphi^{(i)}(\bullet)$ implies the i[th] derivative of the scaling function. The connection coefficients are actually quadratures of scaling functions and their derivatives in bounded domains. The calculation of connection coefficients depends only on the scaling function and consequently the order of the wavelet. Their use is imposed by the lack explicit formulation for the DB wavelets; this induces major difficulties and makes the calculations of Eq. (2) a rather challenging task. An intuitive, but rather simplistic, approach is numerical integration. However, the highly oscillatory nature of wavelets requires an ultra-fine discretization of the wavelet domain, which can be extremely demanding in computational resources. Additionally, numerical experiments showed that using less than 13 significant digits can compromise convergence. Such accuracy requires tremendous amounts of RAM and CPU power and computations become practically impossible as the order of the wavelet increases.

It is, therefore, necessary to implement analytical solutions and derive exact results. Fortunately, there are quite a lot of works in the open literature that may be exploited as a guideline for the differentiation and integration of scaling functions in bounded intervals [14] [16] [17] [18] [19]. Provided solutions usually require minor modifications in order to be adapted to problem specific conditions. A typical approach is to start from the dilation equation – Eq. (1) – and build the required connections coefficients by successive differentiation and integration. This reduces calculations to typical eigenvalue problems. A major drawback, however, of this approach is that eigenvalue problems are by definition singular and normalization conditions are required. Normalization can be accomplished by exploiting the fact that a DB wavelet of order $L$ can accurately approximate polynomials up to order $m$ = L/2 -1:

$$x^m = \sum_{k=-(L-2)}^{0} a_k \cdot \varphi\left(x - k\right) \tag{3}$$

In this way $m$ additional inhomogeneous equations can be obtained and alleviate the singularity of the eigenvalue problem.

### 2.3 SHEAR BEAM ELEMENT FORMULATION

The introduced method is illustrated for the case of guided waves in beams. A beam structure of total length $L_R$ is subdivided in a uniform grid of $N$ segments of length $L_S$ using $N+1$ nodes (Figure 1).

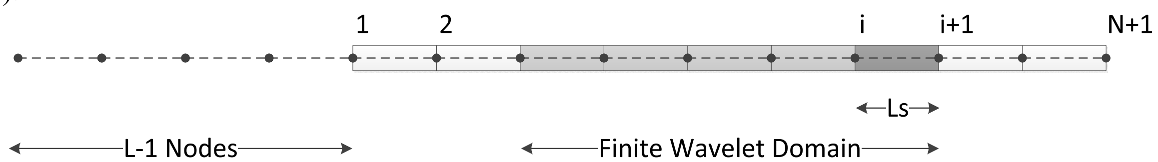



Figure 1 The spatial finite wavelet domain discretization.

Each segment can be treated as a specialty FWD Element. Any arbitrary element defined be nodes (i, i+1) expands in the domain [$x_1$, $x_2$] in a global Coordinate System (CS) or [0, 1] in its normalized local CS (Figure 2); the relation between the local and global coordinates is

$$\xi = \frac{x - x_1}{x_2 - x_1} \ . \tag{4}$$

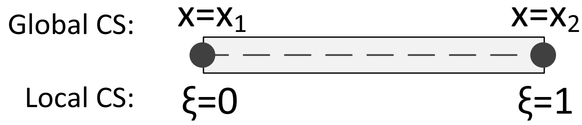


Figure 2 The Rod FWD Element.

According to Timoshenko beam theory cross-sections do not necessarily remain perpendicular to the longitudinal axis during deformation. Thus, the axial ($u$) and transverse ($w$) displacements are respectively obtained from

$$u = u_0 + \beta_x \cdot z, \quad w = w_0 \tag{5}$$

where $u_0$ and $w_0$ are the axial and transverse displacement at the midplane of the beam, and $z$ is the cross-sectional local coordinate and $\beta_x$ is the slope with respect to the axial direction. Derivation of displacement provides axial ($\varepsilon_x$) and shear ($\varepsilon_{xz}$) strain respectively:

$$\varepsilon_x = u_{0,x} + \beta_{x,x} \cdot z, \quad \varepsilon_{xz} = \beta_x + w_{0,x} \tag{6}$$

Incorporation of Hooke's law into Virtual Works Principle leads to

$$-\int_0^1 \left(\delta\varepsilon_x \cdot \sigma_x - \delta\varepsilon_{xz} \cdot \sigma_{xz}\right) \cdot A \; d\xi + \int_0^1 \left(\delta\mathbf{u}^{\mathrm{T}} \cdot \rho \cdot \ddot{\mathbf{u}} + \delta\mathbf{w}^{\mathrm{T}} \cdot \rho \cdot \ddot{\mathbf{w}}\right) \cdot A \; d\xi + \int_0^1 w \cdot q \, dz \tag{7}$$

where $A$ is the cross-sectional area and $q(\xi)$ is the load applied along the element.
According to wavelet theory, the axial displacement ($u$) along the rod can be approximated in its local CS using a DB scaling function $\varphi$ of order $L$ as:

$$u(\xi, t) = \sum_{k=-(L-2)}^{0} u_k(t) \cdot \varphi(\xi - k) \tag{8}$$

where $u_k$ are coefficients to be determined and represent the generalized Degrees of Freedom (DOFs) for node $k$ in wavelet space. This formulation implies that each element also uses information from its preceding domain, termed as Domain of Influence (DI). In traditional FEA, the addition of an element would include a whole set of $L$-1 nodes. On the other hand, the FWD method allows for overlapping DIs (Figure 3), thus, the computational load increases only by one node per additional element. The size of the DI is directly connected to the order $L$ of the wavelet; thus, the amount of information captured by each element can be increased simply by choosing a wavelet of higher order.

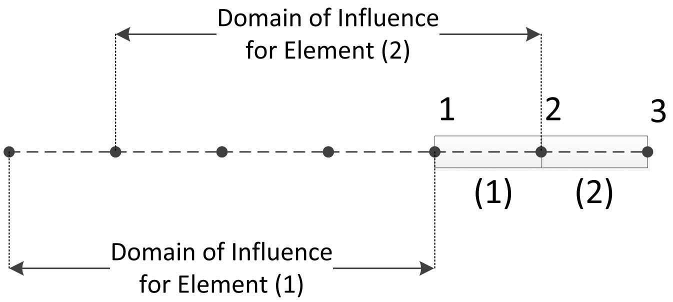


Figure 3 The Domains of Influence of each element for an assembly of two FWD Rod Elements.

Based on Eq. (8), the respective strains in Eq. (7) can be derived:

$$u_{0,\xi} = \sum_k \beta_k \cdot \varphi_{,\xi}\left(\xi - k\right) \cdot z \tag{9}$$

$$\varepsilon_{xz} = \sum_k \beta_k \cdot \varphi_{,\xi}\left(\kappa - k\right) + \sum_k w_k \cdot \varphi_{,\xi}\left(\xi - k\right) \tag{10}$$

Then, incorporation of Eqs. (9)-(10) into (7) leads to

$$\mathbf{M} \cdot \ddot{\mathbf{u}}_l(t) + \mathbf{K} \cdot \mathbf{u}_l(t) = \mathbf{F}(t) \tag{11}$$

where **[M]** is the mass matrix, **[K]** is the stiffness matrix, **u** is the displacement field and **F** is the vector of external forces; all quantities expressed in wavelet space:

$$\left[K^{kl}\right]=\begin{bmatrix} EA\cdot\Gamma_{kl}^{11} & 0 & 0 \\ & GA\cdot\Gamma_{kl}^{11} & GA\cdot\Gamma_{kl}^{01} \\ sym & & GA\cdot\Gamma_{kl}^{00}+EI\cdot\Gamma_{kl}^{11} \end{bmatrix} \tag{12}$$

$$\left[M^{kl}\right]=diag\left(\rho\cdot A\cdot\Gamma_{kl}^{00}\right) \tag{13}$$

$$F^{k}\left(t\right)=\int_{0}^{1} f\left(\xi,t\right)\cdot\varphi\left(\xi-k\right)d\xi \tag{14}$$

In Eq. (14), $f(\xi,t)$ denotes the time varying load along the element in local CS.
Considering time response, the same approach may be followed employing time dependent scaling functions, i.e.

$$u\left(\xi,t\right)=\sum_{i}{}^{i}u_{k}\left(\xi\right)\cdot\varphi\left(t-i\right) \tag{15}$$

$$\ddot{u}\left(\xi,t\right)=\sum_{i}{}^{i}u_{k}\left(\xi\right)\cdot\ddot{\varphi}\left(t-i\right) \tag{16}$$

Incorporation of Eqs. (15)-(16) into Eq. (11) finally yields

$$\begin{aligned}&\mathbf{M}\cdot\ddot{\mathbf{u}}_{l}\left(t\right)+\mathbf{K}\cdot\mathbf{u}_{l}\left(t\right)=\mathbf{F}\left(t\right)\\ &\Rightarrow\left[M^{kl}\right]\cdot\sum_{i}{}^{i}u_{l}\left(\xi\right)\cdot\ddot{\varphi}\left(t-i\right)+\left[K^{kl}\right]\cdot\sum_{i}{}^{i}u_{l}\left(\xi\right)\cdot\varphi\left(t-i\right)=\sum_{i}{}^{k}F_{i}\left(x\right)\cdot\varphi\left(t-i\right)\end{aligned} \tag{17}$$

which can be iteratively solved with usual explicit time integration schemes. It has to be noted that the mass matrix in Eq. (13) is diagonal due to the orthogonality of scaling functions; this will boost the speed of time integration in Eq. (17).
In the following section, benchmark cases will be presented for rod and beam elements. The formulation of a rod element can be derived from Eqs. (5)-(17) by neglecting all transverse and shear terms. Alternatively, the same procedure can be applied on the equation of motion for an elastic rod:

$$u_{,xx}=\frac{\rho}{E}\ddot{u} \tag{18}$$

where $\rho$ is the density and $E$ is the Young modulus of the rod. In any case, Eqs. (11) and (17) remain the same, but the mass and stiffness matrices are reduced to

$$\left[M^{kl}\right]=\rho A\cdot\int_{0}^{1}\varphi\left(\xi-k\right)\cdot\varphi\left(\xi-l\right)d\xi=\rho A\cdot\left[\Gamma_{kl}^{00}\right] \tag{19}$$

$$\left[K^{kl}\right]=EA\cdot\int_{0}^{1}\varphi_{,\xi}\left(\xi-k\right)\cdot\varphi_{,\xi}\left(\xi-l\right)d\xi=EA\cdot\left[\Gamma_{kl}^{11}\right]\ . \tag{20}$$

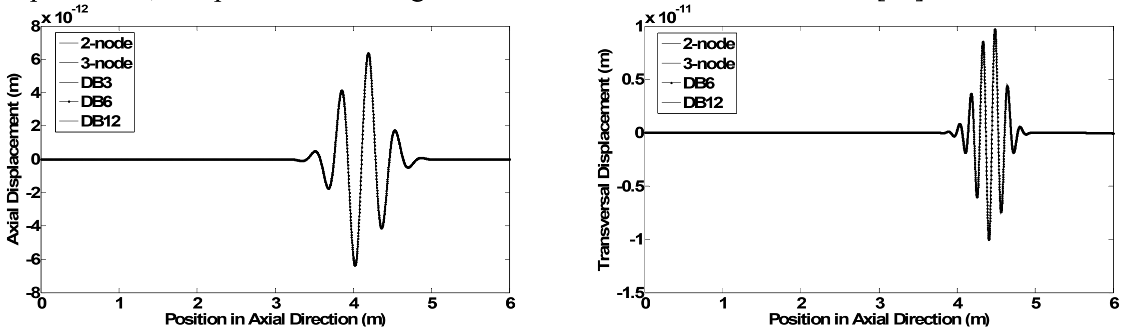
## 3 IMPLEMENTATION AND RESULTS

Implementation of the introduced FWD method demonstrates the accuracy and computational efficiency of the proposed approach. Two typical benchmark cases have been selected: i) a clamped rod; and ii) a clamped beam structure. These have been modeled using the introduced FWD Elements as well as traditional 2- and 3-node elements typically used in commercial FE codes. Specimen geometry and material properties are enlisted in Table 1. A concentrated force was applied on the structures' free end and the transient response for a 5-cycle 15 kHz pulse was predicted.

Table 1. Properties of Specimens for the Benchmark simulations.

| Specimen Geometry | | Material Properties | |
|---|---|---|---|
| Length: | 6m | Young Modulus: | 70GPa |
| Width: | 1m | Poisson's ratio: | 0.34 |
| Thickness: | 0.06m | Density: | 2700kg.m$^{-3}$ |

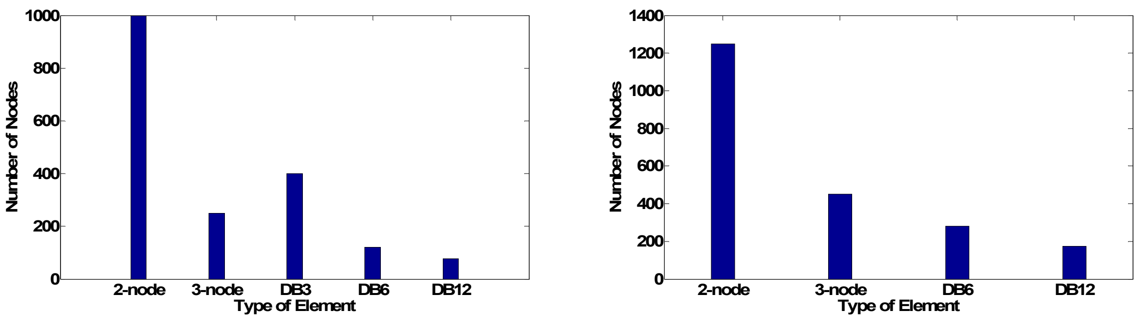
Figure 4 depicts a snapshot of the first antisymmetric mode ($A_0$) of the propagating wave; (a) refers to rod, (b) refers to beam structure. Curves are overlapping since all models converge to the very same prediction; this prediction is in agreement with other validated methods [11].




(a) (b)

Figure 4 Validations of Predictions for the $A_0$ mode in (a) the rod and (b) the beam structures. All FEs converge to the same solution.

In order to prove the computational efficiency of the proposed elements, a convergence study has been conducted for 2-, 3-node FEs and various types of FWD elements. The number of nodes in a model is directly related to the amount of required computational resources and the number of equations that need to be solved. Thus, the minimum number of nodes required for convergence has been selected as the efficiency criterion. Convergence is assumed when the RMS error with respect to the reference solution becomes lower the 2%. Figure 5 summarizes the results. The quality of elements follows the same trend in both cases.

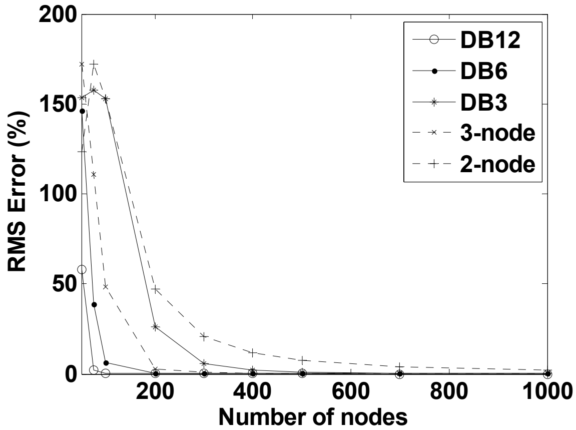



(a) (b)

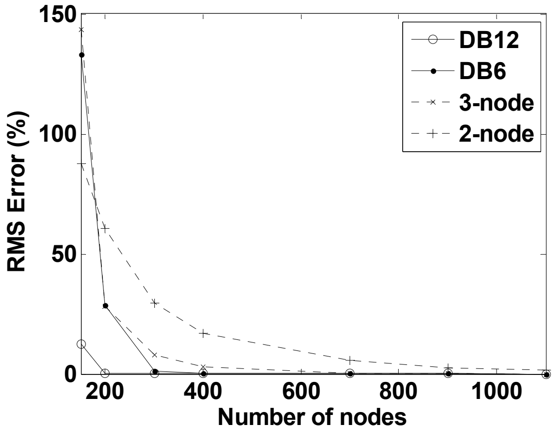
Figure 5 Minimum number of nodes required to achieve convergence with RMS error under 2%. (a) rod; (b) beam structure.

Typical 2-node elements are by far the worst, since maximum number of nodes is required. The 3-node FE seems to be a little better than the DB3 FWD FE, but as the wavelet order increases (DB6, DB12 etc.), the number of required nodes is dramatically reduced. As already stated, the computationally demanding wavelet calculations – Eq. (2) – are problem independent, thus, need to be performed only once. Therefore, increasing the order of the wavelet induces minimum computational cost in the analysis.

Furthermore, the convergence rate of each element has been investigated, i.e. how the RMS error is affected by the number of nodes (Figure 6). When a small number of nodes is used, all elements exhibit significant error. As the number of nodes increases, the RMS error rapidly falls and the models converge to the reference solution, indicated by RMS Error→0%. Convergence is faster for FWD elements except for the DB3 element; however, this is a low order wavelet and was not expected to yield optimal results in the first place. It also noticeable that as the wavelet order increases, convergence speed is dramatically improved.




(a)




(b)

Figure 6 Convergence study for (a) rod and (b) beam elements. As the wavelet order increases, the convergence rate is dramatically improved.

The superiority of the FWD element can be attributed the Domain of Influence (Figure 3); as the order of the wavelet increases, so does the domain of influence, hence, more information is captured by the respective element. This is reflected as faster convergence. Clearly, the FWD method can provide equally accurate results as traditionally employed methods, but in a much more computationally efficient way. Future work will focus on comparisons with high order laminate theories.

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