On the Stability of Sequential Deconvolution

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ABSTRACT

The time domain estimation of inputs from knowledge of the kernel and the outputs is a de-convolution operation. For finite dimensional systems treated in discrete time the operation is tantamount to solving a set of linear equations. When performing a deconvolution an issue that must be dealt with is the fact that the dimension of the system of equations grows with duration and becomes prohibitively large when the inputs are long. For this reason, and sometimes because it is of interest to estimate the inputs with the smallest possible delay, deconvolution must often be implemented on a moving window. This paper shows that a sequential deconvolution is a conditionally stable process and derives the expression that governs numerical stability.

KEYWORDS : input estimation, numerical stability, convolution, deconvolution

1 INTRODUCTION

The problem that underlies this paper is that of reconstructing r inputs of known positions and directions from m measured outputs. The operating premise is that the system that generates the outputs is linear and time invariant and that the input-output map is available either from a Finite Element model or from system identification. With this wealth of information it would appear that the problem should be straightforward, namely, one gets a set of linear equations relating the measured output to the initial condition and the inputs and the task is simply to solve for the inputs. Examination of the literature shows, however, that the problem is typically tagged as ill-posed and ill-conditioned and various methods to address the perceived difficulties, mostly assigned to the so-called non-collocated scenario, where inputs do not have instantaneous effects on the measurements, have been put forth [1,2,3].

We begin, for clarity, by providing some commentary on the issues of posedness and conditioning, terms that are often treated loosely in discussions on the input reconstruction problem. First, ill-posedness is a term that indicates that the constraints on the physical problem have not been adequately translated to the mathematical formulation, or that the question asked is not appropriate. For example, given outputs measured over some time interval find the inputs that generated them. This problem (except in the collocated case) is ill-posed because information transfer takes time and it is impossible to estimate inputs for the same time span as there are measured outputs. As obvious as this statement is, the foregoing is what is done when the deconvolution of inputs from outputs is derived from the finite dimensional state space formulation. There is, of course, the case where a one-time step delay arises because there is no direct transmission term, but this does not change the essence, as the time step and the delay are disconnected. Studies that have tackle the delay issue in input reconstruction have (essentially) tried to eliminate rank deficiency by introducing time shifts [1,2,3]. This strategy however, which does not work in the general case of multiple inputs and multiple outputs, is misguided, as the arbitrariness resulting from the rank deficiency need not be "fixed". It is an arbitrariness that is not global in time (here lies the point to be noted) but restricted to the part of the time span for which outputs do not constrain the inputs. A detailed discussion can be found in [4] from which this paper is essentially extracted.

The other "tag" that is typically used to refer to the deconvolution problem is that it is ill-conditioned. In this regard two comments are opportune. First, since the response for frequencies significantly higher than the high end of the system bandwidth is negligible, it is evident that conditioning for sufficiently high frequencies is always poor. This, however, does not imply that there is trouble, since the bandwidth of the inputs that are to be determined is always (at least approximately) known and thus, by an appropriate spatial discretization, one can always make it fit within the model bandwidth. The second issue pertains to the fact that the bandwidth of the measurement noise is dictated by the sampling frequency and that, as a consequence, this frequency should be selected so the realized noise also fits within the system bandwidth.

A final item deserving commentary is the fact that Finite Dimensional (FD) models have no delay. This assertion can appear a bit surprising when encountered for the first time but it is a well-known result from system theory [5] and it is not difficult to rationalize if one notes that the response of any FD model is the sum of N modes and that the summation of N signals cannot be made identically zero over the infinite points of a pure delay. There are, of course, FD models with added delays used in many applications but the FD model itself, i.e. a finite element discretization of a continuum, only has small values in the delay region of the impulse response functions, not zeroes. In the solution of the forward problem these small values are of no consequence but they are critical in the analytical treatment of the inverse problem since they create the mathematical space for developing schemes, typically referred to as "simultaneous input and state estimators", that offer one-time step delay estimate of the inputs, independently of spatial separation between inputs and outputs and independently of the time step. The noted algorithms are self-consistent and can operate in simulations, where they process the bogus information generated by the FD models, but when applied to real data, unless the sampling time is long compared to the inherent delay, they fail.

2 DISCRETE TIME DECONVOLUTION

The conventional form of the state space model in discrete time is

$$x_{k+1} = A_d x_k + B_d u_k \tag{1}$$

$$\mathrm{y}_\mathrm{k} = \mathrm{C}_\mathrm{d} \mathrm{x}_\mathrm{k} + \mathrm{D}_\mathrm{d} \mathrm{u}_\mathrm{k} \tag{2}$$

The input-output relation can be obtained by following the recurrence in eqs1-2 and one gets

$$y_k = CA_d^k x_0 + \sum_{j=0}^{k} Y_j u_{k-j} \tag{3}$$

Stacking the inputs and outputs in columns

$$\begin{Bmatrix} y_0 \\ y_1 \\ \vdots \\ y_\ell \end{Bmatrix} - \begin{bmatrix} C_d \\ C_d A_d \\ \vdots \\ C_d A_d^\ell \end{bmatrix} x_0 = \begin{bmatrix} Y_0 & 0 & \cdots & 0 \\ Y_1 & Y_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ Y_\ell & Y_{\ell-1} & \cdots & Y_0 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ \vdots \\ u_\ell \end{Bmatrix} \tag{4}$$

or

$$y_{[0,\ell]} - Ob_\ell \cdot x_0 = Hu_{[0,\ell]} \tag{5}$$

where

$$Y_0 = D_d \quad Y_j = C_d A_d^{j-1} B_d \tag{6}$$

with $H \in R^{(\ell_1 m)x(\ell_1 \mathrm{r})}$, Ob_ℓ is the observability block of order ℓ , $\ell =$ total number of time steps and $\ell_1 = \ell + 1$ is the total number of time stations. The general solution of eq.5 is

$$u_{[0,\ell]} = H^{-*} y_{[0,\ell]} - H^{-*} Ob_\ell \cdot x_0 + Z \cdot h \tag{7}$$

where -* stands for pseudo-inversion, $Z = N(H)$ is the null space of H and h is an arbitrary vector of appropriate dimension. Define Q_p as

$$Q_p = \begin{bmatrix} I_{pr} & 0_{prx(\ell_1 - p)} \end{bmatrix} \tag{8}$$

where I_{pr} is the identity of order p x r. Pre-multiplication of eq.7 by eq.8 selects the first p values of the input vector so one has

$$u_{[0,p]} = Q_p H^{-*} y_{[0,\ell]} - Q_p H^{-*} Ob_\ell \cdot x_0 + Q_p Z \cdot h \tag{9}$$

Assuming the initial condition is known the necessary and sufficient condition for the inputs $u_{[0,p]}$ to be unique is thus

$$Q_p Z = 0 \tag{10}$$

and these inputs are

$$\overline{u}_p = Q_p H^{-*}\left(y_{[0,\ell]} - Ob_\ell \cdot x_0\right) \tag{11}$$

2.1 Sequential Deconvolution

Application of eq.11 to long duration sequences is computationally problematic because the dimensions of H grow with the number of time steps. The obvious solution is to implement the deconvolution in a window that slides along the time axis - two parameters have to be selected: the size of the window $\lambda \cdot \Delta t$ and the rate of advance, $p \cdot \Delta t$, where this "p" is the "p" in eq.11. Assume for now that λ and p are selected so eq.10 is satisfied. Given these inputs one can call on eq.1 to evaluate the initial condition at time station p+1; the window can then be shifted forward p-time steps and the process continued until the full duration of interest is covered.

3 STABILITY

Numerical stability imposes a constraint when selecting the {λ,p} pair. Namely, the pair must be selected such that the estimation errors that inevitably arise in each segment do not grow as the solution advances. The notation used next is illustrated in fig.1.

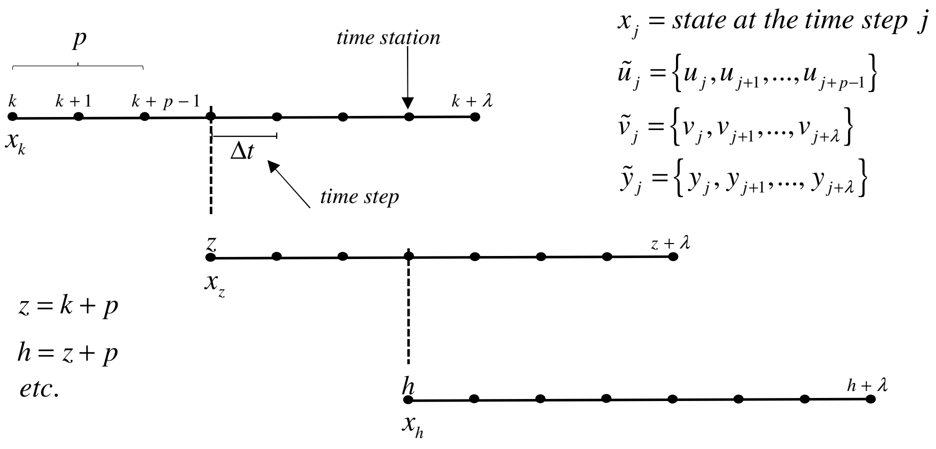

Fig.1 Schematic illustration of sequential deconvolution with notation.

From the notation in fig.1 and eq.11 one can write

$$\tilde{u}_k + \delta\tilde{u}_k = Q_p H^{-*}\left(\tilde{y}_k + \tilde{v}_k - Ob_\lambda \cdot (x_k + \delta x_k)\right) \tag{12}$$

and therefore

$$\delta\tilde{u}_k = Q_p H^{-*}\tilde{v}_k - Q_p H^{-*} Ob_\lambda \cdot \delta x_k \tag{13}$$

Following the recurrence of eq.1 the state at time station k+p can be expressed in terms of the state at k and the intervening inputs, namely

$$x_{k+p} = A_d^p x_k + \begin{bmatrix} A_d^{p-1}B_d & A_d^{p-2}B_d & \cdots & B_d \end{bmatrix} \begin{Bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+p-1} \end{Bmatrix} \tag{14}$$

which, noting that the row block matrix is the controllability (in reversed order) writes

$$x_{k+p} = A_d^p x_k + Rc_{p-1}\tilde{u}_k \tag{15}$$

and thus

$$\delta x_{x+p} = A_d^p \delta x_k + Rc_{p-1}\delta\tilde{u}_k \tag{16}$$

Substituting eq.16 into eq.15 writes

$$\delta x_{k+p} = P_p \delta x_k + Z_p \tilde{v}_k \tag{17}$$

where

$$P_p = A_d^p - Rc_{p-1} Q_p H^{-*} Ob_\lambda \tag{18}$$

and

$$Z_p = Rc_{p-1} Q_p H^{-*} \tag{19}$$

As one gathers from fig.1, if $p \neq 1$ the index k does not contain all the integers but advances as $k = (j-1)p+1$ where j = 1,2,3 …Substituting the previous expression in eq.17 and tracking the recurrence one finds that

$$\delta x_{j\cdot p+1} = P_p^j \delta x_1 + \sum_{i=1}^{j} P_p^{i-1} Z_p \tilde{v}_{(j-i)p+1} \tag{20}$$

which shows, given that the matrix P_p is raised to a power that increases as the window shifts along the time axis, that the requirement for stability is

$$\rho = \left|\eta_j\right|_{\max} \leq 1 \tag{21}$$

where η = eigenvalues of P_p.

4 NUMERICAL ILLUSTRATION

Let the system be a prismatic rod of length L_r area A, density ρ and modulus of elasticity E. We think of this rod as being in the horizontal plane with one end free and the other fixed. The wave propagation speed is $c_0 = \sqrt{E/\rho}$ and thus, if the distance from an input to the location of an output is χ the dead time is χ $/c_0$. In some consistent set of units (with time in seconds) we take A = 4, L_r = 10, E = 100 and $\rho = 1$, so c_0 = 10. An estimate of the maximum significant frequency in the spectra of the unknown inputs would typically be available. Let's assume that in this case 10 Hz is a reasonable cut. Using a uniform chain as a FD model of the rod one finds that for the frequency of the last mode to be no less than 10Hz it is necessary to have at least 31 discrete masses. On this ground we select a model with 40-DOF for which the first two and the last two modal frequencies are (0.247, 0.74,….,12.69, 12.73)Hz. To keep the measurement noise within the bandwidth we have $\Delta t \geq 0.5/12.73$ =0.039sec; we formulate the DT model using $\Delta t = 0.05\,\text{sec}$.

Inputs

We consider two inputs at coordinates #5 and #40 and three accelerometers at coordinates {1, 10 and 30}. The time histories are

$$u_5(t) = 10e^{-0.5t}\{(1-\cos 2\pi t)\sin 6\pi t\} \qquad u_{40}(t) = 10e^{-0.25t}\{(1-\cos 2\pi t)\sin 6\pi t\}$$

Rayleigh damping with 2% in the first and the last mode is used.

Stability

Fig.2 plots the spectral radius ρ vs the observation time (window size) for two values of the forward shift, p. As can be seen, the stability limit for λ (around 30 and 35) is not very different for the two values of p.

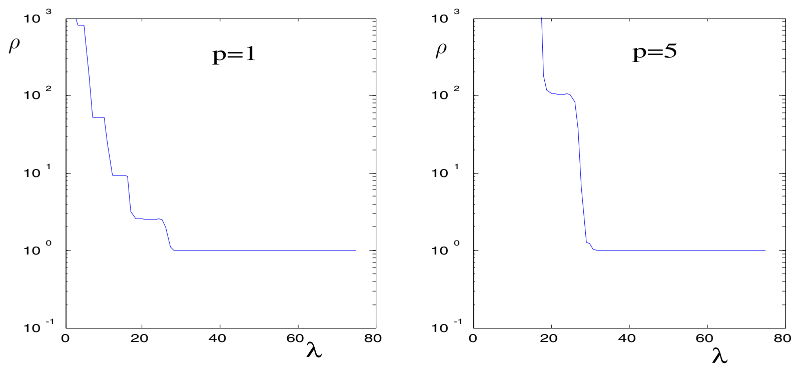

Fig.2 Spectral radius vs. observation window size

Results

Fig.3 illustrates results with a NSR of 3% included. Since results for both inputs are qualitatively the same we show only the ones on the 5th coordinate.

- Part (a) illustrates results when $\lambda = 20$ and p = 1. Since this window size is on the unstable region (i.e., ρ >1 in fig.2), on expects divergence and this is what is observed.
- Part (b) shows results for $\lambda = 50, p = 1$, which is a stable combination. As can be seen the solution is in this case stable and the results are accurate.

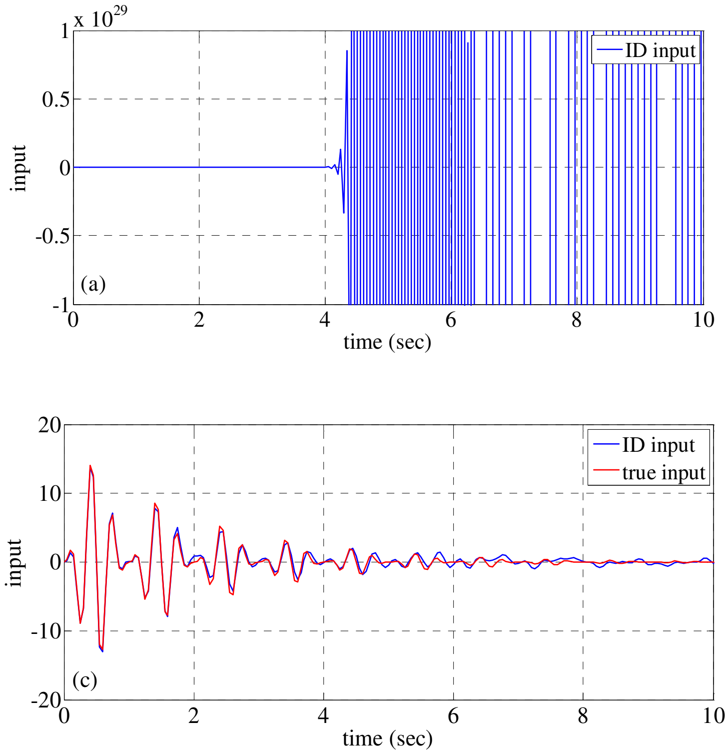

Fig.3 SDR reconstructions: (a) $\lambda = 20, p = 1$, (b) $\lambda = 50, p = 1$

5 Conclusions

It is shown that the sequential deconvolution of inputs from measured outputs is a conditionally stable process and the condition for stability is clarified. We close by re-stating that there are a number of algorithms presently in the literature, (typically referred to as simultaneous input and state estimators) that operate in the non-collocated case on a one time step delay. These algorithms rely on the supra-luminous information transfer of the FD model and cannot operate with real data, except, of course, in those cases where the physical delay is small compared to the time step size selected.

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6 REFERENCES

[1] L. J.L. Nordstrom, T. P. Nordberg, A time delay method to solve non-collocated input estimation problems, Mech. Syst. Signal Process. 18 (2004) 1469-1483.

[2] A. D. Steltzner, D. C. Kammer, Input force estimation using an inverse structural filter, IMAC, (1999) 954-960.

[3] M. S. Allen, T. G. Carne, Delayed, multi-step inverse structural filter for robust force identification, Mech. Syst. Signal Process. 22 (2008) 1036-1054.

[4] Bernal D and Ussia A "Sequential deconvolution input reconstruction", Mechanical Systems and Signal Processing, (in press).

[5] P. Antsaklis, A. N. Michel, Linear Systems, McGraw-Hill, New York, 1997.