

# Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

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# Hamilton-Jacobi equations on networks as limits of singularly perturbed problems

Yves Achdou

joint work with N. Tchou

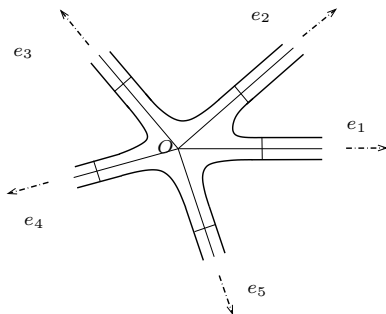
Laboratoire J-L Lions, Université Paris Diderot

# Setting: a thick version of a network

Consider a domain  $\Omega \subset \mathbb{R}^2$ :

- star-shaped w.r.t. the origin  $O$
- $\partial\Omega$  is smooth
- Far enough from the origin  $O$ ,  $\Omega$  coincides with the union of  $N$  non-intersecting semi-infinite strips directed by the vectors  $e_i$ ,  $i = 1, \dots, N$

The domain  $\Omega$



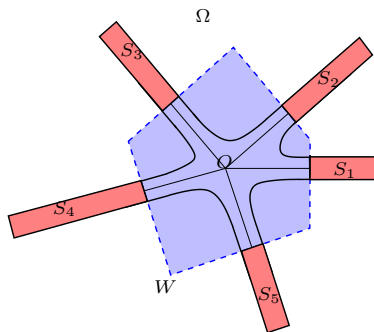
## Setting: a more precise definition

- for some  $r_0 > 0$ , let  $W$  be the polygonal set

$$W = \{x \in \mathbb{R}^2 : x \cdot e_i \leq r_0, \forall i = 1, \dots, N\}$$

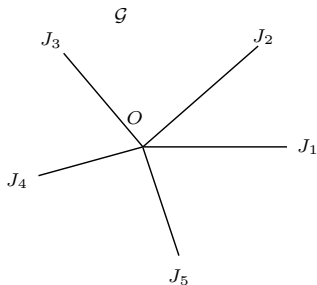
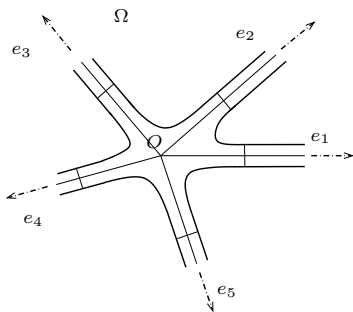
- Then  $\Omega \setminus W = \bigcup_{i=1}^N S_i$ , where  $S_i$  is the half-strip

$$S_i = \left\{ x_i e_i + x_i^\perp e_i^\perp, \quad x_i > r_0, \quad |x_i^\perp| < 1 \right\}$$



$\Omega_\epsilon = \epsilon\Omega$  “tends” to a network  $\mathcal{G}$  as  $\epsilon \rightarrow 0$

$$\mathcal{G} = \{O\} \cup \bigcup_{i=1}^N J_i,$$
$$J_i = \{x_i e_i, x_i > 0\}.$$



# State constrained control problems in $\overline{\Omega_\epsilon}$

$$u_\epsilon(x) = \inf_{\alpha} \int_0^{\infty} \ell_\epsilon(y_\epsilon(t; x, \alpha), \alpha(t)) e^{-\lambda t} dt$$

subject to

$$\begin{cases} \dot{y}_\epsilon(t; x, \alpha) &= \alpha(t), & \text{a.a. } t > 0, \\ y_\epsilon(0; x, \alpha) &= x, \end{cases}$$

with  $\alpha : \mathbb{R}_+ \rightarrow A$  measurable, under the state constraint

$$y_\epsilon(t) \in \overline{\Omega_\epsilon}, \quad \forall t \geq 0.$$

Controlability :  $A$  is a compact subset of  $\mathbb{R}^2$  such that  $B(0, r) \subset A$  for some  $r > 0$ .

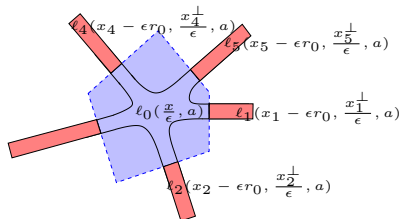
# Assumptions on the running costs

- The function  $\ell_\epsilon : \Omega_\epsilon \rightarrow \mathbb{R}$  is bounded and continuous
- 

$$\ell_\epsilon(x, a) = \begin{cases} \ell_i \left( x_i - \epsilon r_0, \frac{x_i^\perp}{\epsilon}, a \right) & \text{in } \epsilon S_i \\ \ell_0 \left( \frac{x}{\epsilon}, a \right) & \text{in } \epsilon W \cap \Omega_\epsilon \end{cases}$$

$$\ell_i : [0, +\infty) \times [-1, 1] \times A \rightarrow \mathbb{R}$$

$$\ell_0 : (W \cap \Omega) \times A \rightarrow \mathbb{R}$$



$\ell_i$  and  $\ell_0$  match properly in order to ensure the continuity of  $\ell_\epsilon$

# Assumptions on the running costs

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## Remark

In  $\epsilon S_i$ ,  $\ell_\epsilon$  varies slowly w.r.t.  $x_i$  and fast w.r.t.  $x_i^\perp$ .

- It is not restrictive to assume that  $\ell_0 \geq 0$ .



## The value function $u_\epsilon$

$u_\epsilon$  is bounded uniformly with respect to  $\epsilon$ , continuous, and is the unique viscosity solution of

$$\begin{aligned}\lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\geq 0 && \text{in } \overline{\Omega}_\epsilon, \\ \lambda u_\epsilon(x) + H_\epsilon(x, Du_\epsilon) &\leq 0 && \text{in } \Omega_\epsilon,\end{aligned}$$

where

$$H_\epsilon(x, p) = \max_{a \in A} \left( -p \cdot a - \ell_\epsilon(x, a) \right).$$

### Questions

- Asymptotic behavior of  $u_\epsilon$  as  $\epsilon \rightarrow 0$ ?
- Can we define an effective problem in  $\mathcal{G}$ ?

# Background and additional new difficulties

- Existing results:
  - Singularly perturbed control problems in thin domains around smooth manifolds: Bensoussan, Alvarez-Bardi, Arztein-Gaitsgory, Gaitsgory-Leisarowitz, Terrone,...
  - Comparison for viscosity solutions on networks: YA-Oudet-Tchou and Imbert-Monneau
  - Very recent results of Lions-Souganidis on homogenization of HJB equations with defects
  
- Several additional new difficulties:
  - Identification of the effective problem
  - Will the effective problem keep track of  $\ell_0$  near  $O$ ? How?
  - In the perturbed test-function method of Evans, we need to construct correctors in unbounded domains.

# Main result

## Theorem

*Under a further technical assumption,  $u_\epsilon$  converges locally uniformly to the bounded viscosity solution  $u : \mathcal{G} \rightarrow \mathbb{R}$  of*

$$\left\{ \begin{array}{l} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 \quad x \in J_i, \\ \lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(O), \dots, \frac{du}{dx_N}(O)) \right) = 0, \end{array} \right.$$

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with

$$\overline{H}_i(x_i, p_i) = \sup_{\mu \in \mathcal{Z}_i} \left( \int_{[-1,1] \times A} (-p_i a_i - \ell_i(x_i, y, a)) d\mu(y, a) \right)$$

$\mathcal{Z}_i$  is a compact and convex set of Radon probability measures on  $[-1, 1] \times A$

$\mathcal{Z}_i$ : limiting relaxed controls.

# Main result

## Theorem

*Under a further technical assumption,  $u_\epsilon$  converges locally uniformly to the bounded viscosity solution  $u : \mathcal{G} \rightarrow \mathbb{R}$  of*

$$\left\{ \begin{array}{l} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 \quad x \in J_i, \\ \lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(O), \dots, \frac{du}{dx_N}(O)) \right) = 0, \end{array} \right.$$

with

–  $E$  : an effective cost at the junction

# Main result

## Theorem

Under a further technical assumption,  $u_\epsilon$  converges locally uniformly to the bounded viscosity solution  $u : \mathcal{G} \rightarrow \mathbb{R}$  of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0)) \right) = 0, \end{cases}$$

with  $\overline{H}(O, p_1, \dots, p_N) := \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$ ,

$$\overline{H}_i^+(0, p_i) = \sup_{\mu \in \mathcal{Z}_i^+} \left( \int_{[-1,1] \times A} (-p_i a_i - \ell_i(0, y, a)) d\mu(y, a) \right),$$

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}.$$

What is the meaning of

$$\begin{cases} \lambda u(x) + \overline{H}_i(x_i, \frac{du}{dx_i}(x)) = 0 & x \in J_i, \\ \lambda u(O) + \max \left( E, \overline{H}(O, \frac{du}{dx_1}(0), \dots, \frac{du}{dx_N}(0)) \right) = 0 \end{cases} \quad ? \quad (1)$$

See YA-Camilli-Cutri-Tchou(2013), Imbert-Monneau-Zidani(2013), YA-Oudet-Tchou(2014), and Imbert-Monneau(2014).

### Definition (Test functions)

- $\phi : \mathcal{G} \rightarrow \mathbb{R}$  is an admissible test-function if
  - $\phi$  is continuous on  $\mathcal{G}$
  - for any  $j \in 1, \dots, N$ ,  $\phi|_{\overline{J}_j} \in \mathcal{C}^1(\overline{J}_j)$
- $\mathcal{R}(\mathcal{G})$ : set of the admissible test-functions

- An usc function  $u : \mathcal{G} \rightarrow \mathbb{R}$  is a sub-solution of (1) if for any  $x \in \mathcal{G}$  and  $\phi \in \mathcal{R}(\mathcal{G})$  s.t.  $u - \phi$  has a local maximum in  $x$ , then

$$\begin{aligned} \lambda u(x) + \overline{H}_i(x, \frac{d\phi}{dx_i}(x)) &\leq 0 && \text{if } x \in J_i, \\ \lambda u(O) + \max \left( E, \overline{H} \left( \frac{d\phi}{dx_1}(O), \dots, \frac{d\phi}{dx_N}(O) \right) \right) &\leq 0. \end{aligned}$$

- A lsc function  $u : \mathcal{G} \rightarrow \mathbb{R}$  is a super-solution of (1) if for any  $x \in \mathcal{G}$  and  $\phi \in \mathcal{R}(\mathcal{G})$  s.t.  $u - \phi$  has a local minimum in  $x$ , then

$$\begin{aligned} \lambda u(x) + \overline{H}_i(x, \frac{d\phi}{dx_i}(x)) &\geq 0 && \text{if } x \in J_i, \\ \lambda u(O) + \max \left( E, \overline{H} \left( \frac{d\phi}{dx_1}(O), \dots, \frac{d\phi}{dx_N}(O) \right) \right) &\geq 0. \end{aligned}$$



# The effective Hamiltonians away from the junctions

Notation: for  $x \geq 0, y \in [-1, 1], p \in \mathbb{R}^2$ ,

$$H_i(x, y, p) = \max_{a \in A} (-p \cdot a - \ell_i(x, y, a))$$

Theorem (Alvarez-Bardi 2000)

$\forall x_i \geq 0, \forall p_i \in \mathbb{R}$ , there is a unique  $\bar{H}_i(x_i, p_i)$  s.t. the cell problem

$$H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) \geq \bar{H}_i(x_i, p_i) \quad y \in [-1, 1],$$

$$H_i(x_i, y, p_i e_i + D_y \chi_i(y) e_i^\perp) \leq \bar{H}_i(x_i, p_i) \quad y \in (-1, 1)$$

has a viscosity solution  $\chi_i \in \text{Lip}([-1, 1])$ .

## Limiting relaxed controls and limit control problem

For  $s > 0$ , the *occupational measure*  $\mu_s$  generated by  $(y(t), \alpha(t))$  is the Radon probability measure defined on  $[-1, 1] \times A$  by

$$\mu_s = \frac{1}{s} \int_0^s \delta_{(y(t), \alpha(t))} dt$$

where  $\delta_{(y(t), \alpha(t))}$  is the Dirac mass concentrated at  $(y(t), \alpha(t))$ .

$\mathcal{Z}(s; i, y_0)$ : set of the occupational measures generated by the trajectories  $(y(t), \alpha(t))$  up to time  $s$ .

# Limiting relaxed controls and limit control problem

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$\mathcal{Z}(s; i, y_0)$ : set of the occupational measures generated by the trajectories  $(y(t), \alpha(t))$  up to time  $s$ .

## Theorem (Gaitsgory and Leisarovitz (1999))

*There exists a set  $\mathcal{Z}_i \subset \mathcal{P}([-1, 1] \times A)$  s.t*

$$\forall y_0 \in [-1, 1], \quad \lim_{s \rightarrow \infty} \pi_H(\mathcal{Z}(s; i, y_0), \mathcal{Z}_i) = 0,$$

*( $\pi_H$  : Prokhoroff distance).*

*$\mathcal{Z}_i$  is convex and compact for the weak-\* topology.*

By using results from Terrone-2011 and Alvarez-Bardi-2000

- 

$$\mathcal{Z}_i \subset \left\{ \mu \in \mathcal{P}([-1, 1] \times A), \int_{[-1, 1] \times A} a_i^\perp d\mu(y, a) = 0 \right\}$$

- $\mathcal{Z}_i$  coincides with the set of limiting relaxed controls, i.e.

$$\begin{aligned} & \bar{H}_i(x_i, p_i) \\ &= \sup_{\mu \in \mathcal{Z}_i} \left( -p_i \int_{[-1, 1] \times A} a_i d\mu(y, a) - \int_{[-1, 1] \times A} \ell_i(x_i, y, a) d\mu(y, a) \right) \end{aligned}$$

## Interpretation

Away from the junction, the effective Hamiltonian corresponds to a control problem with controls in  $\mathcal{Z}_i$ .

# The effective Hamiltonian at the junction

## Definition

The effective Hamiltonian at the junction is  $\max(E, \overline{H}(O, \cdot))$  where  $E$  is a suitable constant to be defined and

$$\overline{H}(O, p_1, \dots, p_N) := \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$$

$$\begin{aligned} & \overline{H}_i^+(0, p_i) \\ &= \sup_{\mu \in \mathcal{Z}_i^+} \left( -p_i \int_{[-1,1] \times A} a_i d\mu(y, a) - \int_{[-1,1] \times A} \ell_i(0, y, a) d\mu(y, a) \right) \end{aligned}$$

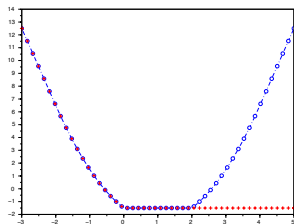
and

$$\mathcal{Z}_i^+ = \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) \geq 0 \right\}$$

Lemma: links between  $\overline{H}_i(0, p)$  and  $\overline{H}_i^+(0, p)$

$$\mathcal{Z}_i^0 := \left\{ \mu \in \mathcal{Z}_i \text{ s.t. } \int_{[-1,1] \times A} a_i d\mu(y, a) = 0 \right\}$$

$$\operatorname{argmin} \overline{H}_i(0, \cdot) = \left[ \underline{p}_i, \bar{p}_i \right]$$



The graphs of  
 $p \mapsto \overline{H}_i(0, p)$  and  
of  $p \mapsto \overline{H}_i^+(0, p)$

- $p < \underline{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$  is achieved for  $\mu \in \mathcal{Z}_i^+ \setminus \mathcal{Z}_i^0$
- $\underline{p}_i < p < \bar{p}_i \Rightarrow \overline{H}_i(0, p) = \overline{H}_i^+(0, p)$  is achieved for  $\mu \in \mathcal{Z}_i^0$
- $p > \bar{p}_i \Rightarrow \overline{H}_i(0, p) > \overline{H}_i^+(0, p) = \min \overline{H}_i(0, \cdot)$

# The constant $E$

## Zoom near the junction point $O$

We extend the function  $\ell_0$  to the whole domain  $\bar{\Omega}$  by setting

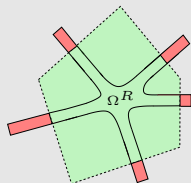
$$\ell_0(z, a) = \ell_i(0, z_i^\perp, a), \quad \text{if } z_i \geq r_0, |z_i^\perp| \leq 1.$$

## Ergodic constants in bounded subdomains

Define  $\Omega^R := \Omega \cap W_R$  and the ergodic constant  $E^R$ , which is the unique number s.t.

$$\begin{aligned} H_0(z, Dw^R(z)) &\geq E^R && \text{in } \bar{\Omega}^R, \\ H_0(z, Dw^R(z)) &\leq E^R && \text{in } \Omega^R, \end{aligned}$$

has a viscosity solution  $w^R$ .



# The constant $E$

## Lemma

$\exists C > 0$  s.t.

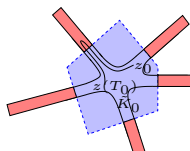
$$\begin{aligned} |E^R| &\leq C && \forall R, \\ \|Dw^R\|_\infty &\leq C && \forall R, \\ R \leq R' &\Rightarrow && E^R \leq E^{R'}. \end{aligned}$$

## Definition

$$E := \lim_{R \rightarrow \infty} E^R.$$



# The constant $E$



## Theorem

$\exists C > 0$  s.t.  $\forall z_0 \in \tilde{K}_0, \forall \alpha \in \mathcal{A}_{z_0}, \forall T > 0,$

if  $z(s) := z_0 + \int_0^s \alpha(\theta) d\theta$  satisfies  $z(T) \in \tilde{K}_0$ , then

$$\int_0^T \ell_0(z(s), \alpha(s)) ds \geq -ET - C.$$

## Lemma

Under the assumptions,  $\forall i = 1, \dots, N,$   $\min_{p_i} \bar{H}_i(0, p_i) \leq E.$

# Proof of the convergence result

The main step is to prove the following theorem

## Theorem

Consider the relaxed semilimits, of  $u_\epsilon$ : for all  $x \in \mathcal{G}$ ,

$$\underline{u}(x) = \liminf_{\epsilon \rightarrow 0+, x' \rightarrow x, x' \in \Omega_\epsilon} u_\epsilon(x'); \quad \bar{u}(x) = \limsup_{\epsilon \rightarrow 0+, x' \rightarrow x, x' \in \Omega_\epsilon} u_\epsilon(x').$$

Then, under the assumptions,  $\underline{u}$  is a bounded supersolution of

$$\begin{aligned} \lambda \underline{u}(x) + \overline{H}(x, D\underline{u}(x)) &\geq 0, & \text{if } x \in \mathcal{G} \setminus \{O\}, \\ \lambda \underline{u}(O) + \max(E, \overline{H}(O, D\underline{u}(O))) &\geq 0, & \text{if } x = O, \end{aligned}$$

and  $\bar{u}$  is a bounded subsolution of

$$\begin{aligned} \lambda \bar{u}(x) + \overline{H}(x, D\bar{u}(x)) &\leq 0, & \text{if } x \in \mathcal{G} \setminus \{O\}, \\ \lambda \bar{u}(O) + \max(E, \overline{H}(O, D\bar{u}(O))) &\leq 0, & \text{if } x = O. \end{aligned}$$

then use comparison results on  $\mathcal{G}$ , [AOT2014] or [IM2014]

## Some ideas of the proof

- Not standard only at the junction
  - Try to use Evans' method of perturbed test-functions
  - The test-functions  $\phi$  have  $N$  slopes at  $O : p = (p_1, \dots, p_N)$
  - Evans' method requires the construction of bounded correctors in unbounded domains, (ideally  $\Omega$ )
  - Bounded correctors may not exist in the full domain  $\Omega$ , because, given a set of slopes at  $O$ , i.e.  $p = (p_1, \dots, p_N)$ ,

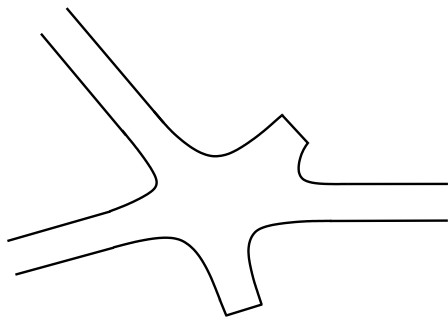
$$\overline{H}(O, p_1, \dots, p_n) = \max_{i=1, \dots, N} \overline{H}_i^+(0, p_i)$$

is achieved by  $i \in \mathcal{I}(p) \subset \{1, \dots, N\}$ . If  $j \notin \mathcal{I}(p)$ , then the optimal trajectories starting from  $z \in S_j$  should leave  $S_j$ : this leads to the fact that the corrector should not be bounded when  $|z| \rightarrow \infty$ ,  $z \in S_j$ .

- Hence, we construct the correctors in subdomain obtained by truncating the half-strips  $S_j$  if  $j \notin \mathcal{I}(p)$ .

The corrector (when  $\overline{H}(O, p_1, \dots, p_N) > E$ )

The truncated domain  $\Omega_p$



### Theorem

For  $p \in \mathbb{R}^N$ :  $\overline{H}(O, p) > E$ . Let  $\psi_p$  be a smooth function such that  $D\psi_p = p_i e_i$  in  $S_i$ . Under the assumptions,  $\exists \chi_p$ , a bounded and Lipschitz viscosity solution of

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \geq 0 \quad \text{in } \overline{\Omega}_p,$$

$$H_0(z, D\psi_p + D\chi_p) - \overline{H}(O, p) \leq 0 \quad \text{in } \Omega_p.$$

# The further assumption

## Assumption

1) For any real number  $p_i$  such that  $p_i < \underline{p}_i$ ,

there exist two constants  $L_i \geq 0$  and  $C_i > 0$  such that

$\forall y_0 \in [-1, 1], \forall t > 0$ , there exists a control law  $\tilde{\alpha} \in \mathcal{A}_{i, y_0}$  with

$$\int_0^s \tilde{\alpha}_i(\tau) d\tau \geq -L_i, \quad \forall 0 \leq s \leq t,$$

$$\int_0^t (p_i \tilde{\alpha}_i(s) + \ell_i(0, y(s), \tilde{\alpha}(s))) ds \leq v_i(0, p_i, y_0, t) + C_i,$$

where  $y(t) = y_0 + \int_0^t \tilde{\alpha}_i^\perp(s) ds$ .

Recall that

$$v_i(x_i, p_i, y_0, t) = \inf_{\alpha \in \mathcal{A}_{i, y_0}} \left\{ \int_0^t p_i \alpha_i(s) + \ell_i(x_i, y(s), \alpha(s)) ds \right\}$$