Optimal control of stochastic processes via probability density distribution function control

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Collaboration and projects

- Collaboration with Alfio Borzì at the Würzburg Universität.
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Outline

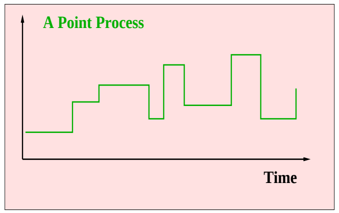
Preliminary: stochastic modeling

- Adding the gaussian noise W_t, or its variants, to deterministic equations.

$$\boldsymbol{dX_t = b(X_t, u(t))\, dt + \sigma dW_t}$$

Randomness acts on all time scales, with highly irregular fluctuations: $dW_t \propto \sqrt{dt}$

- Point processes: the fluctuations affect the motion only at isolated time points. Application to queue theory, stochastic hybrid systems, anomalous diffusion, filtering of random telegraph processes.



Statistical characterization

The dynamics of the state X_t is not deterministic:
the time evolution of the state changes, even if the same initial condition is hold for each experiment.

The state of the system can be described statistically by the probability density function $f(x,t)$ related to the state X_t. It gives the probability that X_t assumes a certain range of values.

- Pathwise integration/simulation (Monte Carlo)
- Spectral Techniques: Laplace, Fourier, Generating functions,...
- 'Direct Methods'
- Chapman Kolmogorov Equation (Fokker-Planck, Liouville Master Equation,...)

A Stochastic Modeling Tool

Example with the well-known Itō Equation

$$dX_t = b(X_t,t)\,dt + \sigma(X_t,t)\,dW_t$$

and the associated Fokker-Planck equation

$$\partial_t f(x,t) - \frac{1}{2}\partial_x^2(\sigma(x,t)^2 f(x,t)) + \partial_x(b(x,t)f(x,t)) = 0$$

The probability density function (PDF)

$f(x,t)$ is the PDF for X_t. It determines the statistical state of the system.

State of the art: The Average Objective

The optimal control problem

$$\boxed{\min_u J(X,u)}$$

where u is the control function
Since X_t is random, a deterministic objective will result to be a random variable .
The average makes the objective deterministic

$$J(X,u) = \mathbb{E}\left[\Psi[X_T] + \int_0^T \mathcal{L}(X_t, u(t))\,dt\right]$$

The average $\mathbb{E}[\cdot]$ of functionals of X is the basic ingredient of stochastic optimal control problems considered in the literature.

Alternative approaches with deterministic objective

The state of a stochastic process can be characterized by the shape of its statistical distribution represented by the probability density function (PDF).

In some works, control schemes were proposed, where the deterministic objective depends on the PDF of the stochastic state variable.
Examples are objectives defined by the Kullback-Leibler distance or the square distance between the state PDF and a desired one. Nevertheless, stochastic governing models were used and the state PDF is obtained by averaging or by approximation.

PDF optimal control

Correspondence between the basic, unnoised, evolutory optimal control problem and the proposed PDF optimal control problem for noised systems.

basic unnoised

- $X(t)$
- $\dot{X}(t) = b(X(t), u(t))$
- $J(X, u)$
- $\min_u J(X, u)$

PDF control

- $f(x, t)$ (PDF)
- PDE for $f(x, t)$ and $u(t)$
- $J(f, u)$
- $\min_u J(f, u)$

The proposed framework for the optimal control of time dependent problems with noise is a PDE constrained optimization problem.

A new approach based on the Fokker-Planck equation

The evolution of the PDF given by $f(x, t)$ associated to the stochastic process is modeled by the Fokker-Planck (FP) equation.

$$\partial_t f - \frac{1}{2}\sum_{i,j=1}^{n} \partial^2_{x_i x_j}\left(a_{ij} f\right) + \sum_{i=1}^{n} \partial_{x_i}\left(b_i(u) f\right) = 0$$

$$f(t_0) = \rho$$

where $a = \sigma\sigma^\top$. Positiveness $f(x, t) \geq 0$ and conservativeness $\int_\Omega f(x, t) dx = 1$ is required.

The formulation of objectives that are functions of the PDF and the Fokker-Planck equation provide a consistent framework for the optimal control of stochastic processes.

A tracking objective

- The control problem is formulated in the time window (t_k, t_{k+1}) with known initial value at time t_k.
- The problem is to determine a piecewise constant control $u(t) \in \mathbb{R}^\ell$ such that the process evolves towards a desired target probability density $f_D(x, t)$ at time $t = t_{k+1}$.
- This objective can be formulated by the following tracking functional

$$J(f, u) := \frac{1}{2}\|f(\cdot, t_{k+1}) - f_D(\cdot, t_{k+1})\|^2_{L^2(\Omega)} + \frac{\nu}{2}|u|^2$$

where $|u|^2 = u_1^2 + \ldots + u_\ell^2$.

A Fokker-Planck optimal control problem

The optimal control problem to find u that minimizes the objective J subject to the constraint given by the FP equation is formulated by the following

Tracking objective

$$\min_u J(f, u) := \frac{1}{2}\|f(\cdot, t_{k+1}) - f_D(\cdot, t_{k+1})\|^2_{L^2(\Omega)} + \frac{\nu}{2}|u|^2$$

Fokker Planck

$$\partial_t f - \frac{1}{2}\sum_{i,j=1}^{n} \partial^2_{x_i x_j}\left(a_{ij} f\right) + \sum_{i=1}^{n} \partial_{x_i}\left(b_i(u) f\right) = 0$$

$$f(t_k) = \rho$$

It is a non linear optimal control problem

A Fokker-Planck optimality system

The first-order necessary optimality condition is characterized as the solution of the following optimality system

$$\textbf{forward:}\quad \partial_t f - \frac{1}{2}\sum_{i,j=1}^n \partial^2_{x_i x_j}(a_{ij}\, f) + \sum_{i=1}^n \partial_{x_i}(b_i(u)\, f) = 0$$

$$\textbf{initial:}\quad f(x, t_k) = \rho(x)$$

$$\textbf{backward:}\quad -\partial_t p - \frac{1}{2}\sum_{i,j=1}^n a_{ij}\,\partial^2_{x_i x_j} p - \sum_{i=1}^n b_i(u)\,\partial_{x_i} p = 0$$

$$\textbf{terminal:}\quad p(x, t_{k+1}) = f(x, t_{k+1}) - f_D(x, t_{k+1})$$

$$\textbf{gradient:}\quad (\nabla \hat{J})_l := \nu\, u_l + \left(\sum_{i=1}^n \partial_{x_i}\left(\frac{\partial b_i}{\partial u_l} f\right), p\right)_{L^2(Q_k)} = 0$$

where $Q_k = \Omega \times (t_k, t_{k+1})$ and $\Sigma_k = \partial\Omega \times (t_k, t_{k+1})$.

Theory of the FP control problem

Proposition Ref. [2]

Assume that $b(x; u) = \gamma(x) + u$, $\gamma \in C^1(\Omega)$ and sufficiently small $\bar{\gamma} = \max_{x\in\Omega}(|\gamma(x)|, |\gamma'(x)|)$, $f_0 \in L^2(\Omega)$, and $u \in U$. Then the FP problem $\mathrm{FP}(f; f_0, u) = 0$ admits a unique solution f, that is for sufficiently small initial condition or sufficiently large ν, a unique optimal control exists

$$\nu\|u_1 - u_2\|_U \leq \left(\frac{1}{s^2}\|f_1(T) - f_D\|_{L^2} + \frac{\alpha_1}{s\sqrt{2}}\|f_0\|_{L^2} + \frac{1}{s^2}\|f_2(T) - f_D\|_{L^2}\right) \|u_1 - u_2\|_U \|f_0\|_{L^2}$$

where s is an appropriate constant related to the model.

The discretization of the FP optimality system

Time discretization

The forward- and adjoint FP equations are discretized using the second-order backward time-differentiation formula (BDF2) as follows

$$\partial^-_{BD} f_{\mathrm{i}}^m := \frac{3f_{\mathrm{i}}^m - 4f_{\mathrm{i}}^{m-1} + f_{\mathrm{i}}^{m-2}}{2\delta t} \qquad \partial^+_{BD} p_{\mathrm{i}}^m := -\frac{3p_{\mathrm{i}}^m - 4p_{\mathrm{i}}^{m+1} + p_{\mathrm{i}}^{m+2}}{2\delta t}.$$

Fokker-Planck in flux form

$\partial_t f = \nabla \cdot F(f)$, where $F = Bf + C\nabla f$,

$$B^i(x, t, u) = \frac{1}{2}\sum_{j=1}^n \partial_{x_j} a_{ij}(x, t) - b_i(x, t; u) \qquad C^i(x, t) = \frac{1}{2} a_{ii}(x, t)$$

Discrete gradient of the flux: $\nabla \cdot F \approx \frac{1}{h}\sum_{i=1}^n (F^i_{\mathrm{i}+1_i/2} - F^i_{\mathrm{i}-1_i/2})$.

The Chang-Cooper (CC) scheme

For spatial-discretization we use the Chang-Cooper (CC) scheme that is stable, second-order accurate, positive, and conservative.
The cell boundary value is a linear spatial combination of f^{n+1} (reconstruction)

$$f^{n+1}_{\mathrm{i}+1_i/2} = (1 - \delta_i)\, f^{n+1}_{\mathrm{i}+1_i} + \delta_i\, f^{n+1}_{\mathrm{i}}, \qquad \delta_i \in [0, 1/2]$$

choosing $\delta_i = \frac{1}{w_i} - \frac{1}{\exp(w_i) - 1}$ where $w_i = h\, B^{i,n}_{\mathrm{i}+1_i/2} / C^{i,n}_{\mathrm{i}+1_i/2}$

The discrete flux in the i-th direction is computed as follows

$$F^i_{\mathrm{i}+1_i/2} = \left[(1 - \delta_i) B^{i,n}_{\mathrm{i}+1_i/2} + \frac{1}{h} C^{i,n}_{\mathrm{i}+1_i/2}\right] f^{n+1}_{\mathrm{i}+1_i} - \left(\frac{1}{h} C^{i,n}_{\mathrm{i}+1_i/2} - \delta_i B^{i,n}_{\mathrm{i}+1_i/2}\right) f^{n+1}_{\mathrm{i}}$$

The transpose discrete operator is used to solve the backward equation (discretize then optimize approach)

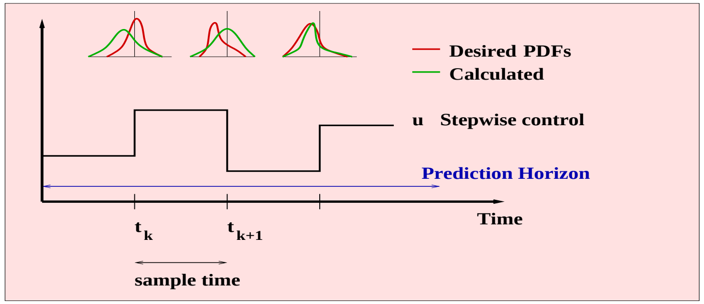
The overall optimization process

1. Solve forward in time the discrete FP equations with given initial condition;
2. Assign the terminal condition;
3. Solve backward in time the discrete adjoint FP equations with terminal condition;
4. Compute the discrete gradient $\nabla_u \hat{J}(u)$;
5. Solve the optimization problem $\min_u J(f,u)$ with a Nonlinear Coniugate Gradient (NCG) scheme, with Dai and Yuan β, and a robust bisection linesearch.
   This requires many repetitions of the steps 1., 2., 3. and 4.
6. End.

Receding Horizon Model Predictive Control

Set $k=0$, set $\rho_0=\rho$;

1. Assign the initial PDFs, $f(x,t_k)=\rho_k(x)$, and the desired targets $f_D(\cdot,t_{k+1})$;
2. In the time window (t_k,t_{k+1}), apply an optimization to solve $\min_u J(f(u),u)$, thus obtain the optimal pair $(\boldsymbol{f},\boldsymbol{u})$;
3. If $t_{k+1}<T$, set $k:=k+1$, $\rho_k=f(\cdot,t_k)$, go to 1.
4. End.



Tracking a trajectory with a limit-cycle model

Noised limit cycle equation with control Ref.[2]

$$\begin{cases} dX_1 = (+X_2 + (1+u_1 - X_1^2 - X_2^2)X_1)\,dt + \sigma dW_{1t} \\ dX_2 = (-X_1 + (1+u_2 - X_1^2 - X_2^2)X_2)\,dt + \sigma dW_{2t} \end{cases}$$

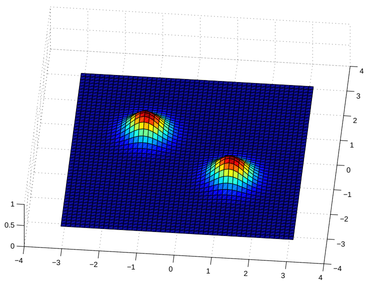
The purpose of the control is to track the target given by a bi-modal multivariate Gaussian PDF

$$f_D = \frac{1}{2}\frac{\exp\left(-\frac{(x_1-\mu_{11})^2}{2\sigma_{11}^2} - \frac{(x_2-\mu_{21})^2}{2\sigma_{21}^2}\right)}{2\pi\sigma_{11}\sigma_{21}} + \frac{1}{2}\frac{\exp\left(-\frac{(x_1-\mu_{12})^2}{2\sigma_{12}^2} - \frac{(x_2-\mu_{22})^2}{2\sigma_{22}^2}\right)}{2\pi\sigma_{12}\sigma_{22}}$$

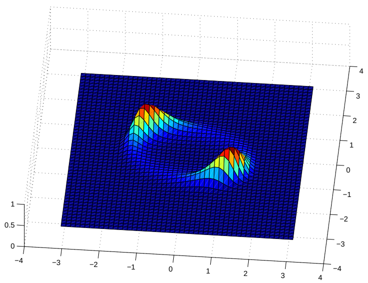
with peaks placed symmetrically with respect to the origin at the points $(\mu_{11},\mu_{21})=(-1.2,0.8)$ and $(\mu_{12},\mu_{22})=(1.2,-0.8)$
We have $T=30$ and the time-window size is $\Delta t=5$.

A controlled noised limit-cycle model

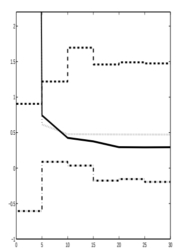
The Fokker-Planck RH-MPC control strategy is able to drive the system to a bi-modal PDF configuration (!) starting from an initial approximate delta-Dirac PDF located at the point $(1.5,1.5)$.



$f_D(x_1,x_2)$



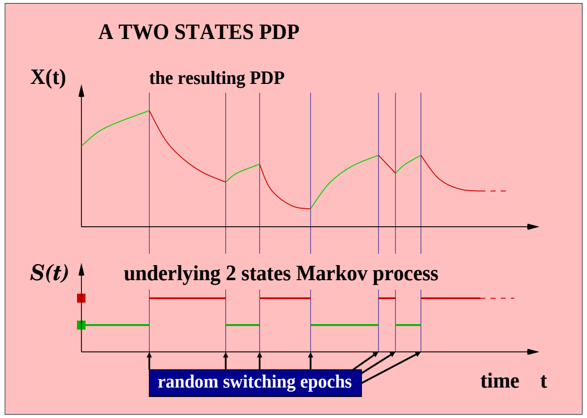
$f(x_1,x_2,T)$



Controls and norm difference

2 states Piecewise deterministic process

1. $\dot{X}(t) = A_{\mathcal{S}(t)}(X, u)$, $\{A_1(x,u), A_2(x,u)\}$
2. $\mathcal{S}(t)$ is a discrete Markov process:
   with exponential transition events $\mu e^{-\mu t}$ and stochastic transition matrix between states: $0 \le q_{ij} \le 1$
   $\sum_{i=1}^{2} q_{ij} = 1$
   $i, j = 1, 2$



$X(t)$ is mostly deterministic, punctuated by random switches at random times

Case study from engineering

Linear filtering of dichotomous noise Ref. [4]

$$\dot{\boldsymbol{X}} = -\gamma\, \boldsymbol{X} + (\boldsymbol{F} + \boldsymbol{u})\, \xi$$

$\xi(t)$ dichotomous noise with values ± 1 and μ parameter of the Poisson random switching events exponentially distributed.
$\gamma = 1$ friction.
$F = 1$ force intensity.

Controlled dynamics $\boldsymbol{A_1(x, u_1) = -x + 1 + u_1}$, $\boldsymbol{A_2(x, u_2) = -(x + 1 + u_2)}$
the forward equation is an hyperbolic PDE system.
The solution is not of travelling wave type:
it evolves toward an equilibrium.

Optimality system (open loop)

forward: $\boldsymbol{\partial_t f_s(x,t) + \partial_x\left(A_s(x,u_s)\, f_s(x,t)\right) = \sum_{j=1}^{S} Q_{sj} f_j(x,t)}$

initial: $\boldsymbol{f_s(x,0) = f_s^0(x)}$

adjoint: $\boldsymbol{-\partial_t p_s(x,t) - A_s(x,u_s)\, \partial_x p_s(x,t) = \sum_{j=1}^{S} Q_{js} p_j(x,t)}$

terminal: $\boldsymbol{p_s(x,T) = -\left(f_s(x,T) - f_{D,s}(x)\right)}$

gradient: $\boldsymbol{\nu\, u_s - \int_0^T \int_\Omega \left(\partial_x p_s(x,t)\right) \frac{\partial A_s(x,u_s)}{\partial u_s}\, f_s(x,t)\, dx\, dt = 0.}$

Control of linear filtering of dichotomous noise

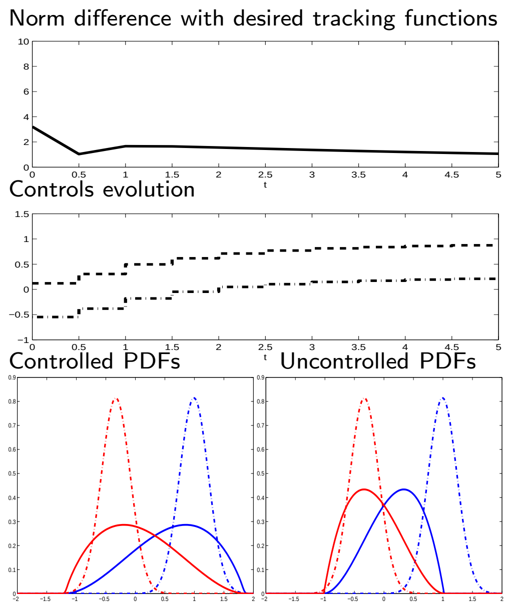
Initial condition: 2 gaussians in $x = 0$ and $\sigma = 0.1$

Transition rate $\mu = 2$

Gaussian tracking functions:
$m_1(t) = 1 - e^{-t}$
$m_2(t) = -m_1(t)/3$
$\sigma_{1,2}(t) = 0.1\sqrt{1+t}$

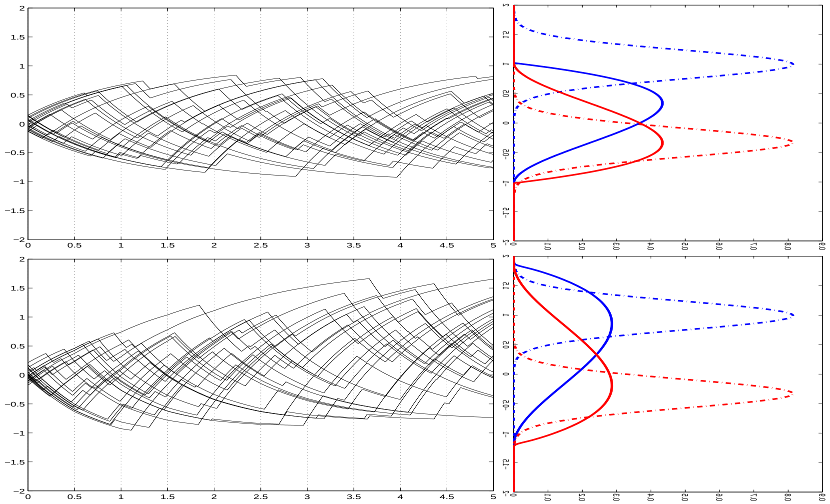
Cost weight: $\nu = 10^{-4}$

Time horizon: $T = 5$ and 10 windows



Monte Carlo simulation

At $t = T$ the controlled trajectories distribute according to the PDFs



Conclusion and thanks

A innovative Kolmogorov-Fokker-Planck control framework for determining controls of multidimensional stochastic processes, including also piecewise deterministic processes.

Optimal controls were obtained minimizing a deterministic PDF objective under the constraint given by the Kolmogorov-Fokker-Planck equation that models the evolution of the probability density function.

A related computational tool for the optimal control of multi-dimensional of stochastic processes was demonstrated to be effective on models of "real-word" applications.

Thank you for your attention

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