Quick reachability and proper extension for problems with unbounded controls

Maria Soledad Aronna, Monica Motta, Franco Rampazzo

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Outline

For a CONTROL SYSTEM of the form

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0, T],$$

$$(x, u)(0) = (\bar{x}, \bar{u}),$$

with $x : [0, T] \to \boldsymbol{R}^n$, $u : [0, T] \to U \subset \boldsymbol{R}^m$, $v : [0, T] \to V \subset \boldsymbol{R}^l$,

we rely on the notion of LIMIT SOLUTION,

and we investigate whether minimum problems with $\mathcal{L}^1-$controls are

PROPER EXTENSIONS

of regular problems with more regular controls (AC or BV).

Motivation: optimality conditions, numerical methods, etc.

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Limit solutions

Consider the Cauchy problem

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{for } t \in [0, T],$$
$$(x, u)(0) = (\bar{x}, \bar{u}).$$

Here $(\bar{x}, \bar{u}) \in R^n \times U$, $u \in \mathcal{L}^1([0, T]; U)$ having $u(0) = \bar{u}$, and $v \in L^1([0, T]; V)$.

We say that $x : [0, T] \to R^n$ is a LIMIT SOLUTION if, for *every* $\tau \in [0, T]$, there exists $(u_k^\tau) \subset AC([0, T]; U)$ such that $u_k^\tau(0) = \bar{u}$ and the corresponding Carathéodory solutions (x_k^τ) are uniformly bounded and satisfy

$$|(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x, u)\|_1 \to 0, \quad \text{when } k \to \infty.$$

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Example: $AC-$reachable set $\neq \mathcal{L}^1-$reachable set

Fix $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$. Observe that the inclusion $\mathcal{R}_{AC} \subseteq \mathcal{R}$ can be strict:

$$\begin{cases} \dot{x}_1 = \dot{u}, \\ \dot{x}_2 = -1 + x_1^2, \\ (x_1, x_2)(0) = (1,1), \ \ u(0) = 1, \end{cases}$$

with $U = [0,1]$, $t \in [0,1]$.

It is trivial to verify that if $\tilde{u}$ is absolutely continuous then the corresponding trajectory $\tilde{x}$ verifies $\tilde{x}_2(1) > 0$. In particular,

$$(0,0,0) \notin \mathcal{R}_{AC}.$$

On the other hand, setting $\hat{u}(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t \in]0,1], \end{cases}$ we get that

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Proper extension of minimum problems

Definition

Let E be a set and let $\mathcal{F} : E \to \mathbb{R}$ be a function. A proper extension of a minimum problem

$$\inf_{e \in E} \mathcal{F}(e),$$

is a new minimum problem

$$\inf_{\hat{e} \in \hat{E}} \hat{\mathcal{F}}(\hat{e})$$

on a set $\hat{E}$ endowed with a limit notion and such that there exists an injective map $i : E \to \hat{E}$ verifying the following properties:

(i) $\hat{\mathcal{F}}(i(e)) = \mathcal{F}(e)$ for all $e \in E$ and, moreover, for every $\hat{e} \in \hat{E}$ there exists a sequence (e_k) in E such that, setting $\hat{e}_k := i(e_k)$, one has

$$\lim_{k \to \infty} (\hat{e}_k, \hat{\mathcal{F}}(\hat{e}_k)) = (\hat{e}, \hat{\mathcal{F}}(\hat{e})),$$

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Proper extension with NO final constraints

For

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consider a cost function $\psi : \mathbf{R}^n \times U \to \mathbf{R}$.

Define the following optimal control problems depending on $(\bar{x},\bar{u})$:

(P) $\inf \psi((x,u)(T))$: $(u,v) \in \mathcal{L}^1 \times L^1$, $u(0) = \bar{u}$, $x \in \Sigma[\bar{x},u,v]$,

(P_{AC}) $\inf \psi((x,u)(T))$: $(u,v) \in AC \times L^1$, $u(0) = \bar{u}$, $x = x[\bar{x},u,v]$,

Theorem

(P) is a proper extension of (P_{AC}) : (i) for every x limit solution associated to (u,v), there exists a sequence $(u_k) \subset AC$, $u_k(0) = 0$, and $x_k := x[\bar{x},u_k,v]$ such that $\|(x,u)-(x_k,u_k)\|_1 \to 0$, and

$$(ii) \quad \inf_{(u,v)\in\mathcal{L}^1\times L^1} \psi((x,u)(T)) = \inf_{(u,v)\in AC\times L^1} \psi((x,u)(T)).$$

Consequently, $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$.

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BV controls - Recall: Graph completions

- For **regular** $u \in AC$ one can reparametrize time $t(s) = \varphi_0(s)$ with $\varphi_0 : [0,1] \to [0,T]$, and set $\varphi(s) := u \circ \varphi_0(s)$.

The SPACE-TIME SYSTEM:

$$\begin{cases} y_0'(s) = \varphi_0'(s), \\ y'(s) = f(y(s), \varphi(s), \psi(s))\varphi_0'(s) + \displaystyle\sum_{\alpha=1}^{m} g_\alpha(y(s))\varphi_\alpha{}'(s), \qquad s \in [0,1]. \\ (y_0, y)(0) = (0, \bar{x}), \end{cases}$$

- For BV controls u, let (φ_0, φ) be a **graph completion** of u : $(\varphi_0, \varphi) : [0,1] \to [0,T] \times U$ Lipschitz continuous such that, $\forall t \in [0,T]$, there exists $s \in [0,1]$ verifying $(t, u(t)) = (\varphi_0, \varphi)(s)$.

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Recall: Graph completion solutions

Given $(\varphi_0, \varphi) : [0,1] \to [0,T] \times U$, let y be the solution of the SPACE-TIME system

Graph completion solution: (possibly) set-valued map $x : [0,T] \rightrightarrows \boldsymbol{R}^n$,

$$t \Longmapsto x(t) := y \circ \varphi_0^{-1}(t).$$

Single-valued graph completion solution:
Let $\sigma : [0,T] \to [0,1]$ be a right-inverse of φ_0 having $u(t) = \varphi \circ \sigma(t)$,

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Optimal control problems with bounded variation controls

(P^K_{gc}) $\inf \psi((x,u)(T)) \,:\, (u,v) \in BV^K \times L^1,\ u(0)=\bar{u},\ x \in \Sigma^{K+T}_{gc}[\bar{x},u,v]$,

Recall: A simple limit solution $x:[0,T] \to R^n$ is a BV-SIMPLE limit solution if (u^τ_k) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

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From [Aronna & Rampazzo, 2013] (Rampazzo's presentation) we know that

$$\Sigma^{K+T}_{gc}[\bar{x},u,v] = \Sigma^{K}_{BVS}[\bar{x},u,v]$$

Theorem

For every initial condition $(\bar{x},\bar{u}) \in R^n \times U$, one has

$$\lim_{K\to\infty} Val(P^K_{gc})(\bar{x},\bar{u}) = \lim_{K\to\infty} Val(P^K_{BVS})(\bar{x},\bar{u}) = V(\bar{x},\bar{u}).$$

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Optimal control problems with bounded variation controls

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Recall: A simple limit solution $x : [0,T] \to \boldsymbol{R}^n$ is a BV-SIMPLE limit solution if (u^τ_k) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

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The problem with final constraints

Let $\mathcal{S} \subset \boldsymbol{R}^n \times U$ be a closed subset. Define the problems

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\begin{aligned}
(P^c)\ \ \inf \psi\big((x,u)(T)\big) : (u,v) \in \mathcal{L}^1 \times L^1,\ u(0) = \bar{u}, \\
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&\inf\ x_2(1),\\
&\quad \dot{x}_1(t) = \dot{u}(t),\\
&\quad \dot{x}_2(t) = |x_1(t)|,\\
&\quad (x_1,\ x_2)(0) = (1,0), \qquad u(0) = 1\\
&\quad (x_1, x_2, u)(1) \in \mathcal{S} := \Big(\{(0,0)\} \cup (\boldsymbol{R} \times [1, +\infty[)\Big) \times [0,1],\\
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For every input $u \in AC$, one has $x_2(1) = \int_0^1 |u(s)| \mathrm{d}s > 0$.

Hence, if $(x_1, x_2, u)(1) \in \mathcal{S} \Longrightarrow x_2(1) \geq 1, \Longrightarrow \mathrm{Val}(P^c_{AC}) \geq 1$.

On the other hand, by implementing the impulsive control

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u(t) := \begin{cases} 1, & t = 0, \\ 0, & t \in]0,1], \end{cases} \Longrightarrow x_2(1) = 0 \Longrightarrow \mathrm{Val}(P^c) \leq 0 < 1 \leq \mathrm{Val}(P^c_{AC}).
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[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

Theorem (A sufficient condition for proper extension in the presence of terminal constraints): Assume that $\mathcal{S}$ is a compact set contained in the interior of $R^n \times U$, and that there exist some positive constants ρ, η such that:

(i) for each $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S},\rho)\setminus\mathcal{S}}$, and $\forall (p_x, p_u) \in D^*\mathrm{d}_{\mathcal{S}}(x,u)$, we have

$$\min_{|w|\leq 1}\left\{\left\langle p_x\,,\,\sum_{\alpha=1}^{m} g_\alpha(x,u)w^\alpha\right\rangle + \langle p_u, w\rangle\right\} < -\eta;$$

Limiting gradient: Let $\Omega \subset R^k$ be an open set, $F:\Omega \to R$ be locally Lipschitz continuous. For $x \in \Omega$ define

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(ii) + viability.

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Sketch of the proof

Let us consider $(u, v) \in \mathcal{L}^1 \times L^1$ and an associated limit solution x feasible for the problem (P^c), i.e. having

$$(x, u)(T) \in \mathcal{S}.$$

By definition of limit solution, there exists a sequence $(\hat{u}_k) \subset AC$ such that $\hat{u}_k(0) = \bar{u}$ and

$$\|(\hat{x}_k, \hat{u}_k) - (x, u)\|_1 + |(\hat{x}_k, \hat{u}_k)(T) - (x, u)(T)| \to 0 \text{ as } k \to +\infty,$$

where $\hat{x}_k \doteq x[\bar{x}, \hat{u}_k, v]$ are the corresponding Carathéodory solutions.

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In general,

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Step 1. We construct (σ_k) such that

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$$\mathcal{T}_{\mathcal{S},\eta}(y) \leq \frac{\mathrm{d}_{\mathcal{S}}(y)}{c(\eta)}.$$

See e.g. [Motta & Rampazzo, 2013]
This way we get $(\tilde{x}_k, \tilde{u}_k) : [T - \sigma_k, T_k] \to \mathbb{R}^n \times U$, having

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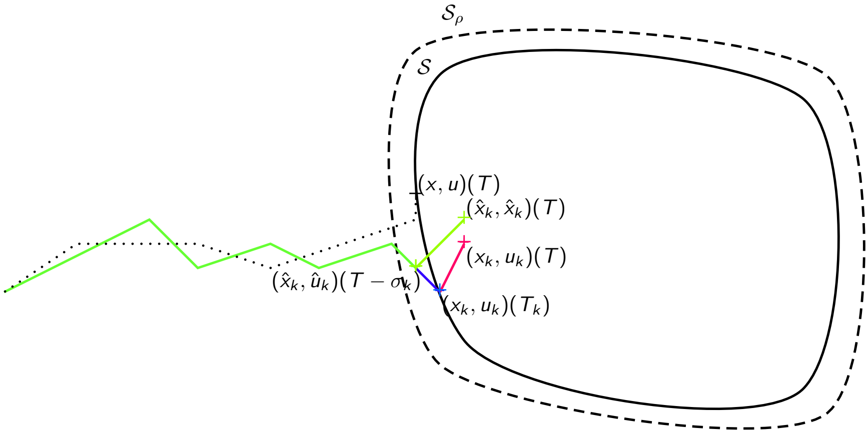

■

Concluding remarks

- The limit solutions naturally provide a proper extension of standard optimal control problems with no final constraints.
- The limit solutions optimal control problem (with $\mathcal{L}^1$ controls and trajectories) with no final constraints is the limit of problems with controls of variation K with $K \to \infty$.
- When constraints are considered, a quick reachability condition + viability guarantee that the impulsive problem (with limit solutions as trajectories) provides a proper extension.

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References

M.S. Aronna & F. Rampazzo. In Proceedings of the 52nd IEEE Conference on Decision and Control, 2013.

M.S. Aronna & F. Rampazzo. $\mathcal{L}^1$ limit solutions for control systems. 2013. Accepted for publication in JDE.

M.S. Aronna & F. Rampazzo. A note on systems with ordinary and impulsive controls. To appear in IMA J. Math. Control Inform, 2014.

M.S. Aronna, M. Motta & F. Rampazzo. Quick reachability and proper extension of unbounded control systems. [In preparation]

THANK YOU FOR YOUR ATTENTION