

# Quick reachability and proper extension for problems with unbounded controls

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# Quick reachability and proper extension for problems with unbounded controls

María Soledad Aronna  
IMPA, Rio de Janeiro, Brazil

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(Joint work with M. Motta & F. Rampazzo, Università di Padova, Italy)

Conference on New Trends in Optimal Control  
June 2014, Tours, France

# Outline

For a CONTROL SYSTEM of the form

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha}, \quad \text{on } [0, T],$$

$$(x, u)(0) = (\bar{x}, \bar{u}),$$

with  $x : [0, T] \rightarrow \mathbf{R}^n$ ,  $u : [0, T] \rightarrow U \subset \mathbf{R}^m$ ,  $v : [0, T] \rightarrow V \subset \mathbf{R}^l$ ,

we rely on the notion of **LIMIT SOLUTION**,

and we investigate whether minimum problems with  $\mathcal{L}^1$ -controls are

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of regular problems with **more regular controls** (AC or BV).

**Motivation:** optimality conditions, numerical methods, etc.

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# Limit solutions

Consider the Cauchy problem

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$$(x, u)(0) = (\bar{x}, \bar{u}).$$

Here  $(\bar{x}, \bar{u}) \in \mathbf{R}^n \times U$ ,  $u \in \mathcal{L}^1([0, T]; U)$  having  $u(0) = \bar{u}$ , and  $v \in L^1([0, T]; V)$ .

We say that  $x : [0, T] \rightarrow \mathbf{R}^n$  is a **LIMIT SOLUTION** if, for every  $\tau \in [0, T]$ , there exists  $(u_k^{\tau}) \subset AC([0, T]; U)$  such that  $u_k^{\tau}(0) = \bar{u}$  and the corresponding Carathéodory solutions  $(x_k^{\tau})$  are uniformly bounded and satisfy

$$|(x_k^{\tau}, u_k^{\tau})(\tau) - (x, u)(\tau)| + \|(x_k^{\tau}, u_k^{\tau}) - (x, u)\|_1 \rightarrow 0, \quad \text{when } k \rightarrow \infty.$$

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# Example: $AC$ –reachable set $\neq \mathcal{L}^1$ –reachable set

Fix  $(\bar{x}, \bar{u}) \in \mathbf{R}^n \times U$ . Observe that the inclusion  $\mathcal{R}_{AC} \subseteq \mathcal{R}$  can be strict:

$$\begin{cases} \dot{x}_1 = \dot{u}, \\ \dot{x}_2 = -1 + x_1^2, \\ (x_1, x_2)(0) = (1, 1), \quad u(0) = 1, \end{cases}$$

with  $U = [0, 1]$ ,  $t \in [0, 1]$ .

It is trivial to verify that if  $\tilde{u}$  is absolutely continuous then the corresponding trajectory  $\tilde{x}$  verifies  $\tilde{x}_2(1) > 0$ . In particular,

$$(0, 0, 0) \notin \mathcal{R}_{AC}.$$

On the other hand, setting  $\hat{u}(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t \in ]0, 1], \end{cases}$  we get that

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# Proper extension of minimum problems

## Definition

Let  $E$  be a set and let  $\mathcal{F} : E \rightarrow \mathbf{R}$  be a function. A **proper extension** of a minimum problem

$$\inf_{e \in E} \mathcal{F}(e),$$

is a new minimum problem

$$\inf_{\hat{e} \in \hat{E}} \hat{\mathcal{F}}(\hat{e})$$

on a set  $\hat{E}$  endowed with a limit notion and such that there exists an injective map  $i : E \rightarrow \hat{E}$  verifying the following properties:

- (i)  $\hat{\mathcal{F}}(i(e)) = \mathcal{F}(e)$  for all  $e \in E$  and, moreover, for every  $\hat{e} \in \hat{E}$  there exists a sequence  $(e_k)$  in  $E$  such that, setting  $\hat{e}_k := i(e_k)$ , one has

$$\lim_{k \rightarrow \infty} (\hat{e}_k, \hat{\mathcal{F}}(\hat{e}_k)) = (\hat{e}, \hat{\mathcal{F}}(\hat{e})),$$

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# Proper extension with NO final constraints

For

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^m g_{\alpha}(x) \dot{u}_{\alpha}, \quad \text{on } [0, T], \quad (x, u)(0) = (\bar{x}, \bar{u}),$$

consider a cost function  $\psi : \mathbf{R}^n \times U \rightarrow \mathbf{R}$ .

Define the following optimal control problems depending on  $(\bar{x}, \bar{u})$  :

$(P)$   $\inf \psi((x, u)(T)) : (u, v) \in \mathcal{L}^1 \times L^1, u(0) = \bar{u}, x \in \Sigma[\bar{x}, u, v],$

$(P_{AC})$   $\inf \psi((x, u)(T)) : (u, v) \in AC \times L^1, u(0) = \bar{u}, x = x[\bar{x}, u, v],$

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$(P)$  is a proper extension of  $(P_{AC})$  : (i) for every  $x$  limit solution associated to  $(u, v)$ , there exists a sequence  $(u_k) \subset AC, u_k(0) = \bar{u}$ , and  $x_k := x[\bar{x}, u_k, v]$  such that  $\|(x, u) - (x_k, u_k)\|_1 \rightarrow 0$ , and

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Consequently,  $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$ .

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# BV controls - Recall: Graph completions

- For **regular**  $u \in AC$  one can **reparametrize time**  $t(s) = \varphi_0(s)$  with  $\varphi_0 : [0, 1] \rightarrow [0, T]$ , and set  $\varphi(s) := u \circ \varphi_0(s)$ .

The **SPACE-TIME SYSTEM**:

$$\begin{cases} y_0'(s) = \varphi_0'(s), \\ y'(s) = f(y(s), \varphi(s), \psi(s))\varphi_0'(s) + \sum_{\alpha=1}^m g_\alpha(y(s))\varphi_\alpha'(s), & s \in [0, 1], \\ (y_0, y)(0) = (0, \bar{x}), \end{cases}$$

- For **BV controls**  $u$ , let  $(\varphi_0, \varphi)$  be a **graph completion** of  $u$  :  
 $(\varphi_0, \varphi) : [0, 1] \rightarrow [0, T] \times U$  Lipschitz continuous such that,  
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# Recall: Graph completion solutions

Given  $(\varphi_0, \varphi) : [0, 1] \rightarrow [0, T] \times U$ , let  $y$  be the solution of the SPACE-TIME system

**Graph completion solution:** (possibly) set-valued map  $x : [0, T] \rightrightarrows \mathbb{R}^n$ ,

$$t \longmapsto x(t) := y \circ \varphi_0^{-1}(t).$$

**Single-valued graph completion solution:**

Let  $\sigma : [0, T] \rightarrow [0, 1]$  be a right-inverse of  $\varphi_0$  having  $u(t) = \varphi \circ \sigma(t)$ ,

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# Optimal control problems with bounded variation controls

$$(P_{gc}^K) \inf \psi((x, u)(T)) : (u, v) \in BV^K \times L^1, u(0) = \bar{u}, x \in \Sigma_{gc}^{K+T}[\bar{x}, u, v],$$

Recall: A simple limit solution  $x : [0, T] \rightarrow \mathbf{R}^n$  is a **BV-SIMPLE limit solution** if  $(u_k^\tau)$  can be chosen independently of  $\tau$  and the approximating inputs  $u_k$  have equibounded variation.

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# The problem with final constraints

Let  $\mathcal{S} \subset \mathbf{R}^n \times U$  be a closed subset. Define the problems

$$(P^c) \inf \psi((x, u)(T)) : (u, v) \in \mathcal{L}^1 \times L^1, u(0) = \bar{u}, \\ x \in \Sigma[\bar{x}, u, v], (x, u)(T) \in \mathcal{S},$$

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$$\dot{x}_1(t) = \dot{u}(t),$$

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$$(x_1, x_2)(0) = (1, 0), \quad u(0) = 1$$

$$(x_1, x_2, u)(1) \in \mathcal{S} := \left( \{(0, 0)\} \cup (\mathbf{R} \times [1, +\infty[)) \times [0, 1], \right.$$

$$\left. u(t) \in U := \left[-\frac{1}{3}, \frac{4}{3}\right] \right.$$

For every input  $u \in AC$ , one has  $x_2(1) = \int_0^1 |u(s)| ds > 0$ .

Hence, if  $(x_1, x_2, u)(1) \in \mathcal{S} \implies x_2(1) \geq 1, \implies \text{Val}(P_{AC}^c) \geq 1$ .

On the other hand, by implementing the impulsive control

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[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

**Theorem (A sufficient condition for proper extension in the presence of terminal constraints):** Assume that  $\mathcal{S}$  is a compact set contained in the interior of  $\mathbb{R}^n \times U$ , and that there exist some positive constants  $\rho, \eta$  such that:

(i) for each  $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S}, \rho)} \setminus \mathcal{S}$ , and  $\forall (p_x, p_u) \in D^*d_{\mathcal{S}}(x, u)$ , we have

$$\min_{|w| \leq 1} \left\{ \left\langle p_x, \sum_{\alpha=1}^m g_\alpha(x, u) w^\alpha \right\rangle + \langle p_u, w \right\rangle \right\} < -\eta;$$

*Limiting gradient:* Let  $\Omega \subset \mathbb{R}^k$  be an open set,  $F : \Omega \rightarrow \mathbb{R}$  be locally Lipschitz continuous. For  $x \in \Omega$  define

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Let us consider  $(u, v) \in \mathcal{L}^1 \times L^1$  and an associated limit solution  $x$  feasible for the problem  $(P^c)$ , i.e. having

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By definition of limit solution, there exists a sequence  $(\hat{u}_k) \subset AC$  such that  $\hat{u}_k(0) = \bar{u}$  and

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$$\mathcal{T}_{\mathcal{S}, \eta}(y) \leq \frac{d_{\mathcal{S}}(y)}{c(\eta)}.$$

See e.g. [Motta & Rampazzo, 2013]

This way we get  $(\tilde{x}_k, \tilde{u}_k) : [T - \sigma_k, T_k] \rightarrow \mathbb{R}^n \times U$ , having

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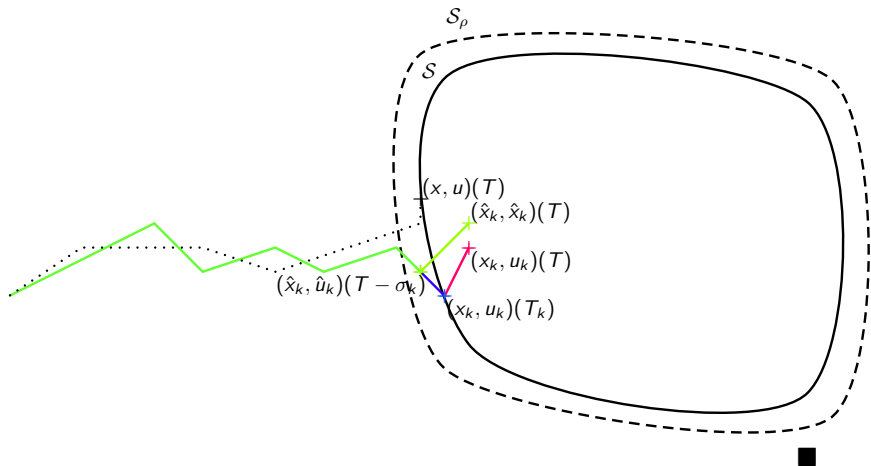
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# Concluding remarks

- The limit solutions naturally provide a proper extension of standard optimal control problems with no final constraints.
- The limit solutions optimal control problem (with  $\mathcal{L}^1$  controls and trajectories) with no final constraints is the limit of problems with controls of variation  $K$  with  $K \rightarrow \infty$ .
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# References

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THANK YOU FOR YOUR ATTENTION