Quick reachability and proper extension for problems with unbounded controls

Maria Soledad Aronna, Monica Motta, Franco Rampazzo

Quick reachability and proper extension for problems with unbounded controls

María Soledad Aronna
IMPA, Rio de Janeiro, Brazil
-
(Joint work with M. Motta & F. Rampazzo, Università di Padova, Italy)

Conference on New Trends in Optimal Control
June 2014, Tours, France

Outline

For a CONTROL SYSTEM of the form

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0, T],$$
$$(x, u)(0) = (\bar{x}, \bar{u}),$$

with $x : [0, T] \to \mathbf{R}^n$, $u : [0, T] \to U \subset \mathbf{R}^m$, $v : [0, T] \to V \subset \mathbf{R}^l$,

we rely on the notion of LIMIT SOLUTION,

and we investigate whether minimum problems with $\mathcal{L}^1-$controls are

PROPER EXTENSIONS

of regular problems with more regular controls (AC or BV).

Motivation: optimality conditions, numerical methods, etc.

Outline

For a CONTROL SYSTEM of the form

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0, T],$$

$$(x, u)(0) = (\bar{x}, \bar{u}),$$

with $x : [0, T] \to \boldsymbol{R}^n$, $u : [0, T] \to U \subset \boldsymbol{R}^m$, $v : [0, T] \to V \subset \boldsymbol{R}^l$,

we rely on the notion of LIMIT SOLUTION,

and we investigate whether minimum problems with $\mathcal{L}^1-$controls are

PROPER EXTENSIONS

of regular problems with more regular controls (AC or BV).

Motivation: optimality conditions, numerical methods, etc.

Outline

For a CONTROL SYSTEM of the form

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0, T],$$
$$(x, u)(0) = (\bar{x}, \bar{u}),$$

with $x : [0, T] \to \boldsymbol{R}^n$, $u : [0, T] \to U \subset \boldsymbol{R}^m$, $v : [0, T] \to V \subset \boldsymbol{R}^l$,

we rely on the notion of LIMIT SOLUTION,

and we investigate whether minimum problems with $\mathcal{L}^1-$controls are

PROPER EXTENSIONS

of regular problems with more regular controls (AC or BV).

Motivation: optimality conditions, numerical methods, etc.

Outline

For a CONTROL SYSTEM of the form

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0, T],$$

$$(x, u)(0) = (\bar{x}, \bar{u}),$$

with $x : [0, T] \to \boldsymbol{R}^n$, $u : [0, T] \to U \subset \boldsymbol{R}^m$, $v : [0, T] \to V \subset \boldsymbol{R}^l$,

we rely on the notion of LIMIT SOLUTION,

and we investigate whether minimum problems with $\mathcal{L}^1-$controls are

PROPER EXTENSIONS

of regular problems with more regular controls (AC or BV).

Motivation: optimality conditions, numerical methods, etc.

Limit solutions

Consider the Cauchy problem

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{for } t \in [0, T],$$
$$(x, u)(0) = (\bar{x}, \bar{u}).$$

Here $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times U$, $u \in \mathcal{L}^1([0, T]; U)$ having $u(0) = \bar{u}$, and $v \in L^1([0, T]; V)$.

We say that $x : [0, T] \to \mathbb{R}^n$ is a LIMIT SOLUTION if, for *every* $\tau \in [0, T]$, there exists $(u_k^\tau) \subset AC([0, T]; U)$ such that $u_k^\tau(0) = \bar{u}$ and the corresponding Carathéodory solutions (x_k^τ) are uniformly bounded and satisfy

$$|(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x, u)\|_1 \to 0, \quad \text{when } k \to \infty.$$

Limit solutions

Consider the Cauchy problem

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{for } t \in [0, T],$$

$$(x, u)(0) = (\bar{x}, \bar{u}).$$

Here $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$, $u \in \mathcal{L}^1([0, T]; U)$ having $u(0) = \bar{u}$, and $v \in L^1([0, T]; V)$.

We say that $x : [0, T] \to \boldsymbol{R}^n$ is a LIMIT SOLUTION if, for *every* $\tau \in [0, T]$, there exists $(u_k^\tau) \subset AC([0, T]; U)$ such that $u_k^\tau(0) = \bar{u}$ and the corresponding Carathéodory solutions (x_k^τ) are uniformly bounded and satisfy

$$|(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x, u)\|_1 \to 0, \quad \text{when } k \to \infty.$$

Limit solutions

Consider the Cauchy problem

$$\dot{x} = f(x, u, v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{for } t \in [0, T],$$
$$(x, u)(0) = (\bar{x}, \bar{u}).$$

Here $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$, $u \in \mathcal{L}^1([0, T]; U)$ having $u(0) = \bar{u}$, and $v \in L^1([0, T]; V)$.

We say that $x : [0, T] \to \boldsymbol{R}^n$ is a LIMIT SOLUTION if, for *every* $\tau \in [0, T]$, there exists $(u_k^\tau) \subset AC([0, T]; U)$ such that $u_k^\tau(0) = \bar{u}$ and the corresponding Carathéodory solutions (x_k^τ) are uniformly bounded and satisfy

$$|(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x, u)\|_1 \to 0, \quad \text{when } k \to \infty.$$

Example: $AC-$reachable set $\neq \mathcal{L}^1-$reachable set

Fix $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$. Observe that the inclusion $\mathcal{R}_{AC} \subseteq \mathcal{R}$ can be strict:

$$\begin{cases} \dot{x}_1 = \dot{u}, \\ \dot{x}_2 = -1 + x_1^2, \\ (x_1,x_2)(0) = (1,1), \ \ u(0) = 1, \end{cases}$$

with $U = [0,1]$, $t \in [0,1]$.

It is trivial to verify that if $\tilde{u}$ is absolutely continuous then the corresponding trajectory $\tilde{x}$ verifies $\tilde{x}_2(1) > 0$. In particular,

$$(0,0,0) \notin \mathcal{R}_{AC}.$$

On the other hand, setting $\hat{u}(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t \in]0,1], \end{cases}$ we get that

$$(0,0,0) \in \mathcal{R}.$$

Example: $AC-$reachable set $\neq \mathcal{L}^1-$reachable set

Fix $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$. Observe that the inclusion $\mathcal{R}_{AC} \subseteq \mathcal{R}$ can be strict:

$$\begin{cases} \dot{x}_1 = \dot{u}, \\ \dot{x}_2 = -1 + x_1^2, \\ (x_1,x_2)(0) = (1,1), \;\; u(0) = 1, \end{cases}$$

with $U = [0,1]$, $t \in [0,1]$.

It is trivial to verify that if $\tilde{u}$ is absolutely continuous then the corresponding trajectory $\tilde{x}$ verifies $\tilde{x}_2(1) > 0$. In particular,

$$(0,0,0) \notin \mathcal{R}_{AC}.$$

On the other hand, setting $\hat{u}(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t \in]0,1], \end{cases}$ we get that

$$(0,0,0) \in \mathcal{R}.$$

Example: $AC-$reachable set $\neq \mathcal{L}^1-$reachable set

Fix $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$. Observe that the inclusion $\mathcal{R}_{AC} \subseteq \mathcal{R}$ can be strict:

$$\begin{cases} \dot{x}_1 = \dot{u}, \\ \dot{x}_2 = -1 + x_1^2, \\ (x_1, x_2)(0) = (1,1), \ u(0) = 1, \end{cases}$$

with $U = [0,1]$, $t \in [0,1]$.

It is trivial to verify that if $\tilde{u}$ is absolutely continuous then the corresponding trajectory $\tilde{x}$ verifies $\tilde{x}_2(1) > 0$. In particular,

$$(0,0,0) \notin \mathcal{R}_{AC}.$$

On the other hand, setting $\hat{u}(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t \in]0,1], \end{cases}$ we get that

$$(0,0,0) \in \mathcal{R}.$$

Example: $AC-$reachable set $\neq \mathcal{L}^1-$reachable set

Fix $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$. Observe that the inclusion $\mathcal{R}_{AC} \subseteq \mathcal{R}$ can be strict:

$$\begin{cases} \dot{x}_1 = \dot{u}, \\ \dot{x}_2 = -1 + x_1^2, \\ (x_1, x_2)(0) = (1, 1), \ u(0) = 1, \end{cases}$$

with $U = [0, 1]$, $t \in [0, 1]$.

It is trivial to verify that if $\tilde{u}$ is absolutely continuous then the corresponding trajectory $\tilde{x}$ verifies $\tilde{x}_2(1) > 0$. In particular,

$$(0, 0, 0) \notin \mathcal{R}_{AC}.$$

On the other hand, setting $\hat{u}(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t \in]0, 1], \end{cases}$ we get that

$$(0, 0, 0) \in \mathcal{R}.$$

Proper extension of minimum problems

Definition

Let E be a set and let $\mathcal{F} : E \to \mathbb{R}$ be a function. A proper extension of a minimum problem

$$\inf_{e \in E} \mathcal{F}(e),$$

is a new minimum problem

$$\inf_{\hat{e} \in \hat{E}} \hat{\mathcal{F}}(\hat{e})$$

on a set $\hat{E}$ endowed with a limit notion and such that there exists an injective map $i : E \to \hat{E}$ verifying the following properties:

(i) $\hat{\mathcal{F}}(i(e)) = \mathcal{F}(e)$ for all $e \in E$ and, moreover, for every $\hat{e} \in \hat{E}$ there exists a sequence (e_k) in E such that, setting $\hat{e}_k := i(e_k)$, one has

$$\lim_{k \to \infty} (\hat{e}_k, \hat{\mathcal{F}}(\hat{e}_k)) = (\hat{e}, \hat{\mathcal{F}}(\hat{e})),$$

(ii) $\inf_{e \in E} \mathcal{F}(e) = \inf_{\hat{e} \in \hat{E}} \hat{\mathcal{F}}(\hat{e})$.

Proper extension of minimum problems

Definition

Let E be a set and let $\mathcal{F} : E \to \mathbb{R}$ be a function. A proper extension of a minimum problem

$$\inf_{e \in E} \mathcal{F}(e),$$

is a new minimum problem

$$\inf_{\hat{e} \in \hat{E}} \hat{\mathcal{F}}(\hat{e})$$

on a set $\hat{E}$ endowed with a limit notion and such that there exists an injective map $i : E \to \hat{E}$ verifying the following properties:

(i) $\hat{\mathcal{F}}(i(e)) = \mathcal{F}(e)$ for all $e \in E$ and, moreover, for every $\hat{e} \in \hat{E}$ there exists a sequence (e_k) in E such that, setting $\hat{e}_k := i(e_k)$, one has

$$\lim_{k \to \infty} (\hat{e}_k, \hat{\mathcal{F}}(\hat{e}_k)) = (\hat{e}, \hat{\mathcal{F}}(\hat{e})),$$

(ii) $\inf\limits_{e \in E} \mathcal{F}(e) = \inf\limits_{\hat{e} \in \hat{E}} \hat{\mathcal{F}}(\hat{e})$.

Proper extension of minimum problems

Definition

Let E be a set and let $\mathcal{F} : E \to \mathbb{R}$ be a function. A proper extension of a minimum problem

$$\inf_{e\in E} \mathcal{F}(e),$$

is a new minimum problem

$$\inf_{\hat{e}\in \hat{E}} \hat{\mathcal{F}}(\hat{e})$$

on a set $\hat{E}$ endowed with a limit notion and such that there exists an injective map $i : E \to \hat{E}$ verifying the following properties:

(i) $\hat{\mathcal{F}}(i(e)) = \mathcal{F}(e)$ for all $e \in E$ and, moreover, for every $\hat{e} \in \hat{E}$ there exists a sequence (e_k) in E such that, setting $\hat{e}_k := i(e_k)$, one has

$$\lim_{k\to\infty} \left(\hat{e}_k, \hat{\mathcal{F}}(\hat{e}_k)\right) = (\hat{e}, \hat{\mathcal{F}}(\hat{e})),$$

(ii) $\inf_{e\in E} \mathcal{F}(e) = \inf_{\hat{e}\in \hat{E}} \hat{\mathcal{F}}(\hat{e})$.

Proper extension with NO final constraints

For

$$\dot{x} = f(x,u,v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0,T], \quad (x,u)(0) = (\bar{x},\bar{u}),$$

consider a cost function $\psi : \boldsymbol{R}^n \times U \to \boldsymbol{R}$.

Define the following optimal control problems depending on $(\bar{x},\bar{u})$:

(P) $\inf \psi((x,u)(T))$: $(u,v) \in \mathcal{L}^1 \times L^1$, $u(0) = \bar{u}$, $x \in \Sigma[\bar{x},u,v]$,

(P_{AC}) $\inf \psi((x,u)(T))$: $(u,v) \in AC \times L^1$, $u(0) = \bar{u}$, $x = x[\bar{x},u,v]$,

Theorem

(P) is a proper extension of (P_{AC}) : (i) for every x limit solution associated to (u,v), there exists a sequence $(u_k) \subset AC$, $u_k(0) = 0$, and $x_k := x[\bar{x},u_k,v]$ such that $\|(x,u) - (x_k,u_k)\|_1 \to 0$, and

$$(ii) \quad \inf_{(u,v)\in\mathcal{L}^1\times L^1} \psi((x,u)(T)) = \inf_{(u,v)\in AC\times L^1} \psi((x,u)(T)).$$

Consequently, $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$.

Proper extension with NO final constraints

For

$$\dot{x} = f(x,u,v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0,T], \quad (x,u)(0) = (\bar{x},\bar{u}),$$

consider a cost function $\psi : \boldsymbol{R}^n \times U \to \boldsymbol{R}$.

Define the following optimal control problems depending on $(\bar{x},\bar{u})$:

(P) $\inf \psi((x,u)(T))$: $(u,v) \in \mathcal{L}^1 \times L^1$, $u(0) = \bar{u}$, $x \in \Sigma[\bar{x},u,v]$,

(P_{AC}) $\inf \psi((x,u)(T))$: $(u,v) \in AC \times L^1$, $u(0) = \bar{u}$, $x = x[\bar{x},u,v]$,

Theorem

(P) is a proper extension of (P_{AC}) : (i) for every x limit solution associated to (u,v), there exists a sequence $(u_k) \subset AC$, $u_k(0) = 0$, and $x_k := x[\bar{x},u_k,v]$ such that $\|(x,u)-(x_k,u_k)\|_1 \to 0$, and

$$(ii) \qquad \inf_{(u,v)\in\mathcal{L}^1\times L^1} \psi((x,u)(T)) = \inf_{(u,v)\in AC\times L^1} \psi((x,u)(T)).$$

Consequently, $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$.

Proper extension with NO final constraints

For

$$\dot{x} = f(x,u,v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0,T], \quad (x,u)(0) = (\bar{x},\bar{u}),$$

consider a cost function $\psi : \boldsymbol{R}^n \times U \to \boldsymbol{R}$.

Define the following optimal control problems depending on $(\bar{x},\bar{u})$:

(P) $\inf \psi((x,u)(T))$: $(u,v) \in \mathcal{L}^1 \times L^1,\ u(0) = \bar{u},\ x \in \Sigma[\bar{x},u,v],$

(P_{AC}) $\inf \psi((x,u)(T))$: $(u,v) \in AC \times L^1,\ u(0) = \bar{u},\ x = x[\bar{x},u,v],$

Theorem

(P) is a proper extension of (P_{AC}) : (i) for every x limit solution associated to (u,v), there exists a sequence $(u_k) \subset AC$, $u_k(0) = 0$, and $x_k := x[\bar{x},u_k,v]$ such that $\|(x,u) - (x_k,u_k)\|_1 \to 0$, and

$$(ii) \quad \inf_{(u,v)\in\mathcal{L}^1\times L^1} \psi((x,u)(T)) = \inf_{(u,v)\in AC\times L^1} \psi((x,u)(T)).$$

Consequently, $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$.

Proper extension with NO final constraints

For

$$\dot{x} = f(x,u,v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0,T], \quad (x,u)(0) = (\bar{x},\bar{u}),$$

consider a cost function $\psi : \boldsymbol{R}^n \times U \to \boldsymbol{R}$.

Define the following optimal control problems depending on $(\bar{x},\bar{u})$:

(P) $\inf \psi((x,u)(T)) : \ (u,v) \in \mathcal{L}^1 \times L^1, \ u(0) = \bar{u}, \ x \in \Sigma[\bar{x},u,v],$

(P_{AC}) $\inf \psi((x,u)(T)) : \ (u,v) \in AC \times L^1, \ u(0) = \bar{u}, \ x = x[\bar{x},u,v],$

Theorem

(P) is a proper extension of (P_{AC}) : (i) for every x limit solution associated to (u,v), there exists a sequence $(u_k) \subset AC$, $u_k(0) = 0$, and $x_k := x[\bar{x},u_k,v]$ such that $\|(x,u)-(x_k,u_k)\|_1 \to 0$, and

$$(ii) \quad \inf_{(u,v)\in\mathcal{L}^1\times L^1} \psi((x,u)(T)) = \inf_{(u,v)\in AC\times L^1} \psi((x,u)(T)).$$

Consequently, $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$.

Proper extension with NO final constraints

For

$$\dot{x} = f(x,u,v) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}_\alpha, \quad \text{on } [0,T], \quad (x,u)(0) = (\bar{x},\bar{u}),$$

consider a cost function $\psi : \boldsymbol{R}^n \times U \to \boldsymbol{R}$.

Define the following optimal control problems depending on $(\bar{x},\bar{u})$:

(P) $\inf \psi((x,u)(T))$: $(u,v) \in \mathcal{L}^1 \times L^1,\ u(0) = \bar{u},\ x \in \Sigma[\bar{x},u,v]$,

(P_{AC}) $\inf \psi((x,u)(T))$: $(u,v) \in AC \times L^1,\ u(0) = \bar{u},\ x = x[\bar{x},u,v]$,

Theorem

(P) is a proper extension of (P_{AC}) : (i) for every x limit solution associated to (u,v), there exists a sequence $(u_k) \subset AC$, $u_k(0) = 0$, and $x_k := x[\bar{x},u_k,v]$ such that $\|(x,u) - (x_k,u_k)\|_1 \to 0$, and

$$(ii) \quad \inf_{(u,v)\in\mathcal{L}^1\times L^1} \psi((x,u)(T)) = \inf_{(u,v)\in AC\times L^1} \psi((x,u)(T)).$$

Consequently, $\overline{\mathcal{R}} = \overline{\mathcal{R}_{AC}}$.

BV controls - Recall: Graph completions

- For **regular** $u \in AC$ one can reparametrize time $t(s) = \varphi_0(s)$ with $\varphi_0 : [0,1] \to [0,T]$, and set $\varphi(s) := u \circ \varphi_0(s)$.

The SPACE-TIME SYSTEM:

$$\begin{cases} y_0'(s) = \varphi_0'(s), \\ y'(s) = f(y(s), \varphi(s), \psi(s))\varphi_0'(s) + \sum_{\alpha=1}^{m} g_\alpha(y(s))\varphi_\alpha{}'(s), \qquad s \in [0,1]. \\ (y_0, y)(0) = (0, \bar{x}), \end{cases}$$

- For BV controls u, let (φ_0, φ) be a **graph completion** of u : $(\varphi_0, \varphi) : [0,1] \to [0,T] \times U$ Lipschitz continuous such that, $\forall t \in [0,T]$, there exists $s \in [0,1]$ verifying $(t, u(t)) = (\varphi_0, \varphi)(s)$.

BV controls - Recall: Graph completions

- For **regular** $u \in AC$ one can reparametrize time $t(s) = \varphi_0(s)$ with $\varphi_0 : [0,1] \to [0,T]$, and set $\varphi(s) := u \circ \varphi_0(s)$.

The SPACE-TIME SYSTEM:

$$\begin{cases} y_0'(s) = \varphi_0'(s), \\ y'(s) = f(y(s), \varphi(s), \psi(s))\varphi_0'(s) + \displaystyle\sum_{\alpha=1}^{m} g_\alpha(y(s)){\varphi_\alpha}'(s), \qquad s \in [0,1]. \\ (y_0, y)(0) = (0, \bar{x}), \end{cases}$$

- For BV controls u, let (φ_0, φ) be a **graph completion** of u :

$(\varphi_0, \varphi) : [0,1] \to [0,T] \times U$ Lipschitz continuous such that,
$\forall t \in [0,T]$, there exists $s \in [0,1]$ verifying $(t, u(t)) = (\varphi_0, \varphi)(s)$.

Recall: Graph completion solutions

Given $(\varphi_0, \varphi) : [0,1] \to [0,T] \times U$, let y be the solution of the SPACE-TIME system

Graph completion solution: (possibly) set-valued map $x : [0,T] \rightrightarrows \boldsymbol{R}^n$,

$$t \Longmapsto x(t) := y \circ \varphi_0^{-1}(t).$$

Single-valued graph completion solution:
Let $\sigma : [0,T] \to [0,1]$ be a right-inverse of φ_0 having $u(t) = \varphi \circ \sigma(t)$,

$$x : [0,T] \to \boldsymbol{R}^n, \quad x(t) := y \circ \sigma(t), \quad \text{for all } t \in [0,T].$$

Recall: Graph completion solutions

Given $(\varphi_0, \varphi) : [0, 1] \to [0, T] \times U$, let y be the solution of the SPACE-TIME system

Graph completion solution: (possibly) set-valued map $x : [0, T] \rightrightarrows \boldsymbol{R}^n$,

$$t \Longmapsto x(t) := y \circ \varphi_0^{-1}(t).$$

Single-valued graph completion solution:
Let $\sigma : [0, T] \to [0, 1]$ be a right-inverse of φ_0 having $u(t) = \varphi \circ \sigma(t)$,

$$x : [0, T] \to \boldsymbol{R}^n, \quad x(t) := y \circ \sigma(t), \quad \text{for all } t \in [0, T].$$

Optimal control problems with bounded variation controls

(P^K_{gc}) $\inf \psi((x,u)(T)) \, : \, (u,v) \in BV^K \times L^1, \; u(0) = \bar{u}, \; x \in \Sigma^{K+T}_{gc}[\bar{x}, u, v],$

Recall: A simple limit solution $x : [0,T] \to R^n$ is a BV-SIMPLE limit solution if (u^τ_k) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

(P^K_{BVS}) $\inf \psi((x,u)(T)) \, : \, (u,v) \in BV^K \times L^1, \; u(0) = \bar{u}, \; x \in \Sigma^{K}_{BVS}[\bar{x}, u, v],$

From [Aronna & Rampazzo, 2013] (Rampazzo's presentation) we know that

$$\Sigma^{K+T}_{gc}[\bar{x}, u, v] = \Sigma^{K}_{BVS}[\bar{x}, u, v]$$

Theorem

For every initial condition $(\bar{x}, \bar{u}) \in R^n \times U$, one has

$$\lim_{K\to\infty} Val(P^K_{gc})(\bar{x}, \bar{u}) = \lim_{K\to\infty} Val(P^K_{BVS})(\bar{x}, \bar{u}) = V(\bar{x}, \bar{u}).$$

Optimal control problems with bounded variation controls

(P^K_{gc}) $\inf \psi((x,u)(T)) : (u,v) \in BV^K \times L^1,\ u(0) = \bar{u},\ x \in \Sigma^{K+T}_{gc}[\bar{x},u,v]$,

Recall: A simple limit solution $x : [0,T] \to \boldsymbol{R}^n$ is a BV-SIMPLE limit solution if (u^τ_k) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

(P^K_{BVS}) $\inf \psi((x,u)(T)) : (u,v) \in BV^K \times L^1,\ u(0) = \bar{u},\ x \in \Sigma^K_{BVS}[\bar{x},u,v]$,

From [Aronna & Rampazzo, 2013] (Rampazzo's presentation) we know that

$$\Sigma^{K+T}_{gc}[\bar{x},u,v] = \Sigma^K_{BVS}[\bar{x},u,v]$$

Theorem

For every initial condition $(\bar{x},\bar{u}) \in \boldsymbol{R}^n \times U$, one has

$$\lim_{K\to\infty} Val(P^K_{gc})(\bar{x},\bar{u}) = \lim_{K\to\infty} Val(P^K_{BVS})(\bar{x},\bar{u}) = V(\bar{x},\bar{u}).$$

Optimal control problems with bounded variation controls

(P^K_{gc}) $\inf \psi((x,u)(T)) \, : \, (u,v) \in BV^K \times L^1, \; u(0) = \bar u, \; x \in \Sigma^{K+T}_{gc}[\bar x, u, v]$,

Recall: A simple limit solution $x : [0,T] \to \boldsymbol{R}^n$ is a BV-SIMPLE limit solution if (u^τ_k) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

(P^K_{BVS}) $\inf \psi((x,u)(T)) \, : \, (u,v) \in BV^K \times L^1, \; u(0) = \bar u, \; x \in \Sigma^{K}_{BVS}[\bar x, u, v]$,

From [Aronna & Rampazzo, 2013] (Rampazzo's presentation) we know that

$$\Sigma^{K+T}_{gc}[\bar x, u, v] = \Sigma^{K}_{BVS}[\bar x, u, v]$$

Theorem

For every initial condition $(\bar x, \bar u) \in \boldsymbol{R}^n \times U$, one has

$$\lim_{K\to\infty} Val(P^K_{gc})(\bar x, \bar u) = \lim_{K\to\infty} Val(P^K_{BVS})(\bar x, \bar u) = V(\bar x, \bar u).$$

Optimal control problems with bounded variation controls

(P^K_{gc}) $\inf \psi((x,u)(T)) \,:\, (u,v) \in BV^K \times L^1,\ u(0) = \bar{u},\ x \in \Sigma^{K+T}_{gc}[\bar{x}, u, v],$

Recall: A simple limit solution $x : [0, T] \to \boldsymbol{R}^n$ is a BV-SIMPLE limit solution if (u^{τ}_k) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

(P^K_{BVS}) $\inf \psi((x,u)(T)) \,:\, (u,v) \in BV^K \times L^1,\ u(0) = \bar{u},\ x \in \Sigma^{K}_{BVS}[\bar{x}, u, v],$

From [Aronna & Rampazzo, 2013] (Rampazzo's presentation) we know that

$$\Sigma^{K+T}_{gc}[\bar{x}, u, v] = \Sigma^{K}_{BVS}[\bar{x}, u, v]$$

Theorem

For every initial condition $(\bar{x}, \bar{u}) \in \boldsymbol{R}^n \times U$, one has

$$\lim_{K\to\infty} Val(P^K_{gc})(\bar{x}, \bar{u}) = \lim_{K\to\infty} Val(P^K_{BVS})(\bar{x}, \bar{u}) = V(\bar{x}, \bar{u}).$$

Optimal control problems with bounded variation controls

(P^K_{gc}) $\inf \psi((x,u)(T))\,:\,(u,v)\in BV^K\times L^1,\ u(0)=\bar u,\ x\in \Sigma^{K+T}_{gc}[\bar x,u,v],$

Recall: A simple limit solution $x:[0,T]\to \boldsymbol{R}^n$ is a BV-SIMPLE limit solution if (u^τ_k) can be chosen independently of τ and the approximating inputs u_k have equibounded variation.

(P^K_{BVS}) $\inf \psi((x,u)(T))\,:\,(u,v)\in BV^K\times L^1,\ u(0)=\bar u,\ x\in \Sigma^{K}_{BVS}[\bar x,u,v],$

From [Aronna & Rampazzo, 2013] (Rampazzo's presentation) we know that

$$\Sigma^{K+T}_{gc}[\bar x,u,v]=\Sigma^{K}_{BVS}[\bar x,u,v]$$

Theorem

For every initial condition $(\bar x,\bar u)\in \boldsymbol{R}^n\times U$, one has

$$\lim_{K\to\infty} Val(P^K_{gc})(\bar x,\bar u)=\lim_{K\to\infty} Val(P^K_{BVS})(\bar x,\bar u)=V(\bar x,\bar u).$$

The problem with final constraints

Let $\mathcal{S} \subset \boldsymbol{R}^n \times U$ be a closed subset. Define the problems

$$(P^c) \ \inf \psi\big((x,u)(T)\big) : (u,v) \in \mathcal{L}^1 \times L^1,\ u(0) = \bar{u}, \\ x \in \Sigma[\bar{x},u,v],\ (x,u)(T) \in \mathcal{S},$$

$$(P^c_{AC}) \ \inf \psi\big((x,u)(T)\big) : (u,v) \in AC \times L^1,\ u(0) = \bar{u}, \\ x = x[\bar{x},u,v],\ (x,u)(T) \in \mathcal{S}.$$

The problem with final constraints

Let $\mathcal{S} \subset \boldsymbol{R}^n \times U$ be a closed subset. Define the problems

$$(P^c)\ \inf \psi\big((x,u)(T)\big) : (u,v) \in \mathcal{L}^1 \times L^1,\ u(0) = \bar{u},$$
$$x \in \Sigma[\bar{x}, u, v],\ (x,u)(T) \in \mathcal{S},$$

$$(P^c_{AC})\ \inf \psi\big((x,u)(T)\big) : (u,v) \in AC \times L^1,\ u(0) = \bar{u},$$
$$x = x[\bar{x}, u, v],\ (x,u)(T) \in \mathcal{S}.$$

Final constraints $\mathcal{S} \subset R^n \times U$

$$
\begin{aligned}
&\inf\ x_2(1),\\
&\qquad \dot{x}_1(t) = \dot{u}(t),\\
&\qquad \dot{x}_2(t) = |x_1(t)|,\\
&\qquad (x_1,\ x_2)(0) = (1,0), \quad u(0) = 1\\
&\qquad (x_1, x_2, u)(1) \in \mathcal{S} := \Big(\{(0,0)\} \cup (\boldsymbol{R} \times [1,+\infty[)\Big) \times [0,1],\\
&\qquad u(t) \in U := [-\frac{1}{3}, \frac{4}{3}]
\end{aligned}
$$

For every input $u \in AC$, one has $x_2(1) = \int_0^1 |u(s)| \mathrm{d}s > 0$.

Hence, if $(x_1, x_2, u)(1) \in \mathcal{S} \Longrightarrow x_2(1) \geq 1, \Longrightarrow \mathrm{Val}(P^c_{AC}) \geq 1$.

On the other hand, by implementing the impulsive control

$$
u(t) := \begin{cases} 1, & t = 0, \\ 0, & t \in]0,1], \end{cases} \Longrightarrow x_2(1) = 0 \Longrightarrow \mathrm{Val}(P^c) \leq 0 < 1 \leq \mathrm{Val}(P^c_{AC}).
$$

Final constraints $\mathcal{S} \subset R^n \times U$

$$
\begin{aligned}
&\inf\ x_2(1),\\
&\qquad \dot{x}_1(t) = \dot{u}(t),\\
&\qquad \dot{x}_2(t) = |x_1(t)|,\\
&\qquad (x_1,\ x_2)(0) = (1,0), \quad u(0) = 1\\
&\qquad (x_1, x_2, u)(1) \in \mathcal{S} := \Big(\{(0,0)\} \cup (\boldsymbol{R} \times [1, +\infty[)\Big) \times [0,1],\\
&\qquad u(t) \in U := [-\frac{1}{3}, \frac{4}{3}]
\end{aligned}
$$

For every input $u \in AC$, one has $x_2(1) = \int_0^1 |u(s)| \mathrm{d}s > 0$.

Hence, if $(x_1, x_2, u)(1) \in \mathcal{S} \Longrightarrow x_2(1) \geq 1, \Longrightarrow \mathrm{Val}(P^c_{AC}) \geq 1$.

On the other hand, by implementing the impulsive control

$$u(t) := \begin{cases} 1, & t = 0, \\ 0, & t \in]0,1], \end{cases} \Longrightarrow x_2(1) = 0 \Longrightarrow \mathrm{Val}(P^c) \leq 0 < 1 \leq \mathrm{Val}(P^c_{AC}).$$

Final constraints $\mathcal{S} \subset R^n \times U$

$$
\begin{aligned}
&\inf x_2(1),\\
&\qquad \dot x_1(t) = \dot u(t),\\
&\qquad \dot x_2(t) = |x_1(t)|,\\
&\qquad (x_1,\ x_2)(0) = (1,0), \quad u(0) = 1\\
&\qquad (x_1, x_2, u)(1) \in \mathcal{S} := \Big(\{(0,0)\} \cup (\boldsymbol{R} \times [1, +\infty[)\Big) \times [0,1],\\
&\qquad u(t) \in U := [-\frac{1}{3}, \frac{4}{3}]
\end{aligned}
$$

For every input $u \in AC$, one has $x_2(1) = \int_0^1 |u(s)| \mathrm{d}s > 0$.

Hence, if $(x_1, x_2, u)(1) \in \mathcal{S} \Longrightarrow x_2(1) \geq 1, \Longrightarrow \mathrm{Val}(P^c_{AC}) \geq 1$.

On the other hand, by implementing the impulsive control

$$
u(t) := \begin{cases} 1, & t = 0, \\ 0, & t \in]0,1], \end{cases} \Longrightarrow x_2(1) = 0 \Longrightarrow \mathrm{Val}(P^c) \leq 0 < 1 \leq \mathrm{Val}(P^c_{AC}).
$$

Final constraints $\mathcal{S} \subset R^n \times U$

$$
\begin{aligned}
&\inf\ x_2(1),\\
&\qquad \dot x_1(t) = \dot u(t),\\
&\qquad \dot x_2(t) = |x_1(t)|,\\
&\qquad (x_1,\ x_2)(0) = (1,0), \quad u(0) = 1\\
&\qquad (x_1, x_2, u)(1) \in \mathcal{S} := \Big(\{(0,0)\} \cup (\boldsymbol{R} \times [1,+\infty[)\Big) \times [0,1],\\
&\qquad u(t) \in U := [-\frac{1}{3}, \frac{4}{3}]
\end{aligned}
$$

For every input $u \in AC$, one has $x_2(1) = \int_0^1 |u(s)| \mathrm{d}s > 0$.

Hence, if $(x_1, x_2, u)(1) \in \mathcal{S} \Longrightarrow x_2(1) \geq 1, \Longrightarrow \mathrm{Val}(P^c_{AC}) \geq 1$.

On the other hand, by implementing the impulsive control

$$
u(t) := \begin{cases} 1, & t = 0,\\ 0, & t \in]0,1], \end{cases} \Longrightarrow x_2(1) = 0 \Longrightarrow \mathrm{Val}(P^c) \leq 0 < 1 \leq \mathrm{Val}(P^c_{AC}).
$$

Final constraints $\mathcal{S} \subset R^n \times U$

[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

Theorem (A sufficient condition for proper extension in the presence of terminal constraints): Assume that $\mathcal{S}$ is a compact set contained in the interior of $R^n \times U$, and that there exist some positive constants ρ, η such that:

(i) for each $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S},\rho)\setminus\mathcal{S}}$, and $\forall (p_x, p_u) \in D^*\mathrm{d}_{\mathcal{S}}(x,u)$, we have

$$\min_{|w|\le 1}\left\{\left\langle p_x\,,\sum_{\alpha=1}^m g_\alpha(x,u)w^\alpha\right\rangle + \langle p_u, w\rangle\right\} < -\eta;$$

Limiting gradient: Let $\Omega \subset R^k$ be an open set, $F:\Omega\to R$ be locally Lipschitz continuous. For $x \in \Omega$ define

$$D^*F(y) := \{w \in R^k : w = \lim \nabla F(y_k),\ y_k \in DIFF\setminus\{y\},\ \lim y_k = y\}.$$

(ii) + viability.

Then (P^c) is a proper extension of (P^c_{AC}).

Final constraints $\mathcal{S} \subset R^n \times U$

[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

Theorem (A sufficient condition for proper extension in the presence of terminal constraints): Assume that $\mathcal{S}$ is a compact set contained in the interior of $R^n \times U$, and that there exist some positive constants ρ, η such that:

(i) for each $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S},\rho)\backslash\mathcal{S}}$, and $\forall(p_x, p_u) \in D^*\mathrm{d}_\mathcal{S}(x,u)$, we have

$$\min_{|w|\le 1}\left\{\left\langle p_x\,,\,\sum_{\alpha=1}^{m} g_\alpha(x,u)w^\alpha\right\rangle + \langle p_u, w\rangle\right\} < -\eta;$$

Limiting gradient: Let $\Omega \subset R^k$ be an open set, $F : \Omega \to R$ be locally Lipschitz continuous. For $x \in \Omega$ define

$$D^*F(y) := \{w \in R^k : w = \lim \nabla F(y_k),\ y_k \in DIFF \backslash\{y\},\ \lim y_k = y\}.$$

(ii) + viability.

Then (P^c) is a proper extension of (P^c_{AC}).

Final constraints $\mathcal{S} \subset \mathbb{R}^n \times U$

[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

Theorem (A sufficient condition for proper extension in the presence of terminal constraints): Assume that $\mathcal{S}$ is a compact set contained in the interior of $\mathbb{R}^n \times U$, and that there exist some positive constants ρ, η such that:

(i) for each $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S},\rho)\backslash\mathcal{S}}$, and $\forall (p_x, p_u) \in D^* \mathrm{d}_{\mathcal{S}}(x,u)$, we have

$$\min_{|w|\le 1} \left\{ \left\langle p_x \, , \, \sum_{\alpha=1}^{m} g_\alpha(x,u) w^\alpha \right\rangle + \langle p_u, w \rangle \right\} < -\eta;$$

Limiting gradient: Let $\Omega \subset \mathbb{R}^k$ be an open set, $F : \Omega \to \mathbb{R}$ be locally Lipschitz continuous. For $x \in \Omega$ define

$$D^* F(y) := \{ w \in \mathbb{R}^k : w = \lim \nabla F(y_k),\ y_k \in DIFF \setminus \{y\},\ \lim y_k = y \}.$$

(ii) + viability.

Then (P^c) is a proper extension of (P^c_{AC}).

Final constraints $\mathcal{S} \subset \boldsymbol{R}^n \times U$

[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

Theorem (A sufficient condition for proper extension in the presence of terminal constraints): Assume that $\mathcal{S}$ is a compact set contained in the interior of $\boldsymbol{R}^n \times U$, and that there exist some positive constants ρ, η such that:

(i) for each $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S},\rho)\backslash\mathcal{S}}$, and $\forall (p_x, p_u) \in D^* \mathrm{d}_{\mathcal{S}}(x,u)$, we have

$$\min_{|w|\leq 1} \left\{ \left\langle p_x \, , \sum_{\alpha=1}^{m} g_\alpha(x,u) w^\alpha \right\rangle + \langle p_u, w \rangle \right\} < -\eta;$$

Limiting gradient: Let $\Omega \subset \boldsymbol{R}^k$ be an open set, $F : \Omega \to \boldsymbol{R}$ be locally Lipschitz continuous. For $x \in \Omega$ define

$$D^* F(y) := \{ w \in \boldsymbol{R}^k : w = \lim \nabla F(y_k), \; y_k \in DIFF \setminus \{y\}, \; \lim y_k = y \}.$$

(ii) + viability.

Then (P^c) is a proper extension of (P^c_{AC}).

Final constraints $\mathcal{S} \subset \mathbf{R}^n \times U$

[M.S. Aronna, M. Motta & F. Rampazzo, 2014]

Theorem (A sufficient condition for proper extension in the presence of terminal constraints): Assume that $\mathcal{S}$ is a compact set contained in the interior of $\mathbf{R}^n \times U$, and that there exist some positive constants ρ, η such that:

(i) for each $x \in \mathcal{S}_\rho \doteq \overline{B(\mathcal{S}, \rho) \backslash \mathcal{S}}$, and $\forall (p_x, p_u) \in D^* \mathrm{d}_{\mathcal{S}}(x, u)$, we have

$$\min_{|w| \leq 1} \left\{ \left\langle p_x \, , \, \sum_{\alpha=1}^{m} g_\alpha(x, u) w^\alpha \right\rangle + \langle p_u, w \rangle \right\} < -\eta;$$

Limiting gradient: Let $\Omega \subset \mathbf{R}^k$ be an open set, $F : \Omega \to \mathbf{R}$ be locally Lipschitz continuous. For $x \in \Omega$ define

$$D^* F(y) := \{ w \in \mathbf{R}^k : w = \lim \nabla F(y_k), \; y_k \in DIFF \backslash \{y\}, \; \lim y_k = y \}.$$

(ii) + viability.

Then (P^c) is a proper extension of (P^c_{AC}).

Sketch of the proof

Let us consider $(u, v) \in \mathcal{L}^1 \times L^1$ and an associated limit solution x feasible for the problem (P^c), i.e. having

$$(x, u)(T) \in \mathcal{S}.$$

By definition of limit solution, there exists a sequence $(\hat{u}_k) \subset AC$ such that $\hat{u}_k(0) = \bar{u}$ and

$$\|(\hat{x}_k, \hat{u}_k) - (x, u)\|_1 + |(\hat{x}_k, \hat{u}_k)(T) - (x, u)(T)| \to 0 \text{ as } k \to +\infty,$$

where $\hat{x}_k \doteq x[\bar{x}, \hat{u}_k, v]$ are the corresponding Carathéodory solutions.

Sketch of the proof

Let us consider $(u, v) \in \mathcal{L}^1 \times L^1$ and an associated limit solution x feasible for the problem (P^c), i.e. having

$$(x, u)(T) \in \mathcal{S}.$$

By definition of limit solution, there exists a sequence $(\hat{u}_k) \subset AC$ such that $\hat{u}_k(0) = \bar{u}$ and

$$\|(\hat{x}_k, \hat{u}_k) - (x, u)\|_1 + |(\hat{x}_k, \hat{u}_k)(T) - (x, u)(T)| \to 0 \text{ as } k \to +\infty,$$

where $\hat{x}_k \doteq x[\bar{x}, \hat{u}_k, v]$ are the corresponding Carathéodory solutions.

Sketch of the proof

In general,

$$(\hat{x}_k, \hat{u}_k)(T) \notin \mathcal{S}.$$

Step 1. We construct (σ_k) such that

$$\sigma_k \to 0, \quad |(\hat{x}_k, \hat{u}_k)(T - \sigma_k) - (\hat{x}_k, \hat{u}_k)(T)| \to 0.$$

Step 2. We modify $(\hat{x}_k, \hat{u}_k)$ from $t = T - \sigma_k$ in the following way: we apply this estimate for the minimum time problem which holds in view of the quick reachability condition:

$$\mathcal{T}_{\mathcal{S},\eta}(y) \leq \frac{\mathrm{d}_{\mathcal{S}}(y)}{c(\eta)}.$$

See e.g. [Motta & Rampazzo, 2013]
This way we get $(\tilde{x}_k, \tilde{u}_k) : [T - \sigma_k, T_k] \to \mathbb{R}^n \times U$, having

$$(\tilde{x}_k, \tilde{u}_k)(T_k) \in \mathcal{S}.$$

Step 3. If $T_k < T$, then we use the viability to remain in $\mathcal{S}$ until time $t = T$.

Sketch of the proof

In general,

$$(\hat{x}_k, \hat{u}_k)(T) \notin \mathcal{S}.$$

Step 1. We construct (σ_k) such that

$$\sigma_k \to 0, \quad |(\hat{x}_k, \hat{u}_k)(T - \sigma_k) - (\hat{x}_k, \hat{u}_k)(T)| \to 0.$$

Step 2. We modify $(\hat{x}_k, \hat{u}_k)$ from $t = T - \sigma_k$ in the following way: we apply this estimate for the minimum time problem which holds in view of the quick reachability condition:

$$\mathcal{T}_{\mathcal{S},\eta}(y) \leq \frac{\mathrm{d}_{\mathcal{S}}(y)}{c(\eta)}.$$

See e.g. [Motta & Rampazzo, 2013]
This way we get $(\tilde{x}_k, \tilde{u}_k) : [T - \sigma_k, T_k] \to R^n \times U$, having

$$(\tilde{x}_k, \tilde{u}_k)(T_k) \in \mathcal{S}.$$

Step 3. If $T_k < T$, then we use the viability to remain in $\mathcal{S}$ until time $t = T$.

Sketch of the proof

In general,

$$(\hat{x}_k, \hat{u}_k)(T) \notin \mathcal{S}.$$

Step 1. We construct (σ_k) such that

$$\sigma_k \to 0, \quad |(\hat{x}_k, \hat{u}_k)(T - \sigma_k) - (\hat{x}_k, \hat{u}_k)(T)| \to 0.$$

Step 2. We modify $(\hat{x}_k, \hat{u}_k)$ from $t = T - \sigma_k$ in the following way: we apply this estimate for the minimum time problem which holds in view of the quick reachability condition:

$$\mathcal{T}_{\mathcal{S},\eta}(y) \leq \frac{\mathrm{d}_{\mathcal{S}}(y)}{c(\eta)}.$$

See e.g. [Motta & Rampazzo, 2013]
This way we get $(\tilde{x}_k, \tilde{u}_k) : [T - \sigma_k, T_k] \to \boldsymbol{R}^n \times U$, having

$$(\tilde{x}_k, \tilde{u}_k)(T_k) \in \mathcal{S}.$$

Step 3. If $T_k < T$, then we use the viability to remain in $\mathcal{S}$ until time $t = T$.

Sketch of the proof

In general,

$$(\hat{x}_k, \hat{u}_k)(T) \notin \mathcal{S}.$$

Step 1. We construct (σ_k) such that

$$\sigma_k \to 0, \quad |(\hat{x}_k, \hat{u}_k)(T - \sigma_k) - (\hat{x}_k, \hat{u}_k)(T)| \to 0.$$

Step 2. We modify $(\hat{x}_k, \hat{u}_k)$ from $t = T - \sigma_k$ in the following way: we apply this estimate for the minimum time problem which holds in view of the quick reachability condition:

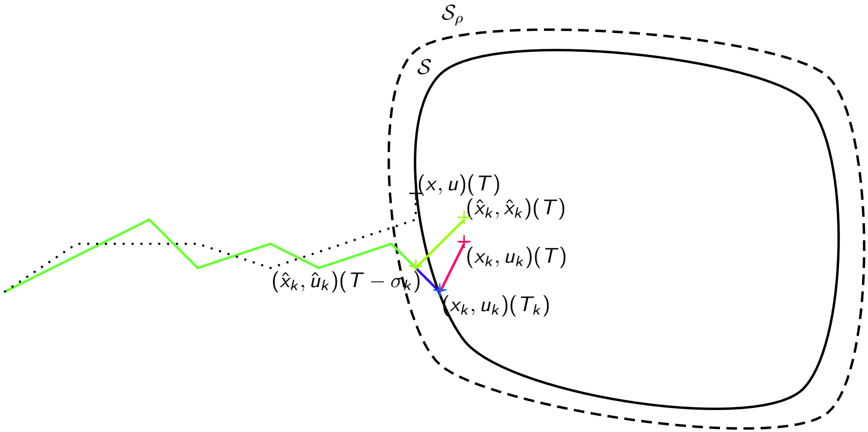
$$\mathcal{T}_{\mathcal{S},\eta}(y) \leq \frac{\mathrm{d}_{\mathcal{S}}(y)}{c(\eta)}.$$

See e.g. [Motta & Rampazzo, 2013]
This way we get $(\tilde{x}_k, \tilde{u}_k) : [T - \sigma_k, T_k] \to \boldsymbol{R}^n \times U$, having

$$(\tilde{x}_k, \tilde{u}_k)(T_k) \in \mathcal{S}.$$

Step 3. If $T_k < T$, then we use the viability to remain in $\mathcal{S}$ until time $t = T$.

Sketch of the proof



■

Concluding remarks

- The limit solutions naturally provide a proper extension of standard optimal control problems with no final constraints.
- The limit solutions optimal control problem (with $\mathcal{L}^1$ controls and trajectories) with no final constraints is the limit of problems with controls of variation K with $K \to \infty$.
- When constraints are considered, a quick reachability condition + viability guarantee that the impulsive problem (with limit solutions as trajectories) provides a proper extension.

Concluding remarks

- The limit solutions naturally provide a proper extension of standard optimal control problems with no final constraints.
- The limit solutions optimal control problem (with $\mathcal{L}^1$ controls and trajectories) with no final constraints is the limit of problems with controls of variation K with $K \to \infty$.
- When constraints are considered, a quick reachability condition + viability guarantee that the impulsive problem (with limit solutions as trajectories) provides a proper extension.

Concluding remarks

- The limit solutions naturally provide a proper extension of standard optimal control problems with no final constraints.
- The limit solutions optimal control problem (with $\mathcal{L}^1$ controls and trajectories) with no final constraints is the limit of problems with controls of variation K with $K \to \infty$.
- When constraints are considered, a quick reachability condition + viability guarantee that the impulsive problem (with limit solutions as trajectories) provides a proper extension.

References

M.S. Aronna & F. Rampazzo. In Proceedings of the 52nd IEEE Conference on Decision and Control, 2013.

M.S. Aronna & F. Rampazzo. $\mathcal{L}^1$ limit solutions for control systems. 2013. Accepted for publication in JDE.

M.S. Aronna & F. Rampazzo. A note on systems with ordinary and impulsive controls. To appear in IMA J. Math. Control Inform, 2014.

M.S. Aronna, M. Motta & F. Rampazzo. Quick reachability and proper extension of unbounded control systems. [In preparation]

THANK YOU FOR YOUR ATTENTION