Differential games with exit costs

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A dynamics-decoupled differential game

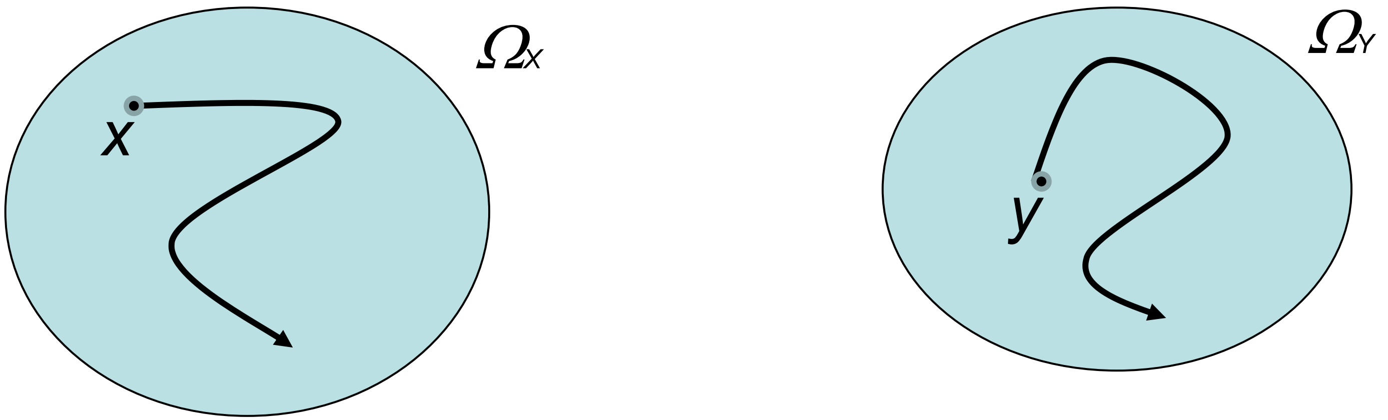
$$
\begin{cases}
X'(t) = f(X(t), \alpha(t)) \quad t > 0 \\
X(0) = x \in \Omega_X \subset \mathbf{R}^n; \\
\\
Y'(t) = g(Y(t), \beta(t)) \quad t > 0 \\
Y(0) = y \in \Omega_Y \subset \mathbf{R}^m
\end{cases}
$$

f, g suitably regular

$\alpha : [0, +\infty[\to A$ measurable control for the first player

$\beta : [0, +\infty[\to B$ measurable control for the second player

Exit time differential games



We run until X or Y exits from a closed subset Ω_X, Ω_Y respectively.

Let τ be the first exit time of the trajectory (X, Y)

from the domain $\Omega_X \times \Omega_Y$

Exit time differential games

$$J(x,y,\alpha,\beta)=\int_0^{\tau} e^{-\lambda s}\ell\big(X(s),Y(s),\alpha(s),\beta(s)\big)dt+e^{-\lambda\tau}\Psi(X(\tau),Y(\tau))$$

ℓ suitably regular running cost, $\lambda>0$, Ψ suitably regular exit cost:

$$\Psi(X,Y)=\begin{cases}\Psi_X(X,Y) \text{ if } X \text{ exits first,}\\ \Psi_{\mathrm{Y}}(X,Y) \text{ if } Y \text{ exits first,}\\ \Psi_{\mathrm{XY}}(X,Y) \text{ if } X \text{ and } Y \text{ simultaneously exit}\end{cases}$$

- A possible reason for considering three different costs will be explained later.
- *X* wants to minimize *J* whereas *Y* wants to maximize

Nonanticipative strategies

$$J(x,y,\alpha,\beta)=\int_0^{\tau} e^{-\lambda s}\ell\big(X(s),Y(s),\alpha(s),\beta(s)\big)ds+e^{-\lambda\tau}\Psi(X(\tau),Y(\tau))$$

γ nonanticipative strategy for X,

ξ nonanticipative strategy for Y,

lower value $\underline{V}(x,y)=\inf_{\gamma}\sup_{\beta} J(x,y,\gamma[\beta],\beta)$,

upper value $\overline{V}(x,y)=\sup_{\xi}\inf_{\alpha} J(x,y,\alpha,\xi[\alpha])$

Nonanticipative strategy for X :

$\gamma:\beta\mapsto\gamma[\beta]$

such that

$\beta_1=\beta_2$ a.e. in $[0,t]\Rightarrow\gamma[\beta_1]=\gamma[\beta_2]$ a.e. in $[0,t]$

We say that the differential game has a value if the lower and the upper values coincide.

Exit time differential games

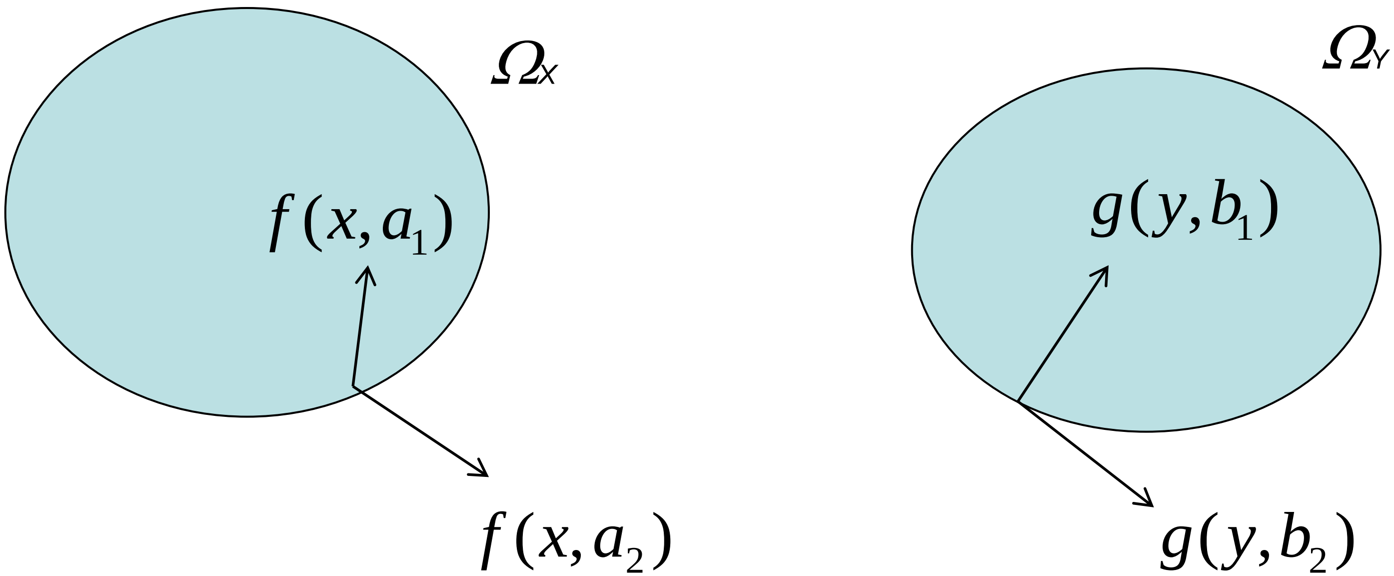
- Goal: use dynamic programming to derive the corresponding Hamilton-Jacobi-Isaacs problems in order to study the properties of the lower and upper values.

In particular, we want that these Isaacs problems are uniquely satisfied by the lower and upper value.

The nature of our particular exit cost will reflect on the boundary conditions.

- Exit time problems have been intensively studied for optimal control problems, but the dynamic programming approach for the corresponding differential game is not yet well studied.
- The result here presented seems to be new.

Controllability conditions



Exit time differential games

- First question: are the lower value and the upper value continuous? (in general they are certainly not differentiable…)
- Besides some classical regularity conditions on dynamics and costs, and controllability hypotheses on the boundaries, a necessary and sufficient condition is

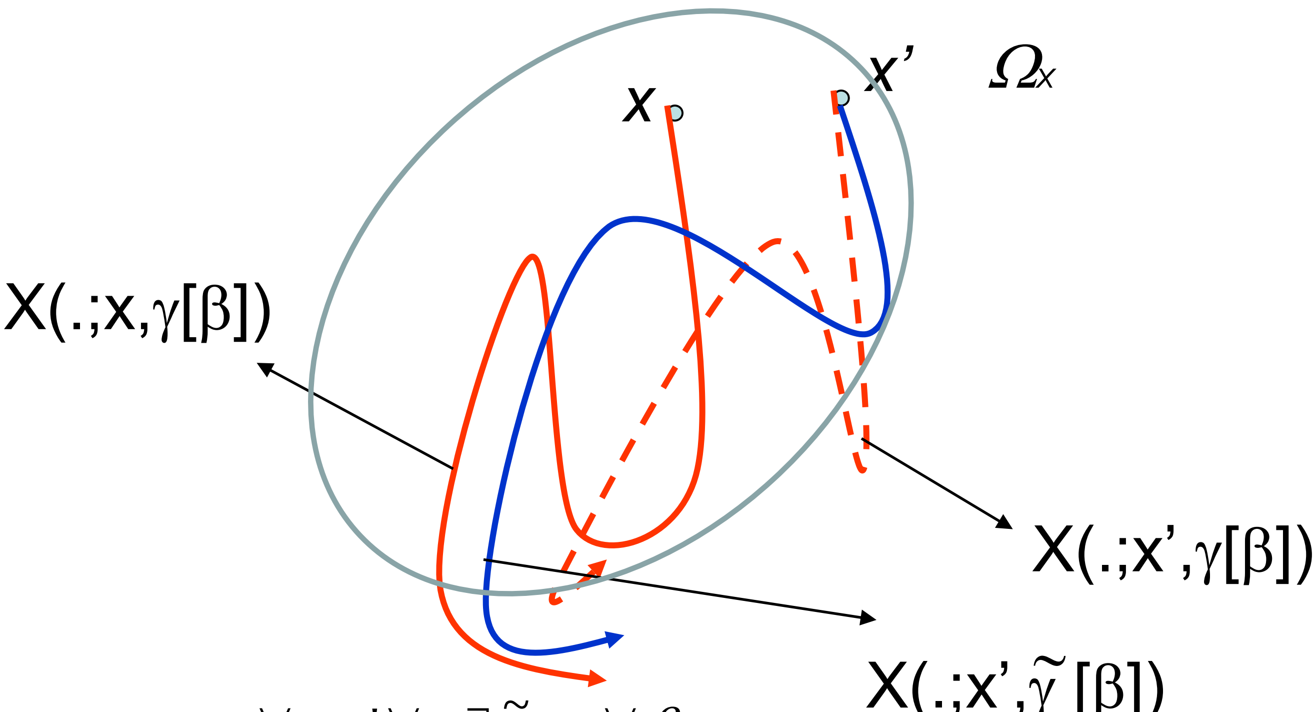
$$\Psi_Y \leq \Psi_{XY} \leq \Psi_X$$

- The *X*-only-exit cost (*X* is the minimizing player) is greater than the *Y*-only-exit costs(*Y* is the maximizing player) and the other cost is in between.

- To prove continuity we also need a careful construction of some suitable nonanticipating strategies which maintain the trajectory inside Ω_X and Ω_Y if necessary.

Continuity of the values

- We need the existence of suitable "exit/constrained" non anticipative strategies.



$\forall\, x, x' \;\forall\, \gamma \;\exists\, \tilde{\gamma}$ s.t. $\forall\, \beta$

the trajectory $X(\cdot; x', \tilde{\gamma}[\beta])$ exits and pays "almost as"

the trajectory $X(\cdot; x, \gamma[\beta])$

On constrained nonanticipative strategies

- The problem of existence and/or construction of suitable constrained non-anticipative strategies has been recently studied, with slightly different goals, by several authors:
- Cardaliaguet-Quincampoix-Saint Pierre, 2001; Bettiol-Cardaliaguet-Quincampoix, 2006; Bettiol-Bressan-Vinter, 2010; Bressan-Facchi, 2011.
- In our case, we can suitably adapt the (first) result by Soner (1986) concerning the construction of constrained optimal control (in an anticipative way…).
- The fact that the dynamics are decoupled seems to play here an essential role.
- Other works on constrained differential games:
  - Koike, 1995;
  - Bardi-Koike-Soravia, 2000.

“Soner” estimate on violation of constraint

$$\varepsilon = \sup_{\alpha}\left(\sup_{0 \le t \le \tau_X(x,\alpha)} dist\left(X(t; x', \alpha), \Omega_X\right) \right)$$

Exit time differential games

- Second question: do the lower and upper values solve (in some suitable sense) a partial differential equation with suitable boundary conditions?
- Third question: if the answer to the second question is YES, are they the unique solutions?

The Hamilton-Jacobi-Isaacs equations

- The lower and the upper values turn out to be the unique viscosity solution of the following (rather new) Dirichlet problems, respectively:

$$\begin{cases} \lambda \underline{V}(x,y) + \inf\limits_{b\in B} \sup\limits_{a\in A} \left\{ -f(x,a)\cdot\nabla_x \underline{V}(x,y) - g(y,b)\cdot\nabla_y \underline{V}(x,y) - \ell(x,y,a,b) \right\} = 0, \\ \underline{V}(x,y) = \Psi_X(x,y) \ \text{ on } \ \partial\Omega_X \times \Omega_Y, \\ \underline{V}(x,y) = \Psi_Y(x,y) \ \text{ on } \Omega_X \times \partial\Omega_Y, \\ \underline{V}(x,y) = \Psi_X(x,y) \ \text{ or } \ \underline{V}(x,y) = \Psi_Y(x,y) \ \text{ on } \ \partial\Omega_X \times \partial\Omega_Y. \end{cases}$$

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- As usual, the boundary conditions have to be also interpreted in the viscosity sense, whose meaning, roughly speaking, is: if the boundary condition is not exactly satisfied at a point of the boundary, then in that point the Isaacs equation holds.

- As it is known, the notion of viscosity solutions requires the treatment of the concepts of sub- and super- solutions.
- For sub-solutions, the boundary conditions

$$u(x,y) = \Psi_X(x,y) \text{ or } u(x,y) = \Psi_Y(x,y) \text{ on } \partial\Omega_X \times \partial\Omega_Y$$

means that: "if $\Psi_Y < u \neq \Psi_X$ then the Isaacs equation holds".

Example of double variable

$$u(x_0, y_0) > v(x_0, y_0)$$

$$(x_0, y_0) \in \partial\Omega_X \times \partial\Omega_Y$$

$$v(x_0, y_0) = \Psi_Y(x_0, y_0) < \Psi_X(x_0, y_0) = u(x_0, y_0)$$

$$\begin{aligned}
&\phi\big((x_1, y_1), (x_2, y_2)\big) = u(x_1, y_1) - v(x_2, y_2) \\
&- \left\| \frac{x_1 - x_2}{\varepsilon} - \eta_X(x_0) \right\|^2 - \left\| x_2 - x_0 \right\|^2 - \left\| \frac{y_2 - y_1}{\varepsilon} - \eta_Y(y_0) \right\|^2 - \left\| y_1 - y_0 \right\|^2 \\
&- (\text{other penalizing terms})
\end{aligned}$$

The Hamilton-Jacobi-Isaacs equations

- Note that the simultaneous exit cost Ψ_{XY} turns out to not play any role in the boundary conditions.
- This is due to our hypothesis on the exit costs and to the particular interpretation of the boundary conditions: if the simultaneous exit is a good choice for both players (an equilibrium) then the cost turns out to be equal to the paid cost supposing that both players keep running inside the sets.

$$V_{run} < \Psi_Y < \Psi_{XY} < \Psi_X$$

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$$\Psi_{XY} \text{ best choice for both players} \Rightarrow \Psi_{XY} = V_{run}$$

- And hence we can still consider the partial differential equation instead of that boundary condition.

- By the way, even in a classical static/strategic min/max game where two players may independently choose to “stay” or to “exit” and the first player wants to minimize, if the utility *u(exit,exit)* is between the utilities *u(stay,exit)≤u(exit,stay),* then the choice *(exit,exit)* is never a Nash (min/max) equilibrium, whichever *u(stay,stay)* is.

Existence of a value

$$\begin{cases} \lambda \underline{V}(x,y) + \inf_{b\in B} \sup_{a\in A}\left\{ - f(x,a)\cdot \nabla_x \underline{V}(x,y) - g(y,b)\cdot \nabla_y \underline{V}(x,y) - \ell(x,y,a,b)\right\} = 0, \\ \\ \underline{V}(x,y) = \Psi_X(x,y) \ \text{ on } \ \partial\Omega_X \times \Omega_Y, \\ \underline{V}(x,y) = \Psi_Y(x,y) \ \text{ on } \Omega_{\mathrm{X}} \times \partial\Omega_Y, \\ \underline{V}(x,y) = \Psi_X(x,y) \ \text{ or } \ \underline{V}(x,y) = \Psi_Y(x,y) \ \text{ on } \ \partial\Omega_X \times \partial\Omega_Y. \end{cases}$$

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- If these two problems are the same problems, then, by uniqueness, the lower and the upper value coincide.

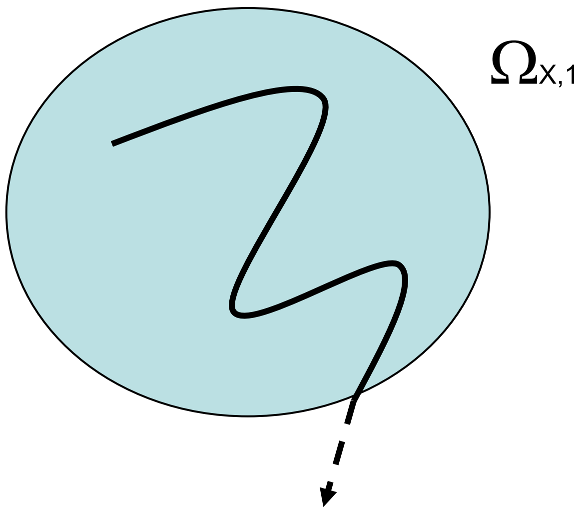
Existence of a value

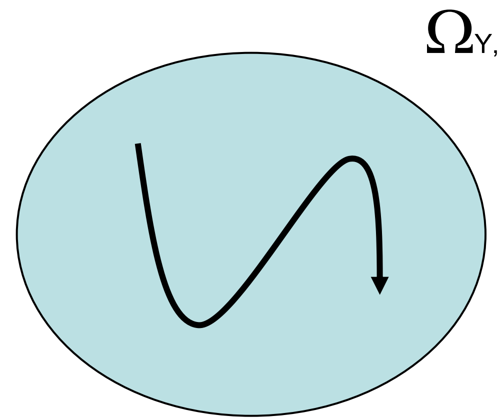
- We then recover the classical Isaacs condition

$$\inf_{b\in B}\sup_{a\in A}\left\{-f(x,a)\cdot p-g(y,b)\cdot q-\ell(x,y,a,b)\right\}=\sup_{a\in A}\inf_{b\in B}\left\{-f(x,a)\cdot p-g(y,b)\cdot q-\ell(x,y,a,b)\right\}$$

for all $x, y, p, q.$

A motivation: switching thermostatic differential game

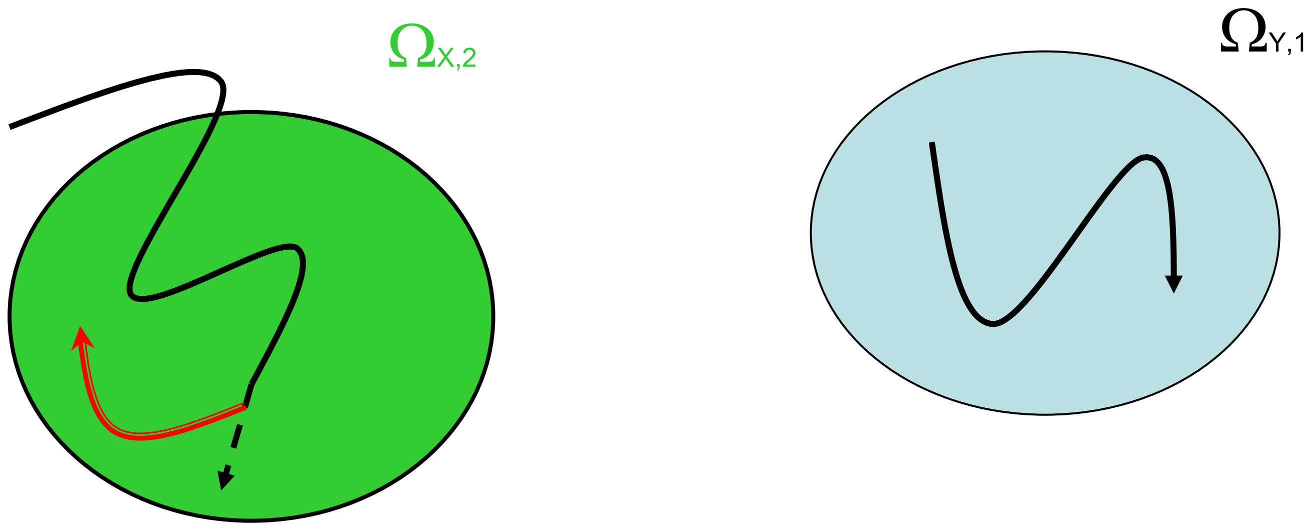




$X' = f_1(X, \alpha)$ $\qquad$ $Y' = g_1(Y, \beta)$

From the blue/blue scenario
we switch on

Switching thermostatic differential games

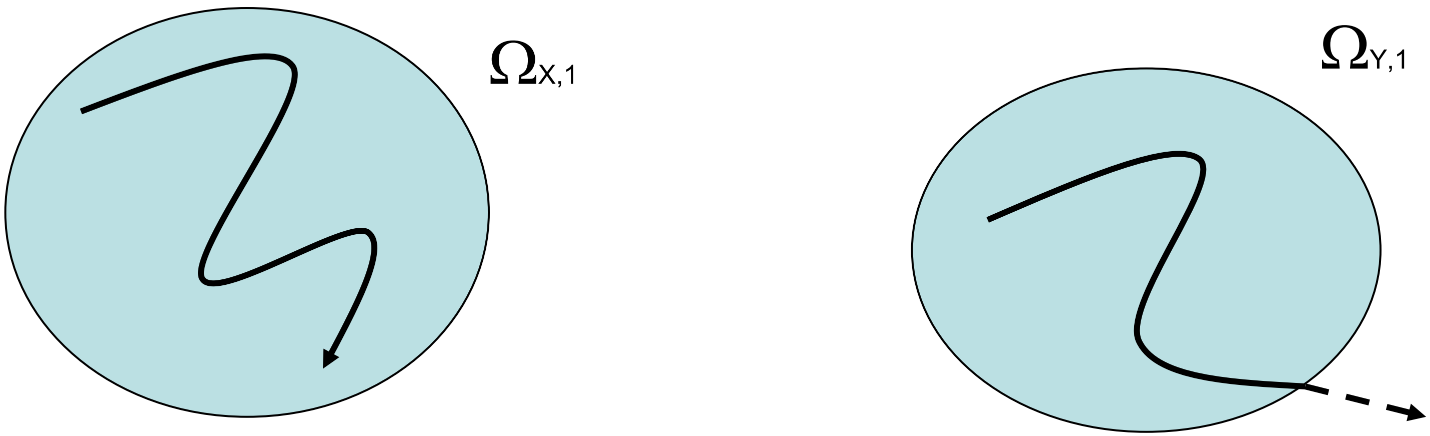


$X'=f_2(X,\alpha)$

$Y'=g_1(Y,\beta)$

the green/blue scenario
and a “new game starts”

Switching thermostatic differential games



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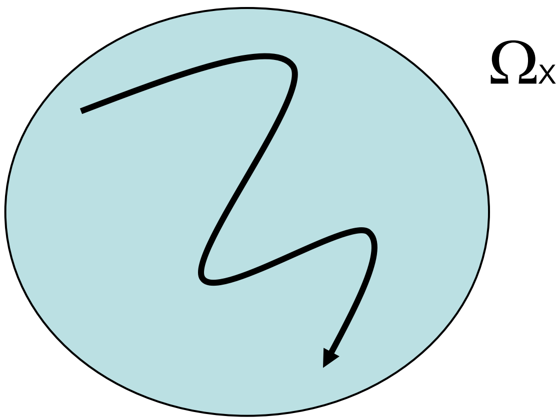

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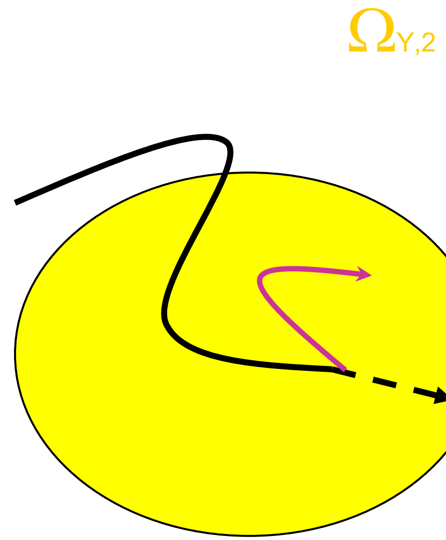
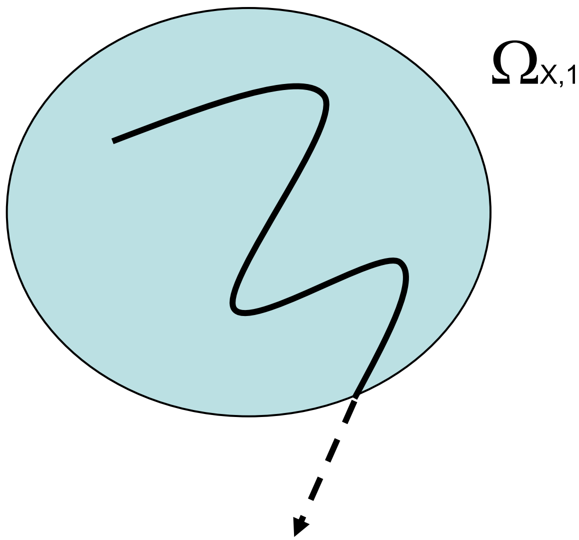
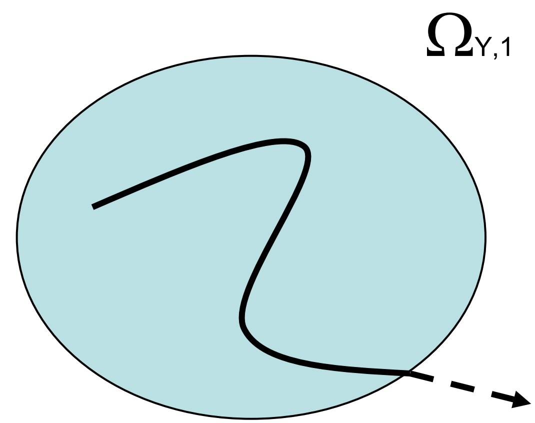


$Y' = g_2(Y, \beta)$

the blue/yellow scenario
and a “new game starts”

Switching thermostatic differential games





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From the blue/blue scenario
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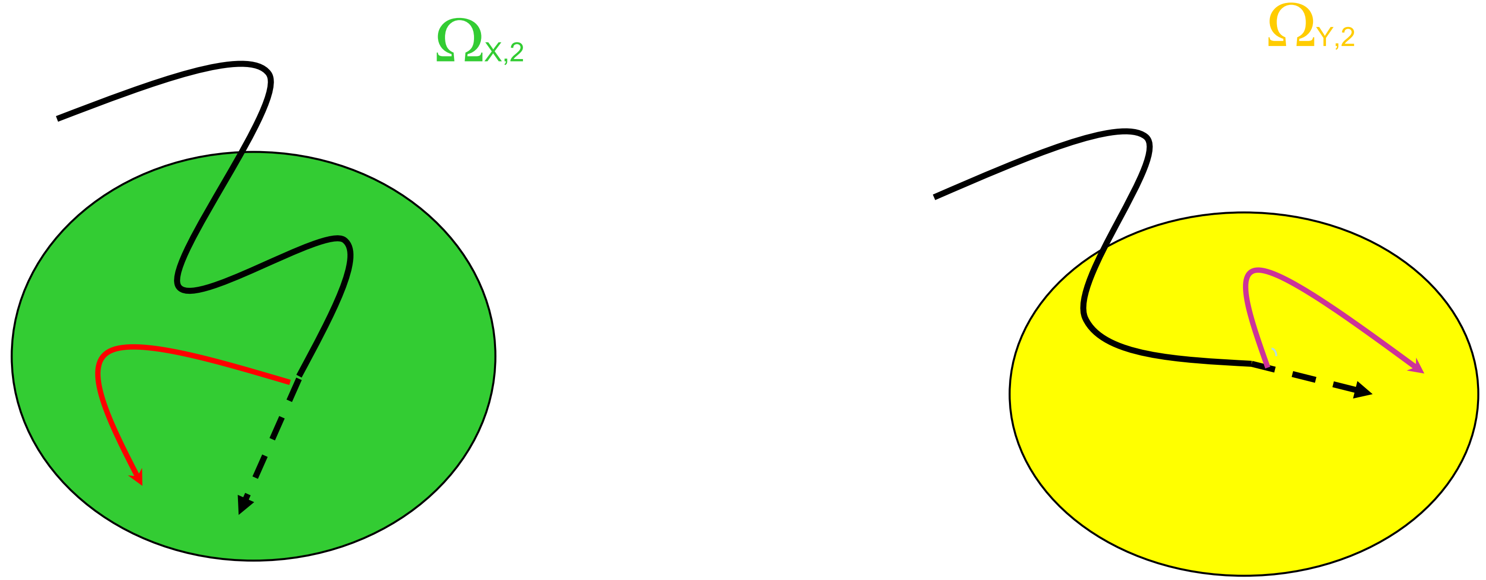
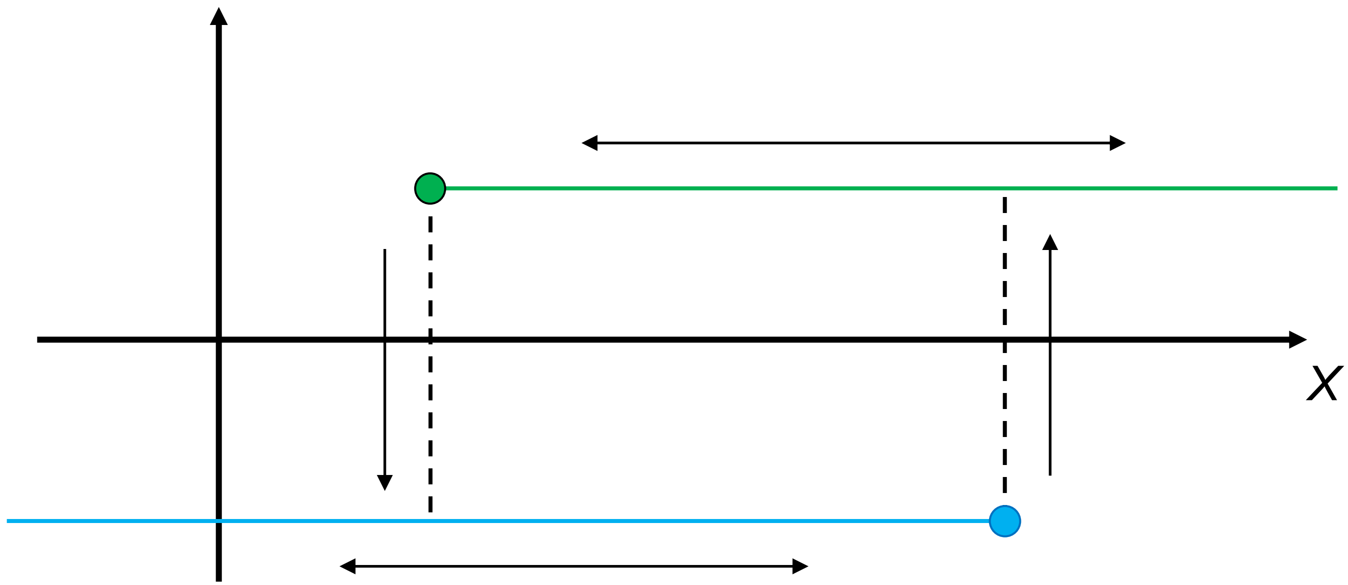
Switching thermostatic differential games

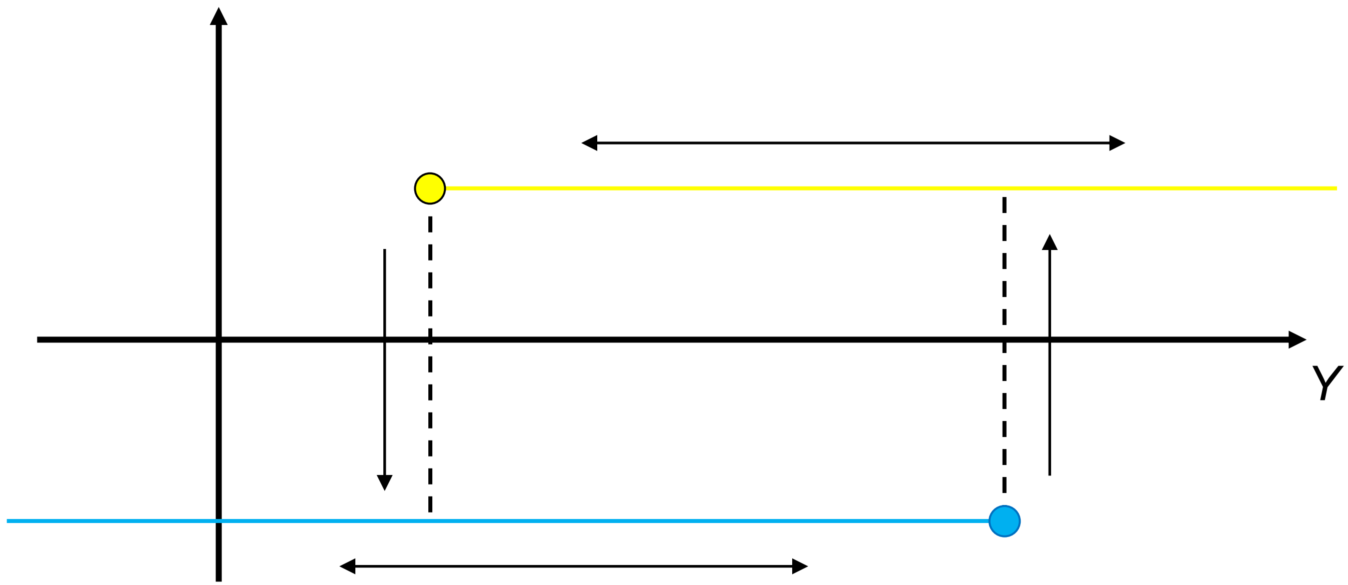


$X'=f_2(X,\alpha)$ the green/yellow scenario and a “new game starts” $Y'=g_2(Y,\beta)$

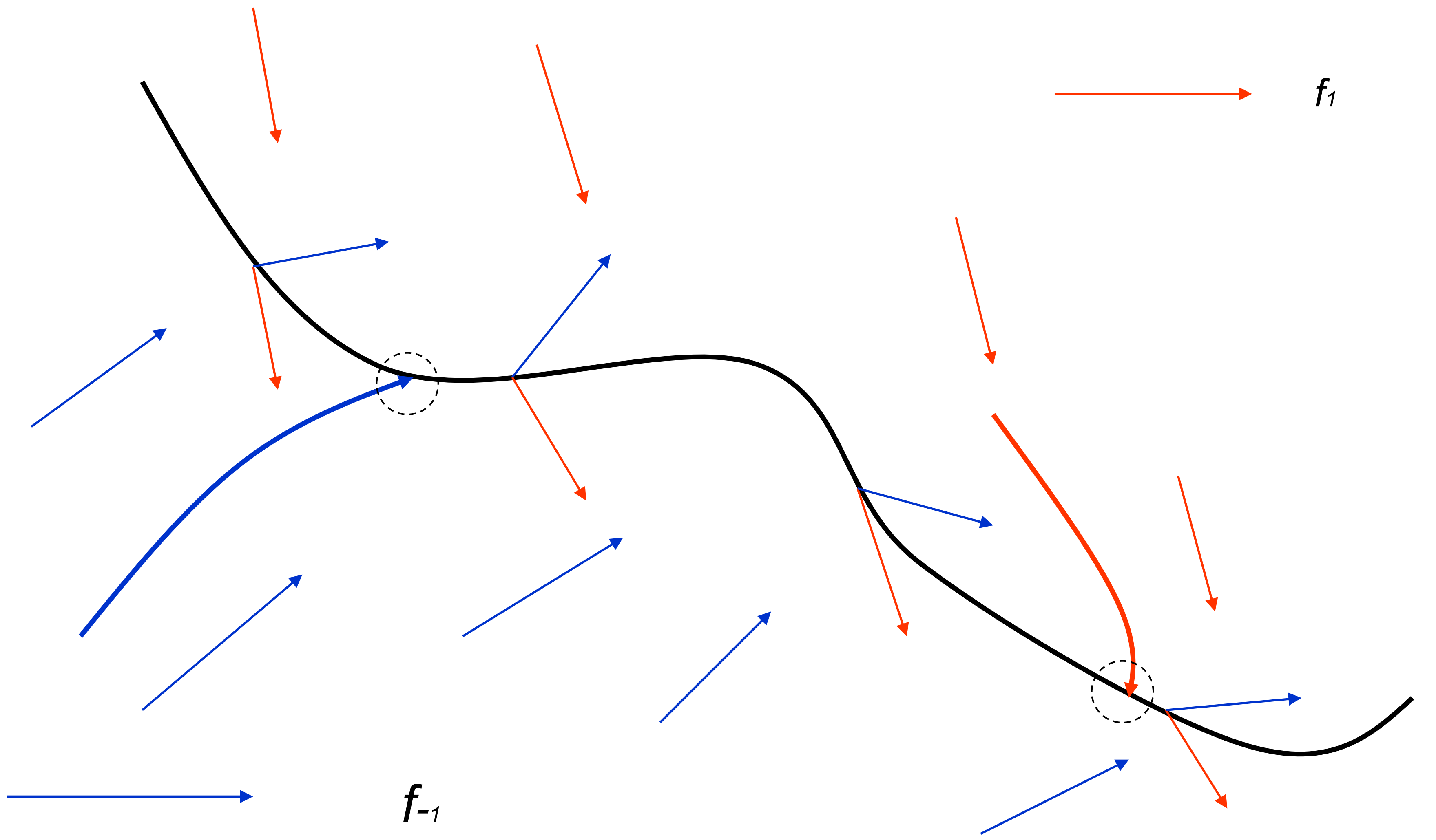
Switching hysteresis (thermostat or delayed relay) for player *X*



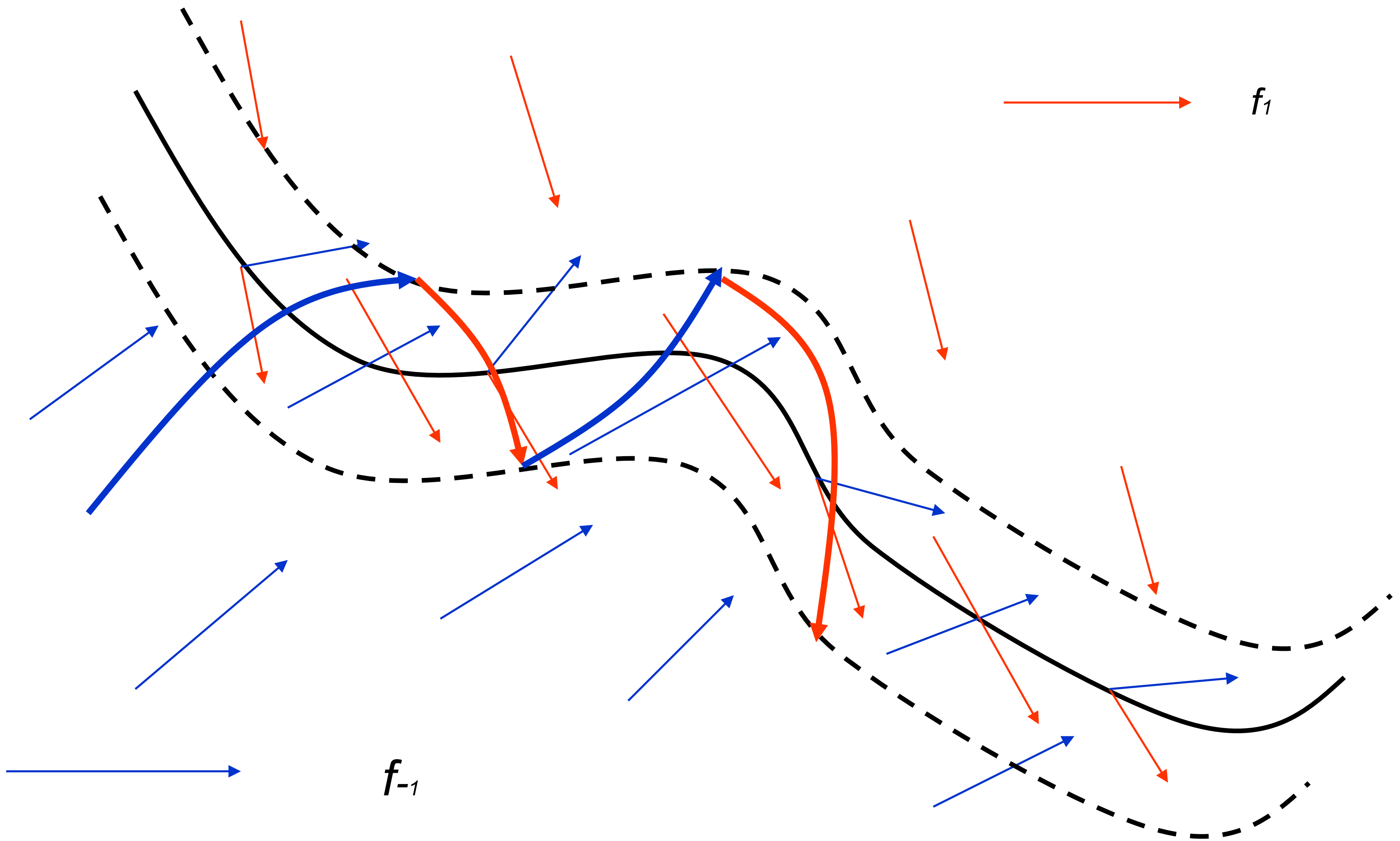
Switching hysteresis (thermostat or delayed relay) for player Y



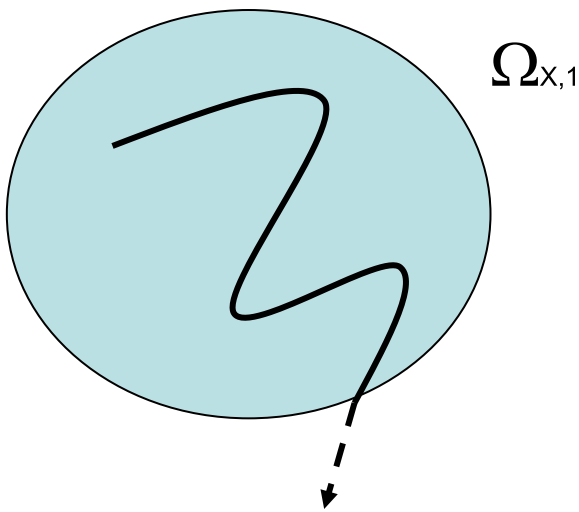
Discontinuous dynamics

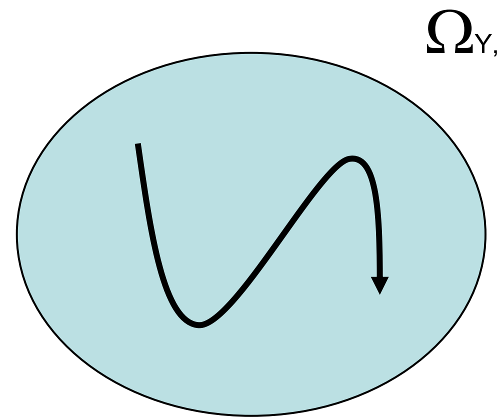


Thermostatic approximation



A motivation: switching thermostatic differential game





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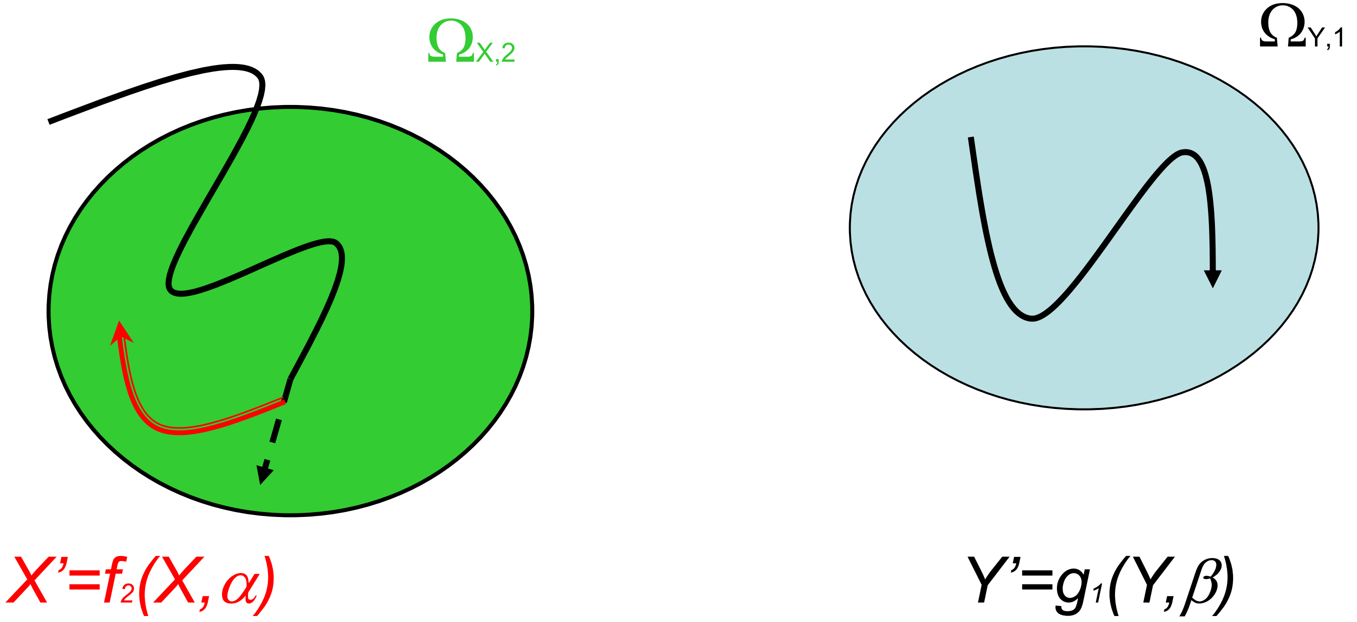
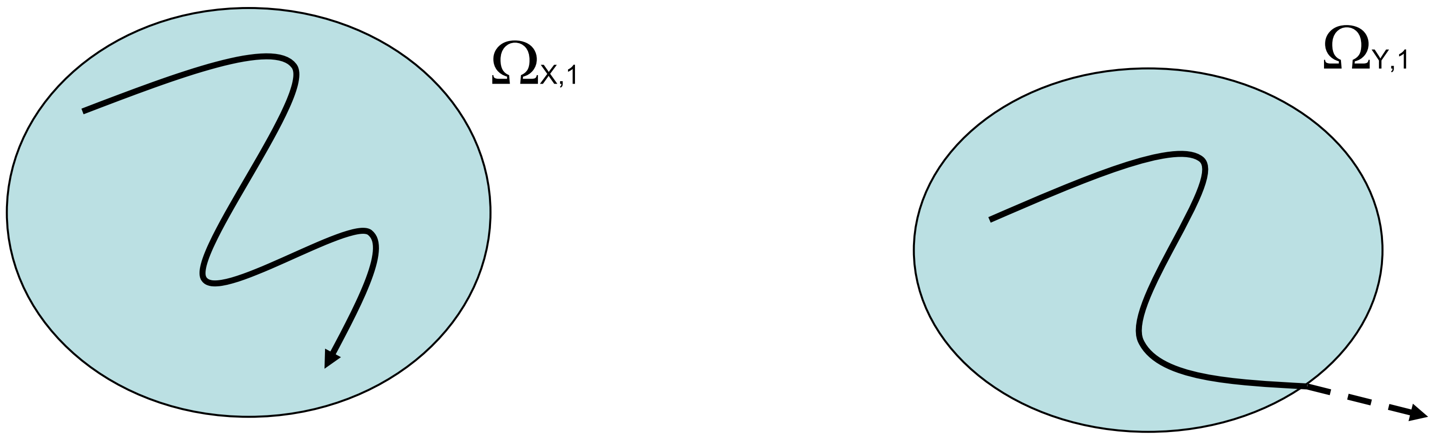
Switching thermostatic differential games



the green/blue scenario
and a "new game starts"

Switching thermostatic differential games



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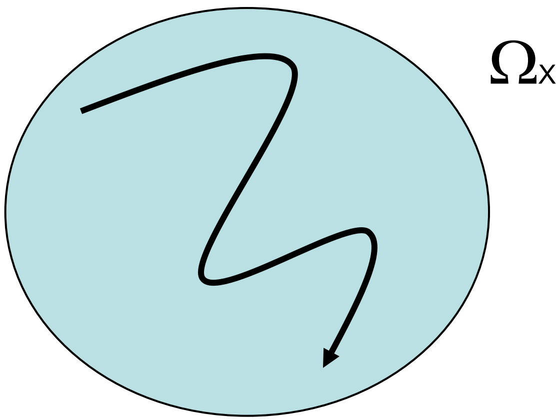

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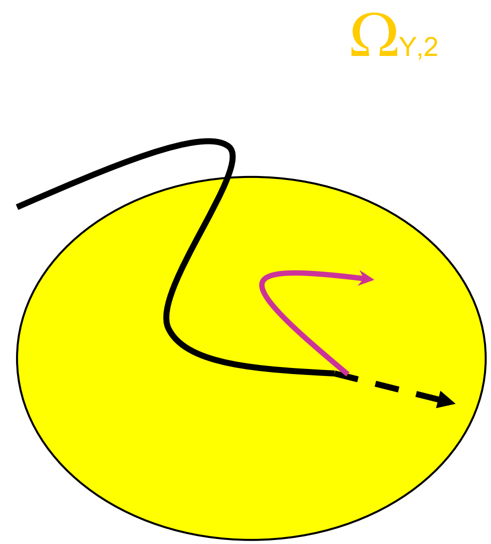
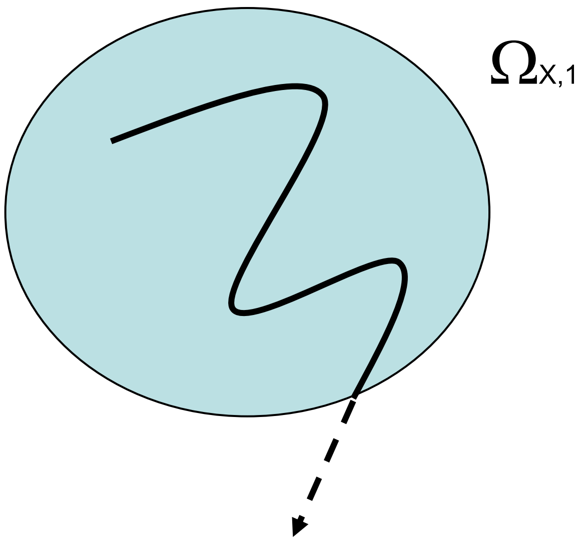
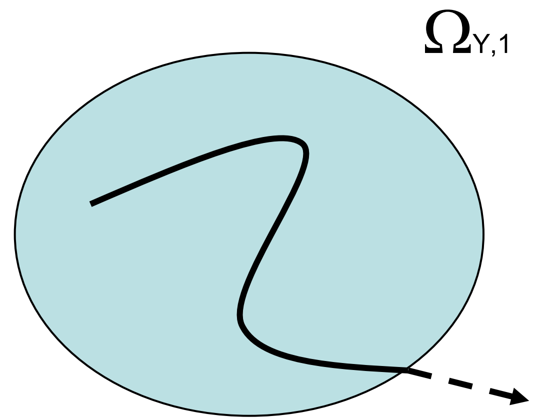


$Y' = g_2(Y, \beta)$

the blue/yellow scenario
and a “new game starts”

Switching thermostatic differential games





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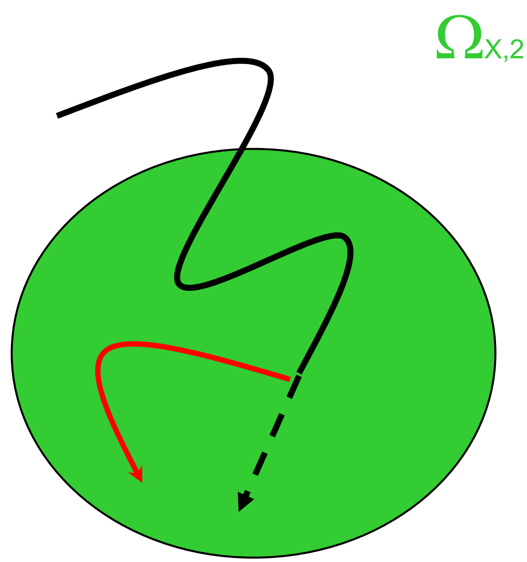

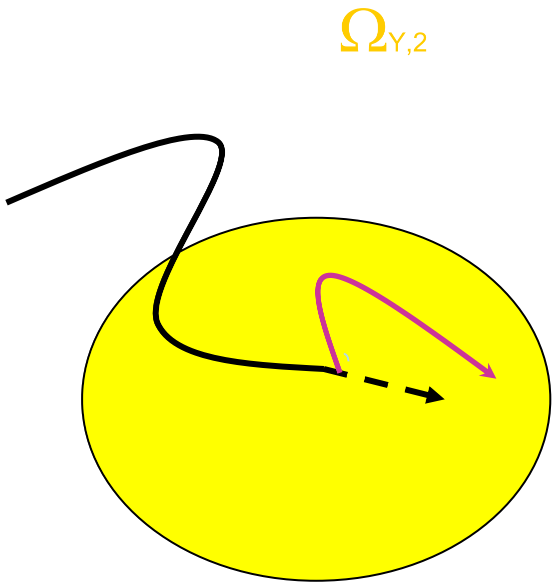

$X' = f_2(X, \alpha)$

the green/yellow scenario
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Switching thermostatic differential games

- We may interpret this problem with switching dynamics as a suitable coupling of problem with exit-time and exit-costs.
- Starting from the blue/blue scenario we run the game (and pay the relative cost) until X or Y hits the boundary (exits from the set). Then we stop running and pay an exit cost which depend on the new scenario (green/blue, blue/yellow or green/yellow).
- Similarly, if we start from another scenario.

- Various kinds of thermostatic optimal control problems already studied via dynamic programming methods (for instance: B.-Danieli, Nonlinear Analysis: Hybrid Systems 2012).
- A related, but different, thermostatic differential game is studied in B.-Bauso, ESAIM COCV 2011)
- Barles-Briani-Chasseigne 2013-2014, Rao-Siconolfi-Zidani, Bressan-Hong 2007
- (Other works on hybrid differential games: Dharmatti-Ramaswamy 2006)
- The results on exit-time differential games here presented are necessary for performing the dynamic programming/Hamilton-Jacobi-Isaacs analysis of thermostatic differential games.
- The latter is still a work in progress.