Viscosity methods for multiscale financial models with stochastic volatility

Martino Bardi, Annalisa Cesaroni, Daria Ghilli, Andrea Scotti

Viscosity methods for multiscale financial models with stochastic volatility

Martino Bardi

joint work with

Annalisa Cesaroni and Daria Ghilli and Andrea Scotti

Department of Mathematics
University of Padua, Italy

NetCO 2014 - Conference on new trends in optimal control
Tours, June 23rd 2014

Plan

Introduction on models

- Financial models and stochastic volatility, Gaussian or with jumps
- Fast stochastic volatility

Part 1

- Control systems with random parameters and multiple scales
- The Hamilton-Jacobi-Bellman approach to Singular Perturbations
  - Tools
  - Assumptions
  - A convergence result
- Applications to finance

Part 2

- Large deviations for small time to maturity:
  see also Daria Ghilli's poster tomorrow

Financial models and stochastic volatility

The evolution of the price of a stock S is described by

$$d \log S_s = \gamma\, ds + \sigma\, dW_s, \qquad s = \text{time}, \;\; W_s = \text{Wiener proc.},$$

and the classical Black-Scholes formula for option pricing and Merton's optimal portfolio are derived assuming the parameters are constants.
In reality the parameters of such models are not constants.
In particular, the volatility σ is not a constant, it rather looks like an ergodic mean-reverting stochastic process, see next slide.

Therefore it has been modeled as $\quad \sigma = \sigma(y_s)$

with y_s either an Ornstein-Uhlenbeck diffusion process,
Refs.: Hull-White 87, Heston 93, Fouque-Papanicolaou-Sircar 2000,...
or by a non-Gaussian ergodic mean-reverting process
Refs.: Barndorff-Nielsen and Shephard 2001.

Financial models and stochastic volatility

The evolution of the price of a stock S is described by

$$d \log S_s = \gamma\, ds + \sigma\, dW_s, \qquad s = \text{time}, \;\; W_s = \text{Wiener proc.},$$

and the classical Black-Scholes formula for option pricing and Merton's optimal portfolio are derived assuming the parameters are constants.
In reality the parameters of such models are not constants.
In particular, the volatility σ is not a constant, it rather looks like an ergodic mean-reverting stochastic process, see next slide.

Therefore it has been modeled as $\quad \sigma = \sigma(y_s)$

with y_s either an Ornstein-Uhlenbeck diffusion process,
Refs.: Hull-White 87, Heston 93, Fouque-Papanicolaou-Sircar 2000,...
or by a non-Gaussian ergodic mean-reverting process
Refs.: Barndorff-Nielsen and Shephard 2001.

Financial models and stochastic volatility

The evolution of the price of a stock S is described by

$$d\log S_s = \gamma\, ds + \sigma\, dW_s, \qquad s = \text{time}, \; W_s = \text{Wiener proc.},$$

and the classical Black-Scholes formula for option pricing and Merton's optimal portfolio are derived assuming the parameters are constants.
In reality the parameters of such models are not constants.
In particular, the volatility σ is not a constant, it rather looks like an ergodic mean-reverting stochastic process, see next slide.

Therefore it has been modeled as $\qquad \sigma = \sigma(y_s)$

with y_s either an Ornstein-Uhlenbeck diffusion process,

Refs.: Hull-White 87, Heston 93, Fouque-Papanicolaou-Sircar 2000,...

or by a non-Gaussian ergodic mean-reverting process

Refs.: Barndorff-Nielsen and Shephard 2001.

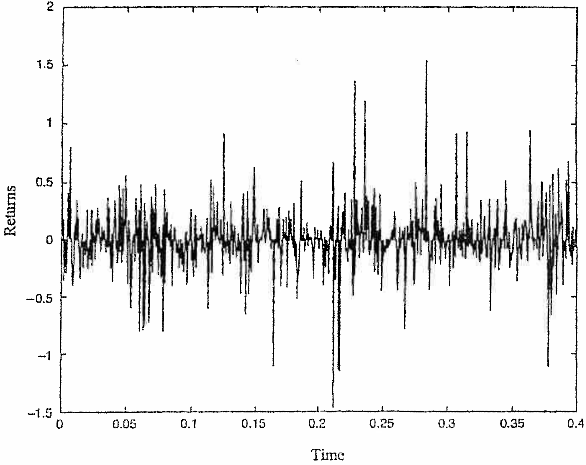

Figure 3.7. 1996 S&P 500 returns computed from half-hourly data.

Diffusion model (Gaussian)

$$dy_s = -y_s\, ds + \tau\, d\tilde{W}_s$$

with $\tilde{W}_s$ a Wiener process possibly correlated with W_s

was used for many papers in finance, see the refs. in the book by Fleming - Soner, 2nd ed., 2006,

for Merton's problem it was studied by Fleming - Hernandez 03.

Non-Gaussian, jump model

$$dy_s = -y_s\, ds + \tau\, dZ_s$$

where Z_s is a pure jump Lévy process with positive increments.

The non-Gaussian model was used for option pricing (Nicolato - Venerdos 03, Hubalek - Sgarra 09, 11)

and for portfolio optimisation by Benth - Karlsen - Reikvam 03.

Diffusion model (Gaussian)

$$d y_s = -y_s\, ds + \tau\, d\tilde{W}_s$$

with $\tilde{W}_s$ a Wiener process possibly correlated with W_s
was used for many papers in finance, see the refs. in the book by Fleming - Soner, 2nd ed., 2006,

for Merton's problem it was studied by Fleming - Hernandez 03.

Non-Gaussian, jump model

$$d y_s = -y_s\, ds + \tau\, dZ_s$$

where Z_s is a pure jump Lévy process with positive increments.

The non-Gaussian model was used for option pricing (Nicolato - Venerdos 03, Hubalek - Sgarra 09, 11)

and for portfolio optimisation by Benth - Karlsen - Reikvam 03.

Fast stochastic volatility

It is argued in the book

Fouque, Papanicolaou, Sircar: Derivatives in financial markets with stochastic volatility, 2000,

that the process y_s also evolves on a faster time scale than the stock prices: this models better the typical bursty behavior of volatility, see previous picture.

The equations for the evolution of a stock S with fast stochastic volatility σ proposed in [FPS] are Gaussian, with $\varepsilon > 0$,

$$d \log S_s = \gamma \, ds + \sigma(y_s) \, dW_s$$

$$dy_s = -\tfrac{1}{\varepsilon} y_s + \tfrac{\tau}{\sqrt{\varepsilon}} d\tilde{W}_s$$

and they study the asymptotics $\varepsilon \to 0$ for many option pricing problems. We'll study also the non-Gaussian volatility

$$dy_s = -\frac{1}{\varepsilon} y_{s^-} ds + dZ_{s/\varepsilon}$$

It is argued in the book

Fouque, Papanicolaou, Sircar: Derivatives in financial markets with stochastic volatility, 2000,

that the process y_s also evolves on a faster time scale than the stock prices: this models better the typical bursty behavior of volatility, see previous picture.

The equations for the evolution of a stock S with fast stochastic volatility σ proposed in [FPS] are Gaussian, with $\varepsilon > 0$,

$$d \log S_s = \gamma\, ds + \sigma(y_s)\, dW_s$$

$$dy_s = -\frac{1}{\varepsilon} y_s + \frac{\tau}{\sqrt{\varepsilon}} d\tilde{W}_s$$

and they study the asymptotics $\varepsilon \to 0$ for many option pricing problems. We'll study also the non-Gaussian volatility

$$dy_s = -\frac{1}{\varepsilon} y_{s^-}\, ds + dZ_{s/\varepsilon}$$

Fast stochastic volatility

It is argued in the book

Fouque, Papanicolaou, Sircar: Derivatives in financial markets with stochastic volatility, 2000,

that the process y_s also evolves on a faster time scale than the stock prices: this models better the typical bursty behavior of volatility, see previous picture.
The equations for the evolution of a stock S with fast stochastic volatility σ proposed in [FPS] are Gaussian, with $\varepsilon > 0$,

$$d \log S_s = \gamma \, ds + \sigma(y_s) \, dW_s$$

$$dy_s = -\frac{1}{\varepsilon} y_s + \frac{\tau}{\sqrt{\varepsilon}} d\tilde{W}_s$$

and they study the asymptotics $\varepsilon \to 0$ for many option pricing problems. We'll study also the non-Gaussian volatility

$$dy_s = -\frac{1}{\varepsilon} y_{s^-} \, ds + dZ_{s/\varepsilon}$$

Two-scale control systems with random parameters

We consider control systems with fast Jump volatility

$$dx_s = f(x_s, y_{s^-}, u_s)\, ds + \sigma(x_s, y_{s^-}, u_s) dW_s \qquad x_s \in \mathbf{R}^n,$$
$$dy_s = -\tfrac{1}{\varepsilon} y_{s^-}\, ds + dZ_{s/\varepsilon} \qquad y_s \in \mathbf{R}$$

Basic assumptions

- f, σ, b, τ Lipschitz in (x, y) (unif. in u) with linear growth
- $Z.$ 1-dim. pure jump Lévy process, independent of $W.$,
  + conditions (later).

Value function is

$$V^\varepsilon(t, x, y) := \sup_{u.} E[e^{c(t-T)} g(x_T) \,|\, x_t = x, y_t = y]$$

with $\quad g : \mathbf{R}^n \to \mathbf{R}$ continuous, $\quad g(x) \leq K(1 + |x|^2), \quad c \geq 0.$

HJB equation

The value V^ε solves the integro-differential HJB equation in $(0,T) \times \mathbf{R}^n \times \mathbf{R}$

$$-\frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon] + cV^\varepsilon = 0,$$

$$\mathcal{H}(x,y,p,M) := \min_{u \in U}\left\{-\mathrm{tr}(\sigma\sigma^T M)/2 - f \cdot p\right\}$$

$$\mathcal{L}[y,v] := -yv_y(y) + \int_0^{+\infty} (v(z+y) - v(y) - v_y(y) z 1_{z \le 1}) d\nu(z)$$

is the generator of the unscaled volatility process $dy_s = -y_{s^-} ds + dZ_s$, ν is the Lévy measure associated to the jump process $Z_\cdot$:

$$\nu(B) = E(\#\{s \in [0,1],\ Z_s - Z_{s^-} \ne 0,\ Z_s - Z_{s^-} \in B\})$$

$=$ expected number of jumps of a certain height in a unit-time interval.

HJB equation

The value V^ε solves the integro-differential HJB equation in $(0,T) \times \mathbf{R}^n \times \mathbf{R}$

$$-\frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon] + cV^\varepsilon = 0,$$

$$\mathcal{H}(x,y,p,M) := \min_{u \in U}\left\{-\mathrm{tr}(\sigma\sigma^T M)/2 - f \cdot p\right\}$$

$$\mathcal{L}[y,v] := -yv_y(y) + \int_0^{+\infty} (v(z+y) - v(y) - v_y(y) z 1_{z \le 1}) d\nu(z)$$

is the generator of the unscaled volatility process $dy_s = -y_{s^-} ds + dZ_s$, ν is the Lévy measure associated to the jump process $Z.$:

$$\nu(B) = E(\#\{s \in [0,1],\ Z_s - Z_{s^-} \neq 0,\ Z_s - Z_{s^-} \in B\})$$

= expected number of jumps of a certain height in a unit-time interval.

HJB equation

The value V^ε solves the integro-differential HJB equation in $(0,T)\times\mathbf{R}^n\times\mathbf{R}$

$$-\frac{\partial V^\varepsilon}{\partial t}+\mathcal{H}\left(x,y,D_xV^\varepsilon,D^2_{xx}V^\varepsilon\right)-\frac{1}{\varepsilon}\mathcal{L}[y,V^\varepsilon]+cV^\varepsilon=0,$$

$$\mathcal{H}(x,y,p,M):=\min_{u\in U}\left\{-\mathrm{tr}(\sigma\sigma^T M)/2-f\cdot p\right\}$$

$$\mathcal{L}[y,v]:=-yv_y(y)+\int_0^{+\infty}(v(z+y)-v(y)-v_y(y)z1_{z\le 1})d\nu(z)$$

is the generator of the unscaled volatility process $dy_s=-y_{s^-}ds+dZ_s$,

ν is the Lévy measure associated to the jump process Z. :

$$\nu(B)=E(\#\{s\in[0,1],\ Z_s-Z_{s^-}\neq 0,\ Z_s-Z_{s^-}\in B\})$$

= expected number of jumps of a certain height
in a unit-time interval.

HJB equation

The value V^ε solves the integro-differential HJB equation in $(0, T) \times \mathbf{R}^n \times \mathbf{R}$

$$-\frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon] + cV^\varepsilon = 0,$$

$$\mathcal{H}(x, y, p, M) := \min_{u \in U}\left\{-\mathrm{tr}(\sigma\sigma^T M)/2 - f \cdot p\right\}$$

$$\mathcal{L}[y, v] := -yv_y(y) + \int_0^{+\infty} (v(z+y) - v(y) - v_y(y)z1_{z\le 1})d\nu(z)$$

is the generator of the unscaled volatility process $dy_s = -y_{s^-}ds + dZ_s$, ν is the Lévy measure associated to the jump process $Z.$:

$$\nu(B) = E(\#\{s \in [0,1],\ Z_s - Z_{s^-} \neq 0,\ Z_s - Z_{s^-} \in B\})$$

$=$ expected number of jumps of a certain height in a unit-time interval.

PDE approach to the singular limit $\varepsilon \to 0$

Search an *effective Hamiltonian* $\overline{H}$ such that

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{as } \varepsilon \to 0,$$

V solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} -\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) + cV = 0 & \text{in } (0,T) \times \mathbf{R}^n, \\ V(T,x) = g(x) \end{cases}$$

Then, if possible, intepret the effective Hamiltonian $\overline{H}$ as the Bellman Hamiltonian for a new effective optimal control problem in $\mathbf{R}^n$,

which is therefore a variational limit of the initial $n+m$-dimensional problem.

PDE approach to the singular limit $\varepsilon \to 0$

Search an *effective Hamiltonian* $\overline{H}$ such that

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{as } \varepsilon \to 0,$$

V solution of

$$(\overline{\text{CP}}) \qquad \begin{cases} -\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) + cV = 0 & \text{in } (0,T) \times \mathbf{R}^n, \\ V(T,x) = g(x) \end{cases}$$

Then, if possible, intepret the effective Hamiltonian $\overline{H}$ as the Bellman Hamiltonian for a new effective optimal control problem in $\mathbf{R}^n$, which is therefore a variational limit of the initial $n+m$-dimensional problem.

Tools

1. Ergodicity of the unscaled volatility process, or fast subsystem, i.e., of

$$dy_s = -y_{s^-}\, ds + dZ_s$$

Assume conditions such that this process has a unique invariant probability measure μ and it is uniformly ergodic.

By solving an auxiliary (linear) PDE called cell problem we find that the candidate effective Hamiltonian is

$$\overline{H}(x, p, M) = \int_{\mathbf{R}^m} \mathcal{H}(x, y, p, M)\, d\mu(y).$$

Tools

1. Ergodicity of the unscaled volatility process, or fast subsystem, i.e., of

$$dy_s = -y_{s^-}\, ds + dZ_s$$

Assume conditions such that this process has a unique invariant probability measure μ and it is uniformly ergodic.

By solving an auxiliary (linear) PDE called cell problem we find that the candidate effective Hamiltonian is

$$\overline{H}(x, p, M) = \int_{\mathbf{R}^m} \mathcal{H}(x, y, p, M)\, d\mu(y).$$

2. The generator $\mathcal{L}$ has the Liouville property
(based on the Strong Maximum Principle by Ciomaga 2012), i.e.

any bounded sub- or supersolution of $-\mathcal{L}[y, v] = 0$ is constant.

Then the relaxed semilimits

$$\underline{V}(t, x, y) := \liminf_{\varepsilon \to 0, t' \to t, x' \to x, y' \to y} V^{\varepsilon}(t', x', y'),$$

$\overline{V}(t, x, y) := \limsup$ of the same, do not depend on y.

3. Perturbed test function method,
evolving from Evans (periodic homogenisation) and
Alvarez-M.B. (singular perturbations with bounded fast variables),
allows to prove that

$\underline{V}(t, x)$ is supersol., $\overline{V}(t, x)$ is subsol. of limit PDE in $(\overline{CP})$.

2. The generator $\mathcal{L}$ has the Liouville property
(based on the Strong Maximum Principle by Ciomaga 2012), i.e.

any bounded sub- or supersolution of $-\mathcal{L}[y, v] = 0$ is constant.

Then the relaxed semilimits

$$\underline{V}(t, x, y) := \liminf_{\varepsilon \to 0, t' \to t, x' \to x, y' \to y} V^{\varepsilon}(t', x', y'),$$

$\overline{V}(t, x, y) :=$ lim sup of the same, do not depend on y.

3. Perturbed test function method,
evolving from Evans (periodic homogenisation) and
Alvarez-M.B. (singular perturbations with bounded fast variables),
allows to prove that

$\underline{V}(t, x)$ is supersol., $\overline{V}(t, x)$ is subsol. of limit PDE in $(\overline{\text{CP}})$.

2. The generator $\mathcal{L}$ has the Liouville property
(based on the Strong Maximum Principle by Ciomaga 2012), i.e.

any bounded sub- or supersolution of $-\mathcal{L}[y, v] = 0$ is constant.

Then the relaxed semilimits

$$\underline{V}(t,x,y) := \liminf_{\varepsilon\to 0, t'\to t, x'\to x, y'\to y} V^\varepsilon(t', x', y'),$$

$\overline{V}(t,x,y) :=$ lim sup of the same, do not depend on y.

3. Perturbed test function method,
evolving from Evans (periodic homogenisation) and
Alvarez-M.B. (singular perturbations with bounded fast variables),
allows to prove that

$\underline{V}(t,x)$ is supersol., $\overline{V}(t,x)$ is subsol. of limit PDE in $(\overline{\text{CP}})$.

4. Comparison principle

between a subsolution and a supersolution of the Cauchy problem $(\overline{\text{CP}})$ satisfying

$$|V(t,x)| \leq C(1+|x|^2),$$

see Da Lio - Ley 2006. It gives

- uniqueness of solution V of $(\overline{\text{CP}})$
- $\underline{V}(t,x) \geq \overline{V}(t,x)$, then $\underline{V} = \overline{V} = V$ and, as $\varepsilon \to 0$,

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{locally uniformly.}$$

Assumptions

The Lévy measure ν of the jump process $Z.$ satisfies

- $\exists\, C > 0,\ 0 < p < 2,\ 0 < \delta \le 1 \ : \quad \int_{|z| \le \delta} |z|^2 \nu(dz) \ge C\, \delta^{2-p}$
- $\exists\, q > 0 \ : \quad \int_{|z|>1} |z|^q \nu(dz) < +\infty.$

Then the unscaled volatility $dy_s = -y_{s^-} ds + dZ_s$ is uniformly ergodic (Kulik 2009).

If, moreover,

- either $\quad p > 1,$
- or $\quad 0 \in \text{int} \ \ \text{supp}(\nu),$

then the integro-differential generator $\mathcal{L}$ of the process $y.$ has the Liouville property.

The Lévy measure ν of the jump process $Z_.$ satisfies

- $\exists\, C > 0,\ 0 < p < 2,\ 0 < \delta \le 1\ :\quad \int_{|z|\le\delta} |z|^2 \nu(dz) \ge C\,\delta^{2-p}$
- $\exists\, q > 0\ :\quad \int_{|z|>1} |z|^q \nu(dz) < +\infty.$

Then the unscaled volatility $dy_s = -y_{s^-}\,ds + dZ_s$ is uniformly ergodic (Kulik 2009).

If, moreover,

- either $\quad p > 1,$
- or $\quad 0 \in \operatorname{int}\ \operatorname{supp}(\nu),$

then the integro-differential generator $\mathcal{L}$ of the process $y_.$ has the Liouville property.

Examples

α-stable Lévy processes:

- ν symmetric

$$\nu(dz) = \frac{dz}{|z|^{1+\alpha}}, \qquad 0 < \alpha < 2,$$

here $\mathcal{L} = (-\Delta)^{\alpha/2}$ is the fractional Laplacian

- ν not symmetric: no negative jumps

$$\nu(dz) = \frac{dz}{|z|^{1+\alpha}} 1_{\{z \geq 0\}}(z), \qquad 1 < \alpha < 2$$

Tempered α-stable Lévy processes:

- $$\nu(dz) = \frac{e^{-\gamma z} dz}{|z|^{1+\alpha}} 1_{\{z \geq 0\}}(z), \qquad 1 < \alpha < 2, \quad \gamma > 0.$$

Convergence Theorem [M.B. - Cesaroni - Scotti 2014]

Theorem

$\lim_{\varepsilon \to 0} V^\varepsilon(t,x,y) = V(t,x)$ *locally uniformly, V solving*

$$-\frac{\partial V}{\partial t} + \int_{\mathbf{R}^m} \mathcal{H}\left(x, y, D_x u, D^2_{xx} u, 0\right) \, d\mu(y) = 0 \quad in\ (0,T) \times \mathbf{R}^n$$

with $V(T,x) = g(x)$.

Related earlier results for Gaussian ergodic mean-reverting volatility

$$dy_s = \frac{1}{\varepsilon} b(x_s, y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s)\, dW_s \qquad y_s \in \mathbf{R}^m$$

- τ nondegenerate, b, τ independent of x [M.B. - Cesaroni - Manca, SIAM J. Financial Math. 2010],
- $b, \tau \in \mathcal{C}^{1,\alpha}$ bdd. derivatives [M.B. - Cesaroni, Eur. J. Control 11]

Convergence Theorem [M.B. - Cesaroni - Scotti 2014]

Theorem

$\lim_{\varepsilon\to 0} V^\varepsilon(t,x,y) = V(t,x)$ *locally uniformly, V solving*

$$-\frac{\partial V}{\partial t} + \int_{\mathbf{R}^m} \mathcal{H}\left(x, y, D_x u, D^2_{xx} u, 0\right)\, d\mu(y) = 0 \quad \text{in } (0,T)\times \mathbf{R}^n$$

with $\quad V(T,x) = g(x).$

Related earlier results for Gaussian ergodic mean-reverting volatility

$$dy_s = \frac{1}{\varepsilon} b(x_s, y_s)\, ds + \frac{1}{\sqrt{\varepsilon}} \tau(x_s, y_s)\, dW_s \qquad y_s \in \mathbf{R}^m$$

- τ nondegenerate, b, τ independent of x [M.B. - Cesaroni - Manca, SIAM J. Financial Math. 2010],
- $b, \tau \in \mathcal{C}^{1,\alpha}$ bdd. derivatives [M.B. - Cesaroni, Eur. J. Control 11]

Financial examples

In Black-Scholes option pricing model with $\sigma = \sigma(y)$ the limit PDE is

$$-\frac{\partial V}{\partial t} - rxV_x + \int \sigma^2(y)d\mu(y)\, x^2 V_{xx} + cV = 0 \quad \text{in } (0,T) \times \mathbf{R},$$

which is a Black-Scholes PDE with constant volatility

$$\tilde{\sigma}^2 := \int \sigma^2(y)\mu(dy) = \text{mean historical volatility},$$

a linear average of $\sigma^2(\cdot)$.

Merton portfolio optimization problem

Invest u_s in the stock S_s, $1 - u_s$ in a bond with interest rate r. Then the wealth x_s evolves as

$$d\,x_s = (r + (\gamma - r)u_s)x_s\,ds + x_s u_s\,\sigma(y_s)\,dW_s \qquad y_t = y,$$

and want to maximize the expected utility at time T, $\quad E[g(x_T)]$,
for some g increasing and concave.

Financial examples

In Black-Scholes option pricing model with $\sigma = \sigma(y)$ the limit PDE is

$$-\frac{\partial V}{\partial t} - rxV_x + \int \sigma^2(y)d\mu(y)\, x^2 V_{xx} + cV = 0 \quad \text{in } (0, T) \times \mathbf{R},$$

which is a Black-Scholes PDE with constant volatility

$$\tilde{\sigma}^2 := \int \sigma^2(y)\mu(dy) = \text{mean historical volatility},$$

a linear average of $\sigma^2(\cdot)$.

Merton portfolio optimization problem

Invest u_s in the stock S_s, $1 - u_s$ in a bond with interest rate r. Then the wealth x_s evolves as

$$d\,x_s = (r + (\gamma - r)u_s)x_s\,ds + x_s u_s\,\sigma(y_s)\,dW_s \qquad y_t = y,$$

and want to maximize the expected utility at time T, $\quad E[g(x_T)]$, for some g increasing and concave.

Then $V^\varepsilon(t,x,y) := \sup_{u.} E[g(x_T)]$ solves

$$-\frac{\partial V^\varepsilon}{\partial t} - rxV^\varepsilon_x + \frac{[(\gamma - r)V^\varepsilon_x]^2}{\sigma^2(y)2V^\varepsilon_{xx}} = \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon]$$

By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$-\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } (0,T) \times \mathbf{R}.$$

This is the HJB equation of a Merton problem with constant volatility $\overline{\sigma}$ = harmonic average of $\sigma(\cdot)$.

$$\overline{\sigma}^2 := \left(\int \frac{1}{\sigma^2(y)} d\mu(y)\right)^{-1} \leq \tilde{\sigma}^2 = \int \sigma^2(y)\mu(dy)$$

Then if one uses a constant-parameter model as approximation, the nonlinear average $\overline{\sigma}$ is better, it increases the optimal expected utility.

Then $V^\varepsilon(t,x,y) := \sup_{u.} E[g(x_T)]$ solves

$$-\frac{\partial V^\varepsilon}{\partial t} - rxV_x^\varepsilon + \frac{[(\gamma - r)V_x^\varepsilon]^2}{\sigma^2(y)2V_{xx}^\varepsilon} = \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon]$$

By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$-\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } (0,T) \times \mathbf{R}.$$

This is the HJB equation of a Merton problem with constant volatility $\overline{\sigma}$ = harmonic average of $\sigma(\cdot)$.

$$\overline{\sigma}^2 := \left(\int \frac{1}{\sigma^2(y)} d\mu(y)\right)^{-1} \leq \tilde{\sigma}^2 = \int \sigma^2(y)\mu(dy)$$

Then if one uses a constant-parameter model as approximation, the nonlinear average $\overline{\sigma}$ is better, it increases the optimal expected utility.

Then $V^\varepsilon(t,x,y) := \sup_{u.} E[g(x_T)]$ solves

$$-\frac{\partial V^\varepsilon}{\partial t} - rxV^\varepsilon_x + \frac{[(\gamma - r)V^\varepsilon_x]^2}{\sigma^2(y)2V^\varepsilon_{xx}} = \frac{1}{\varepsilon}\mathcal{L}[y, V^\varepsilon]$$

By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$-\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } (0,T) \times \mathbf{R}.$$

This is the HJB equation of a Merton problem with constant volatility $\overline{\sigma}$ = harmonic average of $\sigma(\cdot)$.

$$\overline{\sigma}^2 := \left(\int \frac{1}{\sigma^2(y)} d\mu(y) \right)^{-1} \leq \tilde{\sigma}^2 = \int \sigma^2(y)\mu(dy)$$

Then if one uses a constant-parameter model as approximation, the nonlinear average $\overline{\sigma}$ is better, it increases the optimal expected utility.

Short time and fast volatility: large deviations

For small $\varepsilon > 0$ and $\delta > 0$ look at

$$dX_t = \varepsilon\phi(X_t, Y_t)dt + \sqrt{2\varepsilon}\sigma(X_t, Y_t)dW_t \qquad X_0 = x \in \mathbf{R}^n,$$

$$dY_t = \frac{\varepsilon}{\delta}b(Y_t)dt + \sqrt{\frac{2\varepsilon}{\delta}}\tau(Y_t)dW_t \qquad Y_0 = y \in \mathbf{R}^m,$$

with ϕ, σ, b, τ periodic in Y (for simplicity). Take

$$\delta = \varepsilon^\alpha, \qquad \alpha > 1$$

and $v^\varepsilon(t,x,y) := \varepsilon \log E\left[e^{h(X_t)/\varepsilon}\right]$. It satisfies $v^\varepsilon(0,x,y) = h(x)$ and

$$v_t = |\sigma^T D_x v|^2 + \varepsilon[\mathrm{tr}(\sigma\sigma^T D^2_{xx} v) + \phi \cdot D_x v] + \frac{2}{\varepsilon^{\alpha/2}}(\tau\sigma^T D_x v) \cdot D_y v +$$

$$\frac{1}{\varepsilon^\alpha}|\tau^T D_y v|^2 + \frac{2}{\varepsilon^{\alpha/2-1}}\mathrm{tr}(\sigma\tau^T D^2_{xy} v) + \frac{1}{\varepsilon^{\alpha-1}}[b \cdot D_y v + \mathrm{tr}(\tau\tau^T D^2_{yy} v)]$$

Convergence Theorem [M.B. - Cesaroni - Ghilli 2014]

$\forall\, \alpha > 1\ \exists\, \bar{H}$ continuous such that $\lim_{\varepsilon \to 0} v^\varepsilon(t,x,y) = v(t,x)$ in the sense of weak semilimits, v solving

$$v_t - \bar{H}(x, Dv) = 0 \quad \text{in } (0,T) \times \mathbf{R}^n$$

with $v(0,x) = h(x)$; $\bar{H}$ depends on the three regimes

1. $\alpha > 2$: $\bar{H}(x,p) = \int_{\mathbb{T}^m} |\sigma^T(x,y)p|^2\, d\mu(y)$, convergence is uniform
2. $\alpha < 2$: $\bar{H}$ has deterministic control formula, convergence is uniform; for $\tau\sigma^T = 0$ $\bar{H}(x,p) = \max_{y \in \mathbf{R}^m} |\sigma^T(x,y)p|^2$
3. $\alpha = 2$: $\bar{H}$ has stochastic control formula, convergence uniform if
    - either $\sigma = \sigma(Y_t)$ independent of X_t, ($\bar{H}$ independent of x)
    - or $|\sigma^T(x,y)p|^2 \geq \nu |p|^2$, $\nu > 0$, ($\bar{H}$ coercive)
    - or $\tau\sigma^T = 0$, (independent noise in dX_t and dY_t)

More on this paper in Daria GHILLI's poster tomorrow!

Further results and perspectives

Can treat also

- limit of the optimal feedback in Merton's problem,
- utility depending on y, i.e., $g = g(x, y)$, then the effective terminal condition is $V(T, x) = \int g(x, y) d\mu(y)$,
- problems with two conflicting controllers, i.e., two-person, 0-sum, stochastic differential games,
- systems with more than two scales.

Developments under investigation:

- more general jump processes for the volatility (without the Liouville property...), e.g., "inverse Gaussian",
- jump terms in the stocks dynamics,
- large deviations for short maturity asymptotics with non-Gaussian volatility and/or in Merton's problem.

Further results and perspectives

Can treat also

- limit of the optimal feedback in Merton's problem,
- utility depending on y, i.e., $g = g(x, y)$, then the effective terminal condition is $V(T, x) = \int g(x, y) d\mu(y)$,
- problems with two conflicting controllers, i.e., two-person, 0-sum, stochastic differential games,
- systems with more than two scales.

Developments under investigation:

- more general jump processes for the volatility (without the Liouville property...), e.g., "inverse Gaussian",
- jump terms in the stocks dynamics,
- large deviations for short maturity asymptotics with non-Gaussian volatility and/or in Merton's problem.

Thanks for your attention!