Optimization of running strategies based on anaerobic energy and variations of velocity

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Outline

1 Keller's model

2 Variable energy recreation

3 Bounding the derivative of f

Sources

- Talk based on the joint work [1] with Amandine Aftalion, Lab. Math. Versailles, UVSQ.
- Pioneering reference: J.B. Keller [2].

[1] A. Aftalion and J.F. Bonnans, *Optimization of running strategies based on anaerobic energy and variations of velocity*. SIAM J. Applied Math., to appear (HAL preprint).

[2] Keller, J.B., *Optimal velocity in a race*, Amer. Math. Monthly 81 (1974), 474–480.

1 Keller's model

2 Variable energy recreation

3 Bounding the derivative of f

Keller's model: dynamics

Consider the following **state equation**:

$$\dot{v}(t) = f(t) - \phi(v(t)); \quad \dot{e}(t) = \bar{\sigma} - f(t)v(t), \tag{1.1}$$

- **Cost function**: $-\int_0^T v(t)\mathrm{d}t$.
- **energy recreation**: $\bar{\sigma} > 0$
- **drag function** $\phi(v) = v/\tau$, with $\tau > 0$.
- **Generalized drag function**: ϕ C^2 on $(0, \infty)$; $\phi(0) = 0$; ϕ' positive; $v\phi'(v)$ increasing
  Example: $\phi(v) = av^\beta$, $a > 0$, $\beta \geq 1$.

Keller's model: constraints

- **Control constraint**: $0 \leq f(t) \leq f_M$.
- **Energy constraint**: $e(t) \geq 0$.
- **Maximal force strategy**: optimal for short horizons $T \leq T_c$.
- In the sequel $T > T_c$.

Keller's model: discussion

- **Maximal speed** (never reached): $0 = f_M - v_M/\tau$, i.e.

$$v_M = \tau f_M.$$

- **Constant energy speed**: $f = v/\tau$, $\sigma = fv$, and so:

$$v_E = \sqrt{\tau\sigma}.$$

- Energy variation when constant speed:

$$e(t) \downarrow \text{ if } v > v_E, \text{ and } e(t) \uparrow \text{ otherwise.}$$

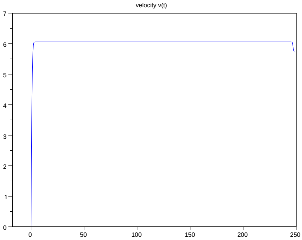
Keller's model: Keller's conjecture

- Optimal strategy has **three arcs**
- First arc: **maximal force** $f = f_M$.
- Second arc: **constant speed** $v > v_E$.
- Last arc: **zero energy** $e = 0$.
- Negative jump of force at junction times
- Unclear proofs in the literature for $\phi(v) = v/\tau$.

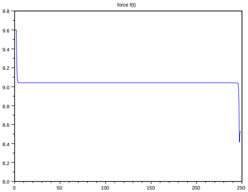
Keller's model: numerical resultsl

- Mechanical parameters: $f_{max} = 9.6m\ s^{-2}$, $\tau = 0.67s$
- Energy parameters: $e^0_{an} = 1400m^2\ s^{-3}$, $\bar{\sigma} = 49m^2\ s^{-3}$.
- $d = 1500m$.
- 2000 discretization steps, i.e. time step close to 0.12s.
- **Free** software **www.bocop.org**.
- Optimal time is 248.21s

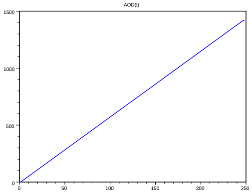
Keller's model: numerical results with bocop.org



Velocity

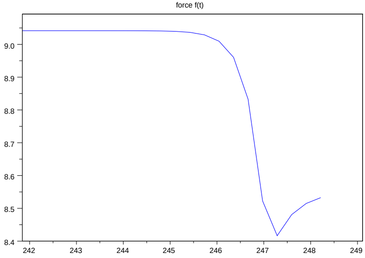


Force

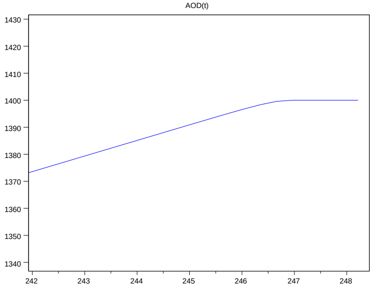


Reverse energy

Zoom on end of race: zero energy arc



Final force



Final reverse energy

Keller's model: main result

So the numerical results agree with Keller's conjecture, and indeed:

Theorem

Keller's conjecture holds, even with the generalized drag function.

We next present the proof of this result.

Keller's model: Hamiltonian and costate

Hamiltonian function:

$$H := -v + p_v(f - \phi(v)) + p_e(\bar{\sigma} - fv).$$

Costate equation:

$$\begin{cases} -\dot{p}_v & = -1 - p_v \phi'(v) - p_e f \\ -\mathrm{d}p_e & = -\mathrm{d}\mu, \end{cases}$$

Final conditions $p_v(T) = p_e(T) = 0$, and so: $p_e = \mu$.
Multiplier to the state constraint: say $\mu(T) = 0$ and

$$\mathrm{d}\mu \geq 0; \quad \mathrm{supp}(\mathrm{d}\mu) \subset \{t;\ e(t) = 0\}.$$

Therefore $p_e = \mu \leq 0$.

Switching function

The **switching function** is

$$\Psi := H_u = p_v - p_e v.$$

By **Pontryagin's principle**, we have:

$$f(t) = \begin{cases} f_M & \text{if } \Psi(t) > 0, \\ 0 & \text{if } \Psi(t) < 0, \end{cases}$$

p_e has negative values

Lemma

p_e has negative values.

Proof.

If $p_e(\tau) = 0$ then over (τ, T):
$p_e = 0$, since it is nondecreasing and $p_e(T) = 0$.
$\dot{p}_v = 1 + p_v \phi'(v)$ (is > 0 if $p_v \geq 0$) and $p_v(T) = 0$
So $p_v < 0 \Rightarrow \Psi = p_v < 0 \Rightarrow f = f_M$
Contradiction by lemma 3.7 of paper.

$\square$

p_v has negative values

Lemma

p_v has negative values.

Proof.

$p_v \geq 0 \Rightarrow \Psi > 0 \Rightarrow f = 0 \Rightarrow \dot{p}_v > 0; \quad$ but $p_v(T) = 0.$ $\square$

Corollary

An optimal trajectory starts with a maximal force arc.

Derivative of switching function

When the state constraint is not active, p_e is constant and so:

$$\Psi := H_u = p_v - p_e v.$$

$$\begin{aligned} \dot{\Psi} &= (1 + p_v \phi'(v) + p_e f) - p_e(f - \phi(v)) \\ &= 1 + p_v \phi'(v) + p_e \phi(v) \end{aligned}$$

and so,

$$\dot{\Psi} - \Psi \phi'(v) = 1 + p_e(\phi(v) + v\phi'(v)).$$

Lemma

On a singular arc, v is constant.

No zero force arc

$$\Delta := \dot{\Psi} - \Psi\phi'(v) = 1 + p_e(\phi(v) + v\phi'(v)).$$

Lemma

No zero force arc occurs.

- Let (t_a, t_b) be such an arc; then $t_a > 0$.
- If $t_b = T$ then $e(T) > 0$: impossible.
- $\Psi \geq 0$ over the arc; $\Psi(t_a) = \Psi(t_b) = 0$.
- We have therefore $\Delta(t_a) \geq 0 \geq \Delta(t_b)$.
- On this arc p_e is constant and v decreases, and so $\Delta(t)$ increases: contradiction.

No second maximal force arc

$$\Delta := \dot{\Psi} - \Psi\phi'(v) = 1 + p_e(\phi(v) + v\phi'(v)).$$

Lemma

No second maximal force arc occurs.

- Let (t_a, t_b) be such an arc; then $t_a > 0$.
- If $t_b < T$, "symmetric argument": v increases, Δ decreases, but $\Delta(t_a) \leq 0 \leq \Delta(t_b)$.
- If $t_b = T$, μ should have a jump at time T, but then

$$\lim_{t \uparrow T} \Psi(t) = -p_e(T_-)v(T) > 0$$

implying that $f = 0$ at the end of the trajectory.

Keller's model: proof of main result

- Let $t_a \in (0, T)$ be the exit point of the maximal force arc.
- We know that $\Psi = 0$ over (t_a, T).
- Let $t_b \in (0, T)$ be the first time at which the energy vanishes (μ has no final jump).
- (t_a, t_b) is a singular arc.
- If the energy is not zero on (t_b, T): there would exist t_c, t_d with $t_b \leq t_c < t_d \leq T$ such that $e(t_c) = e(t_d) = 0$, and $e(t) > 0$, for all $t \in (t_c, t_d)$.
- Then (t_c, t_d) is a singular arc, over which $\dot{e} = \sigma - fv$ is constant: contradiction.

1 Keller's model

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Variable energy recreation

- Less recreation when energy close to its extreme values, and additional recreation when deceleration:

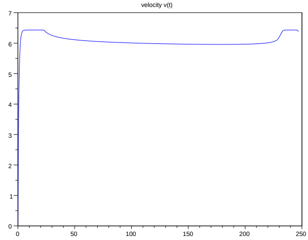
$$\dot{e} = \sigma(e) + \eta(a) - fv$$

where a is the acceleration: $a = \dot{v} = f - \phi(v)$, and η convex:

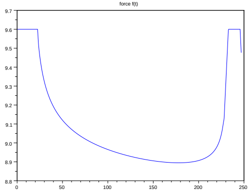
$$\eta(a) \geq 0, \quad \eta(a) = 0 \text{ iff } a \geq 0.$$

- Function $\sigma(e)$ regularization of a piecewise affine and continuous function, value 0 at extreme values, otherwise constant.

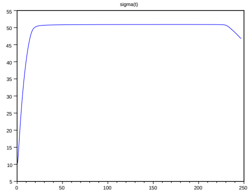
Numerical results with variable $\sigma(e)$ and $\eta(a) = 0$



Velocity



Force



$\sigma(e)$

Hamiltonian and costate with variable $\sigma(e)$

Hamiltonian function:

$$H := -v + p_v(f - \phi(v)) + p_e(\sigma(e) + \eta(f - \phi(v)) - fv).$$

Costate equation:

$$\begin{cases} -\dot{p}_v &= -1 - p_v\phi'(v) - p_e f - p_e\eta'(a)\phi'(v) \\ -\mathrm{d}p_e &= p_e\sigma'(e) - \mathrm{d}\mu, \end{cases}$$

Final conditions $p_v(T) = p_e(T) = 0$.
$\mu(T) = 0$, $\mathrm{d}\mu \geq 0$, $\mathrm{supp}(\mathrm{d}\mu) \subset \{t;\ e(t) = 0\}$.

Minimization of Hamiltonian with variable $\sigma(e)$

- **Hamiltonian function**:

$$H := (p_v - p_e v)f + p_e\eta(f - \phi(v)) + \text{indep. of } f$$

- **Concave function** of the control f
- Minimum attained at extreme values 0 or f_M.
- Need of **relaxed formulation**; expectation of force:

$$f = 0 \times (1-\theta) + \theta f_M$$

- We may as well take f as parameter and then $\theta = f/f_M$.

Relaxed formulation

- Recreation due to deceleration at zero speed:

$$R(v) := \eta(-\phi(v)$$

- State equation

$$\begin{aligned} \dot{v} &= f - \phi(v) \\ \dot{e} &= \sigma(e) - fv + (1 - f/f_M)R(v) \end{aligned}$$

- Hamiltonian

$$H = p_v(f - \phi(v)) + p_e(\sigma(e) - fv + (1 - f/f_M)R(v)).$$

Relaxed formulation: theoretical analysis

- Assume $\sigma(e) = \bar{\sigma}$ and $\eta(a) = \varepsilon\eta'(a)$
- For $\varepsilon > 0$ small enough
- Same optimal structure as for Keller'model:
  maximal force, constant speed, zero energy.

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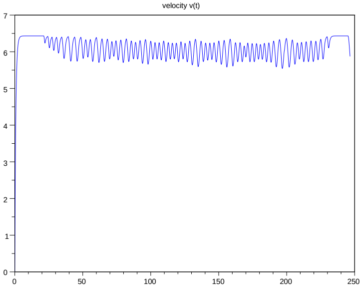
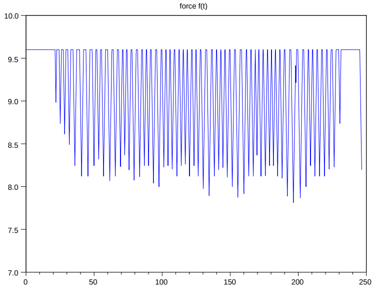
Numerical results when bounding $\dot{f}$

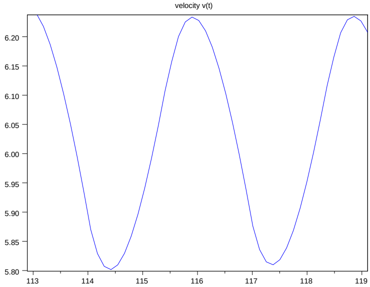


Velocity

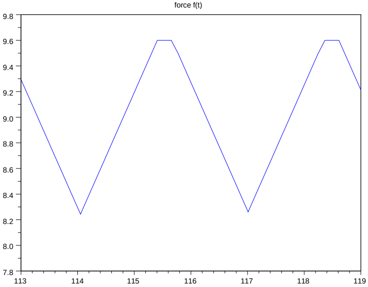


Force

Numerical results: zoom with variable $\sigma(e)$ and $\eta(a) \neq 0$



Zoom on velocity



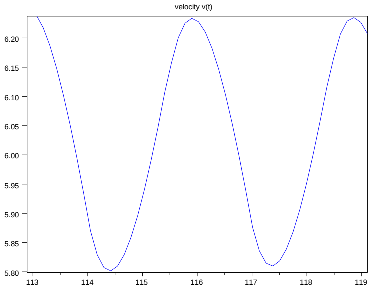
Zoom on force

Periodic problem

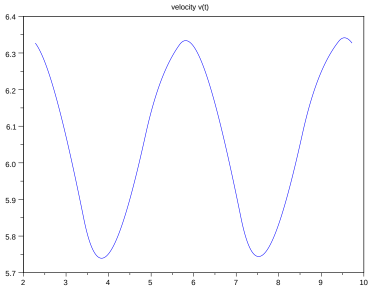
- Maximize average speed: $(1/T)\int_0^T v(t)\mathrm{d}t$.
- Periodic speed and force, loss of energy $e(0) = e(T) + Te_d$.
- State equation and constraints

$$
\begin{cases}
v(0) = v(T); \quad f(0) = f(T). \\
e(0) = e(T) + Te_d; \\
\dot{v} = f - v\tau; \\
\dot{e} = \sigma + \eta(a) - fv; \\
0 \le f \le f_M; \ |\dot{f}| \le 1, \quad \text{for a.e. } t \in (0, T).
\end{cases}
$$

Zoom vs periodic: velocity

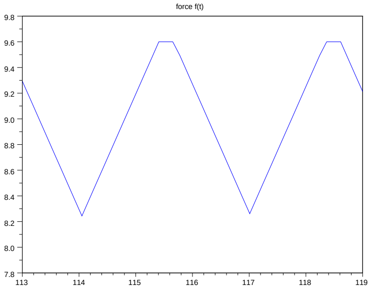


Zoom on velocity

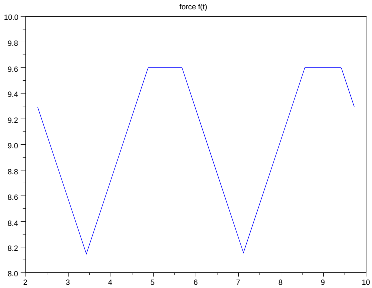


Periodic velocity

Zoom vs periodic: force



Zoom on force



Periodic force

Related work

[1] S. Aronna, J.F. Bonnans and B.S. Goh, *Second order necessary conditions for control-affine problems with state constraints*. Research report, to appear (HAL preprint).

Open questions

- Asymptotic analysis $\eta(a) = \varepsilon\eta'(a)$.
- Shape of oscillations ?
- Expansion of cost function

THE END !