Variational Curvature Flows

Antonin Chambolle, Massimiliano Morini, Marcello Ponsiglione

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Antonin Chambolle

CMAP, Ecole Polytechnique, CNRS, Palaiseau, France

(with M. Morini *(Parma),* M. Ponsiglione *(Roma I))*

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Introduction

- Generalized perimeters (including non-local) and their first variation;
- Properties of the "curvature";
- Existence and Uniqueness of a flow;
- Minimizing movements (variational curvature flows).

Generalized perimeters

In this talk a *perimeter* is a **nonnegative** set function $E \subset \mathbb{R}^N \mapsto J(E) \in [0, +\infty]$ which satisfies a "submodularity inequality"

$$J(E \cup F) + J(E \cap F) \leq J(E) + J(F) \qquad (SUBM)$$

which turns out to imply a sort of ellipticity of its first variation.

Other assumptions

We assume also

- $J(\emptyset) = J(\mathbb{R}^N) = 0$;
- J is l.s.c. with respect to L^1 convergence;
- J is translational invariant;
- J has *some smoothness* (see next slides).

The translational invariance is quite restrictive. Some flows associated to nonlocal perimeters have been studied outside of this context (typical examples: Hele-Shaw, curvature+non-local terms, cf for instance the series of papers of P. Cardaliaguet with E. Rouy, O. Ley).

A first remark on “SUBM”

If J is a “perimeter”, we can associate a “total variation” through the *generalized coarea formula* (Visintin)

$$J(u) \;=\; \int_{-\infty}^{+\infty} J(\{u > s\})\, ds$$

Then: the convexity of J is equivalent to the “submodularity” of the set-function (same as Lovasz’s extension in the discrete setting). (*cf* C.-Giacomini-Lussardi 10)

First variation of the perimeter

We assume that if E is smooth enough[1] and with compact boundary, then $J(E) < +\infty$ and for smooth diffeomorphisms Φ_ε with $(\Phi_\varepsilon - I)/\varepsilon \stackrel{\varepsilon\to 0}{\to} \psi(x)$,

$$\lim_{\varepsilon\to 0} \frac{J(\Phi_\varepsilon(E)) - J(E)}{\varepsilon} = \int_{\partial E} \kappa(E,x)(\psi\cdot\nu)\, d\mathcal{H}^{d-1}$$

for some function κ, which we also assume is **continuous** wr E. (Hadamard shape derivative, *cf* for instance Murat-Simon, 1974/76.)

[1] $C^{\ell,\beta}$ for some $\ell \geq 2$, $\beta \in [0,1]$.

A second remark on “SUBM”

It turns out that $(SUBM)$ also implies the monotonicity of the set function $\kappa(x, E)$ (as the “derivative” of a convex function):

Lemma if $E \subseteq F$ with $x \in \partial E \cap \partial F$ then $\kappa(x, E) \geq \kappa(x, F)$.

Proof: delayed

A subgradient property

The function J defined using the generalized coarea formula is convex, one-homogeneous. Hence it is expected that for "many" u, there exists $p \in \partial J(u)$ and

$$J(v) \geq J(u) + \int_{\mathbb{R}^N} p(v-u)\,dx, \quad J(u) = \int_{\mathbb{R}^N} pu\,dx.$$

In particular, for two smooth sets $E \supset F$, one should have

$$J(E) - J(F) = \int_{E \setminus F} \text{``}\kappa\text{''}\,dx$$

A subgradient property

Proposition $\kappa(x, E)$ (continuous) is a first variation of J if and only if for all ϕ (smooth), if $\nabla\phi \neq 0$ on $\{t_1 \leq \phi \leq t_2\}$, one has

$$J(\{\phi \geq t_1\}) = J(\{\phi \geq t_2\}) + \int_{\{t_1<\phi<t_2\}} \kappa(x, \{\phi \geq \phi(x)\})\, dx.$$

Moreover, in this case, one also has that for any set W such that $\{\phi \geq t_2\} \subset W \subset \{\phi \geq t_1\}$,

$$J(W) \geq J(\{\phi \geq t_2\}) + \int_{W\setminus\{\phi\geq t_2\}} \kappa(x, \{\phi \geq \phi(x)\})\, dx.$$

A subgradient property

More generally, one deduces that $\kappa(x, E)$ satisfies, for any E smooth enough and any perturbation $W \subset B(x, \rho)$:

$$J(E \cup W) \geq J(E) + |W \setminus E|(\kappa(x, E) + o_\rho(1))$$

$$J(E \setminus W) \geq J(E) - |W \cap E|(\kappa(x, E) + o_\rho(1))$$

A second remark on "SUBM" (cont.)

It turns out that $(SUBM)$ also implies the monotonicity of the set function $\kappa(x, E)$:

Lemma if $E \subseteq F$ with $x \in \partial E \cap \partial F$ then $\kappa(x, E) \geq \kappa(x, F)$.

Proof: Now easy:

- First modify sligthly E or F so that $\{x\} = \partial E \cap \partial F$ (use the continuity of κ);
- Then slide slightly E out of F by translating and let $W_\varepsilon = (E + \varepsilon\nu) \setminus F$;
- Use the two previous equations and $(SUBM)$ to find that $\kappa(x, E) \geq \kappa(x, F) + o_\varepsilon(1)$.

Examples

- Of course, the standard Euclidean (or not) perimeter, with its curvature ($J =$ total variation) — *no crystals!*;
- Fractional perimeter: the H^α semi-norm of χ_E ($\alpha < 1$) (Imbert 09/Caffarelli-Souganidis 10/Cardaliaguet 01);
- "Pre-Minkowski" content in image processing, for filtering boundaries without destroying small-scale oscillations (Barchiesi, Kang, Le, Morini, Ponsiglione 10);
- Capacitary potential (Cardaliaguet-Ley 06/Cardaliaguet-Rouy 00/01/04), here an easier translational invariant version, which is covered also in Cardaliaguet'01.

Example: Fractional Perimeter

$$J(E) := [\chi_E]_{H^\alpha} = \left((1-\alpha) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{N+2\alpha}} \, dxdy \right)^{\frac{1}{2}}$$

with curvature

$$\kappa(x, E) = \lim_{\delta \to 0} -2(1-\alpha) \int_{\mathbb{R}^N} (\chi_E(y) - \chi_{\mathbb{R}^N \setminus E}(y)) \rho_\delta(x-y) \, dy,$$

with $\rho_\delta(x) = (1 - \chi_{B(0,\delta)}(x))/|x|^{N+2\alpha}$ (well-defined for $C^{1,1}$ sets, cf Imbert 09)

Example: Pre-Minkowski content

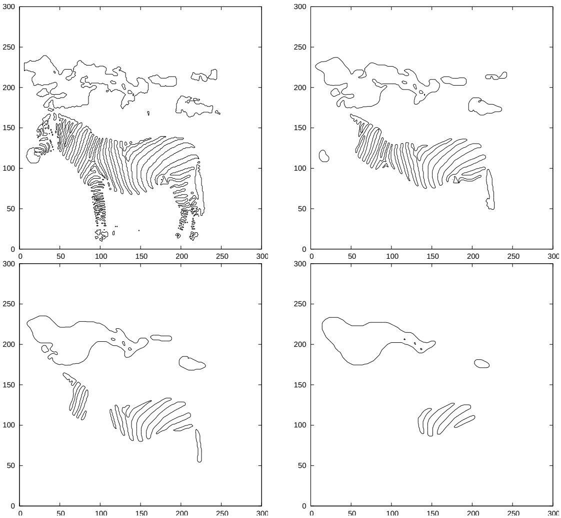
$$J(E) = \mathcal{M}_\rho(E) = \frac{1}{2\rho}|(\partial E)_\rho| = \frac{1}{2\rho}|(\cup_{x\in\partial E} B_\rho(x))|$$

which gives $J(E) \approx Per(E)$ if E is large and smooth wr ρ, but is quite insensitive to perturbations at scales $<< \rho$ near the boundary.
Needs to be smoothed to enter the framework:

$$J_f(E) = \frac{1}{2\rho}\int_{\mathbb{R}^N} f\left(\frac{\operatorname{dist}(x,\partial E)}{\rho}\right) dx$$

with f continuous, nonnegative, compactly supported. Curvature is nonlocal: it is the classical curvature for large smooth sets, but depends on the whole ρ-neighborhood of the point otherwise.

Example: Pre-Minkowski content



Curvature flow of the Pre-Minkowski content

Example: Capacitary potential

$$p-\mathrm{cap}(E) := \inf\left\{ \int_{\mathbb{R}^N} |Dw|^p \, dx : \; w \in C_c^\infty \text{ and } w \geq 1 \text{ a.e. in } E \right\},$$

(for bounded sets). The curvature at x is $-|Dw_E(x)|^p$ where w_E is the capacitary potential. (Related to Hele-Shaw flow, but here $1 < p < N$.)

How to define the associated "curvature flow"?

Of course the "standard" framework (Evans-Spruck, Chen-Giga-Goto) is to look for viscosity solutions of

$$u_t + |\nabla u|\kappa(x, \{u > u(x,t)\}) = 0, \quad u(x,0) = u_0$$

for u_0 appropriately defined. This is in particular to get rid of some non generic nonuniqueness issues.

Definition An USC function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, *constant outside a compact set*, is a viscosity subsolution if $u(0,\cdot) \le u_0$, and for all $z = (x,t)$ and any ϕ smooth enough "admissible" test function (cf Ishii-Souganidis 95), such that $u - \phi$ has a maximum at z, then if the level set $\{\phi(\cdot,t) = \phi(z)\}$ is noncritical we have

$$\phi_t(z) + |D\phi(z)|\,\kappa\,(x, \{y : \phi(y,t) \ge \phi(z)\}) \le 0$$

while if $D\phi(z) = 0$, then $\phi_t(z) \le 0$. (A supersolution is defined in a similar way.)

Existence: (almost) nothing to add...

The existence follows from Perron's method after some (easy) estimates which guarantee that compactly supported (bounded, uniformly continuous) functions should stay so for some time. In fact, to guarantee existence for all time, we need an additional assumption: for all ρ

$$\min_{x \in \partial B_\rho} \min\{\kappa(x, B_\rho), -\kappa(x, \mathbb{R}^N \setminus B_\rho)\} \geq -K > -\infty$$

(otherwise balls may blowup in finite time).

How to define the associated "curvature flow"?

For studying uniqueness the original definition of a solution is (as usual) not practical, since the class of test functions is very restricted. We need to "extend" κ. The natural extension is as follows (2nd order variant of Cardaliaguet '11.):

Envelopes of curvatures

For $F \subseteq \mathbb{R}^N$ with compact boundary, (p, X) second-order superjet of χ_F at x,

$$\kappa_*(x, p, X, F)$$
$$= \sup \{\kappa(x, E) \;:\; E \text{ smooth}, E \supseteq F\,, (p, X) \text{ subjet of } \chi_E \text{ at } x\}$$

and (p, X) a subjet of χ_F at x,

$$\kappa^*(x, p, X, F)$$
$$= \inf \left\{\kappa(x, E) \;:\; E \text{ smooth}, \mathring{E} \subseteq F\,, (p, X) \text{ superjet of } \chi_E \text{ at } x\right\}.$$

These can be shown to be lsc/usc envelopes in an appropriate topology (Hausdorff+stronger condition near x).

Second definition of a solution

Then, it is shown that u is a viscosity subsolution if and only if for all (x, t) in $\mathbb{R}^N \times (0, T)$, if (a, p, X) is a parabolic superjet of u at (x, t) and $p \neq 0$, then

$$a + |p|\kappa_* (x, p, X, \{y : u(y, t) \geq u(x, t)\}) \leq 0.$$

A similar statement holds for supersolutions.

(Similar to Slepčev 03 but needs quite strong assumptions on the Hamiltonian.)

Uniqueness

Requires in general more assumptions, unless the flow is intrinsically first order (in which case the approach of Cardaliaguet '11 applies).

We say that the curvature is a 1st order curvature if for $C^{1,1}$ *sets,* $\kappa_* = \kappa^*$ *and does not depend on the second order term of the sub/superjet ("X").*

This is the case for the non-local flow of the fractional norm (which will also satisfy the 2nd order properties for uniqueness) and for the capacity. In this case, Ilmanen's interposition lemma allows to show that if $F \subset G$ and $x \in \partial F$, $y \in \partial G$ are points of minimal distance ($x \neq y$), then

$$\kappa_*(x, x - y, X, F) \geq \kappa^*(y, x - y, Y, G)\,.$$

Uniqueness 2

For intrinsically 2nd order evolutions we need a stronger continuity property, namely that for any smooth diffeomorphism Φ:

$$|\kappa(x, E) - \kappa(\Phi(x), \Phi(E))| \leq \omega_R(\|\Phi - I\|_\bullet)$$

if E has an interior/exterior ball condition of radius R at x. Then one can transfer a similar property to the κ_*, κ^*, and a comparison property as before.

The proof of uniqueness then is built upon standard arguments, but we need the strong continuity property to put in correspondence points of second differentiability of level sets of a sub and super-solution, obtained by Jensen's Lemma.

Uniqueness 2

The strong continuity property is true for all our examples but, maybe, the capacity. In particular, we get uniqueness for the flow of Barchiesi et al. built upon the “pre-Minkowski content”.

Approximation

By construction, our flows are derived from a potential (J), although in fact the existence/uniqueness result are true also for nonvariational κ's satisfying the correct assumptions on smooth sets (monotonicity, continuity).

On the other hand, if κ *is* a first variation, it turns out that one can show easily that the flow is approximated by a variational minimizing movements scheme (cf Almgren-Taylor-Wang, Luckhaus-Sturzenhecker, 90-95 for perimeters).

Approximation

One fixes a time-step $h > 0$, and given E_0 a (compact) set, one defines E^{n+1} from E^n, $n \geq 0$ by solving

$$\min_E J(E) \; + \; \frac{1}{h} \int_E d_{E^n}$$

where $d_F(x) = \operatorname{dist}(x, F) - \operatorname{dist}(x, F^c)$ is the signed distance to the boundary of F.

Then one defines $E_h(t) := E_{[t/h]}$ and tries to send $h \to 0$. (Done in many situations, most recent work for instance T. Eto-Y. Giga-K Ishii in Anisotropic/unbounded cases, 2012.)

This scheme enjoys a comparison property thanks to $(SUBM)$.

General approximation result

We start from u_0, bounded, uniformly continuous, with compact support. For each level set we apply the scheme $[t/h]$ times to obtain the corresponding level set of $u^h(t,x)$ (thanks to the comparison property). Then:

Theorem As $h \to 0$, u^h converges locally uniformly to u the solution of the evolution starting from u_0.

(Rem: in case uniqueness is unclear, then convergence of subsequences to a solution still holds.)

Remarks

- The fact that u^h remains bounded, uniformly continuous in space is obvious from the translation invariance which guarantee that the level sets of u^h do not get closer.
- The fact u^h is also u.c. in time (in an appropriate sense) follows from the continuity of the curvature, which guarantee that balls do not disappear instantaneously.
- The local "subgradient" property of the curvature seems here essential to show that the limit is a viscosity solution: one considers a contact with a smooth test function, and use this property to transfer the minimality for the level set of u_h into an equation for the level set of the touching test function. (*No crystals!*)
- The figures of slide 15 have been computed using this approximation.

Conclusions

- Quite general assumptions on a "perimeter" and "curvature" are enough to yield existence of an associated geometric flow;
- and uniqueness, with slightly stronger assumptions;
- A "standard" backwards Euler scheme "always" converges;
- Still, the necessary continuity properties of a curvature are unclear (ex: unified framework for $Per(E)$ and $\mathrm{cap}(E)$? — with straight application to $Per(E) + \mathrm{cap}(E)$? Shape Optimization, *cf* Cardaliaguet-Ley'08);
- Nonlocal flows without translational invariance? (ex: Hele-Shaw, Cardaliaguet-Rouy'01, Mellet-Kim'08, Shape Optimization)

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Thank you for your attention