Some old and new perspectives for differential games

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SOME OLD AND NEW PERSPECTIVES FOR DIFFERENTIAL GAMES

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1. ZERO SUM DIFFERENTIAL GAMES

DYNAMICS:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \alpha, \beta) & (0 \leq t \leq \tau) \\ \mathbf{x}(0) = x, \end{cases}$$

CONTROLS:

$$\begin{cases} \alpha : [0, \tau] \to A = \text{ control for the maximizing player} \\ \beta : [0, \tau] \to B = \text{ control for the minimizing player} \end{cases}$$

PAYOFF:

$$J^{\alpha,\beta}(x) = \int_0^\tau r(\mathbf{x}, \alpha, \beta)\, dt + g(\mathbf{x}(\tau)).$$

r is the **running payoff** and g is the **terminal payoff**.

MINIMAX CONDITION:

$$\min_{b\in B}\max_{a\in A}\{\mathbf{f}(x,a,b)\cdot p + r(x,a,b)\} = \max_{a\in A}\min_{b\in B}\{\mathbf{f}(x,a,b)\cdot p + r(x,a,b)\}$$

for all x, p.

VALUE FUNCTION:

$$u(x) = \text{payoff to max player, if each plays optimally}$$

THEOREM

The value function u, if continuous, solves the **Hamilton–Jacobi–Isaacs (HJI)** *equation*

$$H(Du, x) = 0$$

in the viscosity sense, for the **game theory Hamiltonian**

$$H(p,x) = -\min_{b\in B}\max_{a\in A}\{\mathbf{f}(x,a,b)\cdot p + r(x,a,b)\}.$$

References

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- J. Lewin, *Differential Games. Theory and methods for solving game problems with singular surfaces*, Springer, 1994.

- A. Melikyan, *Generalized Characteristics of First Order PDEs. Applications in optimal control and differential games*, Birkhäuser, 1998.

Problems for theory applied to “real” examples

- Hamiltonian $H = H(p, x)$ is nonconvex in p.

- Hamiltonian H is not C^1 (and therefore problems with characteristics = optimal trajectories).

- The value function u need not be continuous.

- Boundary conditions are unclear and/or hard to interpret in viscosity sense.

Failure of minimax condition is usually not a problem, at least for examples in Isaacs' and Lewin's books.

AN (UNNAMED) EXAMPLE OF ISAACS

DYNAMICS:

$$\begin{cases} \dot{x} = c(y) + \cos\alpha \\ \dot{y} = 2\beta + \sin\alpha, \end{cases}$$

where $0 \le \alpha \le 2\pi$ and $-1 \le \beta \le 1$.

$$y \mapsto c(y) \text{ is positive, increasing.}$$

The game is played in the upper half plane $U := \mathbb{R}^2 \cap \{y \ge 0\}$.

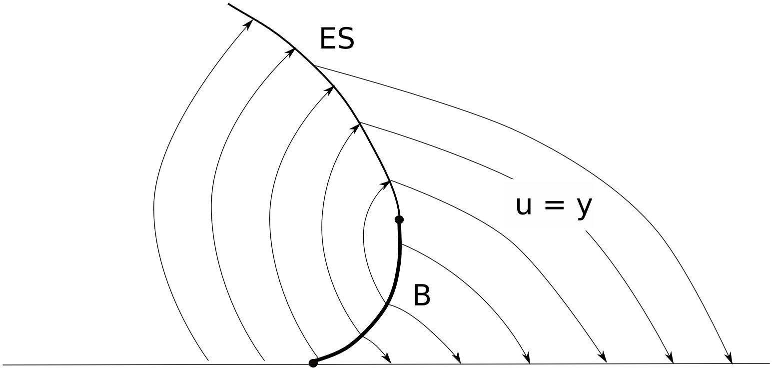
PAYOFF:

$$J^{\alpha,\beta} = \tau = \text{ exit time from } U \text{ along the positive } x\text{-axis.}$$

HJI EQUATION: The value function u ($=$ time to exit) solves

$$H(Du, y) = -c(y)u_x - |Du| + 2|u_y| - 1 = 0 \quad \text{in } U.$$

Boundary condition is $u = 0$ on $\{x \geq 0, y = 0\}$.



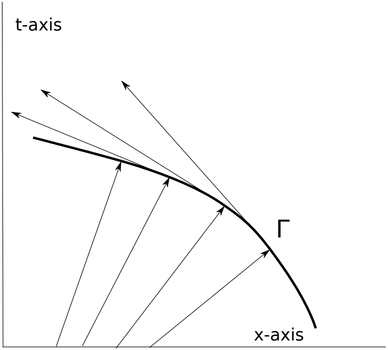
$B =$ **barrier**; $ES =$"**equivocal surface**"

2. NONCONVEX HAMILTON-JACOBI PDE

Study the initial-value problem

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

where H is smooth, but not convex.



The central difficultly for nonconvex Hamilton–Jacobi PDE is that backwards characteristics can hit "shocks", along which the gradient is discontinuous.

New work of F. Rezakhanlou: Backwards characteristics as martingales.

FORMULAS FOR SOLUTIONS:

• If H is convex,

$$u(x,t) := \inf_{y \in \mathbb{R}^n} \left\{ g(y) + tL\left(\tfrac{x-y}{t}\right) \right\} \qquad \text{(Hopf–Lax)}$$

where $L = H^*$

• If g is convex and has linear growth,

$$u(x,t) := \sup_{z \in \mathbb{R}^n} \{ x \cdot z - g^*(z) - tH(z) \} \qquad \text{(Hopf)}$$

THEOREM (ENVELOPE REPRESENTATION FORMULA)

For $\mathcal{L}^{n+1}$-a.e. point $(x,t) \in \mathbb{R}^n \times (0,\infty)$ there exists a Radon probability measure $\gamma_{x,t}$ on $\mathbb{R}^n$ such that

$$u(x,t) = \int_{\mathbb{R}^n} g(y) + (x-y) \cdot Dg(y) - tH(Dg(y))\, d\gamma_{x,t}.$$

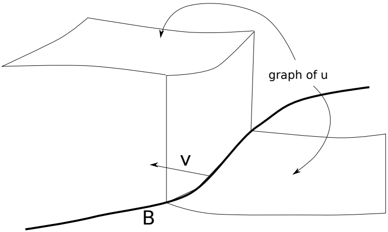
Furthermore,

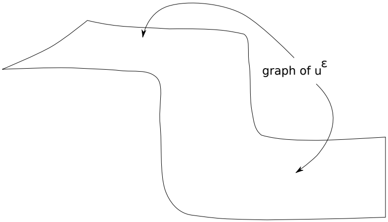
$$u_t(x,t) = -\int_{\mathbb{R}^n} H(Dg(y))\, d\gamma_{x,t}, \quad Du(x,t) = \int_{\mathbb{R}^n} Dg(y)\, d\gamma_{x,t},$$

and

$$H\left(\int_{\mathbb{R}^n} Dg(y)\, d\gamma_{x,t}\right) = \int_{\mathbb{R}^n} H(Dg(y))\, d\gamma_{x,t}.$$

BARRIERS (= surfaces of discontinuity)





$$H(Du, x) = 0 \quad \text{in } U.$$

APPROXIMATION:

$$-\epsilon \Delta u^{\epsilon} + H(Du^{\epsilon}, x) = 0 \quad \text{in } U.$$

Assume:

$$u^{\epsilon} \leq u^{+} + o(1), \ u^{\epsilon} \geq u^{-} - o(1).$$

The **recession function** associated with $H = H(p,x)$ is

$$K(p,x) := \lim_{\lambda\to\infty} \frac{H(q+\lambda p, x)}{\lambda}.$$

THEOREM

(i)

$$\begin{cases} H(Du^{+} + \lambda\nu, x) \leq 0 \\ H(Du^{-} + \lambda\nu, x) \geq 0 \end{cases} \quad \textit{for all } \lambda \geq 0.$$

(ii) *In particular,*

$$K(\nu, x) = 0.$$

This is a PDE for the barrier (Isaacs). We can in principle solve for the location of the discontinuities, before we solve the full HJI equation!

3. NONZERO SUM GAMES: PRINCIPAL-AGENT PROBLEMS

SANNIKOV'S MODEL (simplified)

DYNAMICS:

$$dX = Adt + dB,$$

where B is a one-dimensional Brownian motion.

CONTROLS:

$$\begin{cases} C = \text{ principal's control } (= \text{payments to agent}) \\ A = \text{agent's control of dynamics.} \end{cases}$$

PAYOFFS:

$$\begin{cases} J^P = \text{principal's payoff} = E\left(\int_0^\infty e^{-t}(A - C)\,dt\right) \\ J^A = \text{agent's payoff} = E\left(\int_0^\infty e^{-t}(g(C) - h(A))\,dt\right) \end{cases}$$

How can we make the agent do what we want?

In particular, how can the principal encourage the agent to perform the action A^*?

KEY POINT #1: The agent's actions and the Brownian motion are unobservable by the principal; she can only observe X.

DYNAMICS REINTERPRETED:

$$\begin{aligned} dX &= A^* dt + dB \\ &= A dt + dB^A \end{aligned}$$

KEY POINT #2: B is a Brownian motion under the original probability measure P and B^A is a Brownian motion under a new probability measure P^A.

PAYOFFS REINTERPRETED:

$$\begin{cases} J^P = E^A \left(\int_0^\infty e^{-t}(A - C)\, dt \right) \\ J^A = E^A \left(\int_0^\infty e^{-t}(g(C) - h(A))\, dt \right) \end{cases}$$

NEW DYNAMICS: Given $A^* = a(W)$ and $C^* = c(W)$, solve

$$\begin{aligned} dW &= (W - g(C^*) + h(A^*))dt + h'(A^*)dB \\ &= (W - g(C^*) + h(A^*) - A^* h'(A^*))dt + h'(A^*)dX \\ W(0) &= w. \end{aligned}$$

The principal will therefore pay the agent $C^* = c(W)$, depending upon the values of W, which she can compute in terms of X.

THEOREM (Sannikov optimality condition for agent)

Assume

$$h \text{ is convex.}$$

Then it is optimal for the agent to use the control A^*.

New proof of optimality condition for agent

$$
\begin{aligned}
d(e^{-t}W) &= e^{-t}\left[(-g(C^*) + h(A^*))dt + h'(A^*)dB\right] \\
&= e^{-t}\left[(-g(C^*) + h(A^*) + h'(A^*)(A - A^*))dt + h'(A^*)dB^A\right]
\end{aligned}
$$

Integrate from 0 to ∞:

$$
\begin{aligned}
-w = \int_0^\infty e^{-t}(-g(C^*) + h(A^*) + h'(A^*)(A - A^*))\,dt \\
+ \int_0^\infty e^{-t}h'(A^*)\,dB^A
\end{aligned}
$$

Take expected value with respect to P^A and rewrite:

$$
\begin{aligned}
w = E^A\left(\int_0^\infty e^{-t}(g(C^*) - h(A))\,dt\right) \\
+ E^A\left(\int_0^\infty e^{-t}(h(A) - h(A^*) - h'(A^*)(A - A^*))\,dt\right)
\end{aligned}
$$

By convexity, the last term is nonnegative, and is zero for $A = A^*$. □

EXERCISE: Use dynamic programming to rederive this optimality condition. What is the agent's value function?

THEOREM (Optimality condition for principal)

The principal's value function $u = u(w)$ solves the HJB equation

$$\max_{a,c}\left\{\frac{(h'(a))^2}{2}u'' + (w - g(c) + h(a))u' - u + a - c\right\} = 0$$

Select

$$a(w), c(w)$$

giving the max in the HJB equation.

- I.Ekeland, *How to build stable relationships with people who lie and cheat*, www.ceremade.dauphine.fr/∼ekeland/Articles/13-03-08-ArticleMilano.pdf

Many fascinating PDE/optimality issues remain open.