Necessary Conditions in Dynamic Optimization

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NetCo 2014

New Trends in Optimal Control

Tours, France

23-27 June 2014

NetCo (HJNet + SADCO)

Outline of the talk

- Early Necessary Conditions in Classical C. of V.
- Dynamic Constraints and The Maximum Principle
- Differential Inclusion Problems
- Clarke's Hamiltonian Inclusion for Convex Diff. Inclusions
- Euler Lagrange Inclusion for Non-convex Diff. Inclusions
- Return to the Hamiltonian Inclusion
- Uses of the Hamiltonian Inclusion and Open Questions

Some History

Calculus of Variations:

$$(Q)\begin{cases}\text{Minimize } \int_0^T L(t,x(t),\dot{x}(t))dt \\ \text{over } x(.) \in W^{1,1}[S,T];R^n) \\ \text{satisfying} \\ x(0)=x_0 \text{ and } x(T)=x_1\ .\end{cases}$$

($L : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and points $x_0, x_1 \in \mathbb{R}^n$.)

$W^{1,1}$ $\equiv$ **abs. continuous functions**

Take a minimizer $\bar{x}(.)$.

Assume

- $L(.,.,.)$ is continuous and $L(t,.,.)$ is C^1
- There exist $k(.) \in L^1$ and $\epsilon > 0$ s.t.

$$|\nabla L(x,v)| \leq k(t)$$

for all $(x,v) \in (\bar{x}(t), \dot{\bar{x}}(t)) + \epsilon\mathbb{B}$ $\mathbb{B} \equiv$ **'closed unit ball'.**

Necessary Conditions

Take a minimizer $\bar{x}(.)$ for

Minimize $\{\int_0^T L(t, x(t), \dot{x}(t))dt \mid x(.) \in W^{1,1},\ x(0) = x_0, x(T) = x_1\}$

There exists an absolutely continuous $p(.) : [0, 1] \to R^n$ such that

- $(\dot{p}(t), p(t)) \ = \ \nabla L(t, \bar{x}(t), \dot{\bar{x}}(t))$ a.e

  (The Euler-Lagrange Condition)

- $p(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \max_{v \in \mathbb{R}^n}\{p(t) \cdot v - L(t, \bar{x}(t), v)\}$ a.e.

  (TheWeierstrass condition)

- If $L(t, x, v)$ is indep. of t,

  $p(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \text{const.}$ a.e.

  (Constancy of the Hamiltonian)

Hamilton's Equations

Assume

- $L(t,.,.)$ is C^2
- $L_{vv}(t,x,v)$ is invertible.

Hamiltonian $H(t,x,p) : [0,T] \times R^n \times R^n \to R$:

$H(t,x,p) = p \cdot v(t,x,p) - L(t,x,v(t,x,p))$ (Legendre transformation)

$$(v(t,x,p) := \text{ ‘solution function of } p \cdot v - L_v(t,x,v) = 0\text{’.})$$

Hamilton's equations:

$$(-\dot{p}(t), \dot{\bar{x}}(t)) = \nabla H(t, \bar{x}(t), p(t))$$

Permits study of minimizers via systems of differential equations.

The Optimal Control Problem with Dynamic Constraints

Aerospace applications ('optimal flight trajectories') motived:

Pontryagin Formulation

$$\begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(.) \in W^{1,1}([0,T];\mathbb{R}^n) \\ \qquad\qquad \text{and measurable functions } u(.) : [0,T] \to \mathbb{R}^m \text{ s.t.} \\ \dot{x}(t) \in f(t, x(t), u(t)) \quad a.e. \\ u(t) \in U(t) \quad a.e. \\ (x(0), x(T)) \in C \end{cases}$$

Data: $f(.,.,.) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $U(.) : [0,T] \rightsquigarrow \mathbb{R}^m$, $C \subset \mathbb{R}^n \times \mathbb{R}^n$.

L S Pontryagin

Dynamic constraint, cont.

Take minimizer $(\bar{x}(.), \bar{u}(.))$. Assume

- $f(., ., .)$ is continuous, $f(t, ., u)$ is C^1, C are closed
- Gr $U(.)$ is a Borel set
- there exist $c(.) \in L^1$ and $k_f(.) \in L^1$ such that

$$f(t,x,u) - f(t,x',u)| \leq k(t)|x - x'|,$$

$$|f(t,x,u)| \leq c(t)$$

for all $x, x' \in \bar{x}(t) + \epsilon B$, $u \in U$, a.e. $t \in [0, T]$.

Write $H., ., ., .) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$H(t,x,p,u) \;=\; p \cdot f(t,x,u)\,.$$

(Un-maximized Hamiltonian)

Maximum Principle (L S Pontragin et al.) 1959

Take a minimizer $(\bar{x}(.), \bar{u}(.))$.

There exist $\lambda \geq 0$ and $p(.) \in W^{1,1}$ such that

- $(\lambda, p(.)) \neq (0,0)$ (Nontriviality of multipliers)
- $-\dot{p}(t) = H_x(t, \bar{x}(t), \bar{u}(t))$ a.e. (Costate Equation)
- $H(t, \bar{x}(t), \bar{u}(t)) = \max\{H(t, \bar{x}(t), u) \,|\, u \in U\}$ a.e.

  (Weierstrass Condition)
- $(p(0), -p(T)) = \lambda g_x(\bar{x}(0), \bar{x}(T)) + \xi$

  where ξ is 'normal' to C at $(\bar{x}(0), \bar{x}(T))$.

  (Transversality Condition)

- $H(t, \bar{x}(t), p(t), \bar{u}(t)) = \max\{H(t, \bar{x}(t), p(t), u) \,|\, u \in U(t)\}$ a.e.

  (Constancy of Maximized Hamiltonian)

Problem Reformulation

The dynamic constraint

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ u(t) \in U(t) \end{cases}$$

is a constraint on possible velocities for a give state, and can be captured as a differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

where $$F(t, x) \; := \; \{f(t, x, u) \,|\, u \in U\} \, .$$

The optimal control problem can be expressed directly in terms of $F(.)$:

$$\begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(.) \in W^{1,1}([0, T] : \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

Reformulation cont.

Since 1970's: interest in deriving necessary conditions based on a 'differential inclusion' formulation of the dynamic constraint:

Goal: Express both hypotheses and necessary conditions directly in terms of $F(.)$ instead of ingredients of parameterization:

$$F(t,x) := f(t,x,U)\ .$$

Reasons:

- New kinds of necessary conditions
- Improvements on the Maximum Principle (validity under reduced hypotheses)
- Better tools for studying properties of optimal trajectories, value functions, etc.

First Ideas (R T Rockafellar)

First Ideas (R T Rockafellar)

Re-write the 'differential inclusion' problem

$$\begin{cases} \text{Minimize } \int_0^T L(t, x(t), \dot{x}(t))dt + g(x(0), x(T)) \\ \text{over } x(.) \in W^{1,1}([0,T], \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

as a **Generalized Problem of Bolza:**

$$\begin{cases} \text{Minimize } \tilde{g}(x(0), x(T)) \\ \text{over } x(.) \in W^{1,1}([0,T], \mathbb{R}^n) \end{cases}$$

in which ('hard penalization')

$$L(t,x,v) := \begin{cases} 0 & \text{if } v \in F(t,x) \\ +\infty & \text{otherwise} \end{cases}, \quad \tilde{g}(x_1) = \begin{cases} g(x_0, x_1) & \text{if } (x_0, x_1) \in C \\ +\infty & \text{otherwise} \end{cases}$$

('Generalized' because it involves extended valued functions)

Optimality Conditions for Fully Convex Problems

Theorem (Rockafellar 1970, Fully Convex Case)

Let $\bar{x}(.)$ be a solution to the **Generalized Bolza Problem**. Assume

- $\tilde{L}(t,.,.)$ and $\tilde{g}(.,.)$ are proper convex functions. ('full convexity')
- $\tilde{L}(t,x,.)$ has superlinear growth, $g(.,.)$ linearly bounded below.
- rel interior $\{\tilde{C}\} \cap$ rel interior $\{\mathcal{R}\} \neq \emptyset$ ('controllability')

$(\tilde{C} = \{(x_0,x_1)\,|\,\tilde{g}(x_0,x_1) < \infty\}$ and $\mathcal{R} =$ 'reachable set').

Then there exist $p(.) \in AC$ such that

- $(\dot{p}(t),p(t)) \in \partial L(t,\bar{x}(t),\dot{\bar{x}}(t))$ a.e. (Euler-Lagrange Condition)
- $(p(0),-p(T)) \in \partial \tilde{g}(\bar{x}(0),\bar{x}(T))$

$\partial f(\bar{x})$ is subdifferential of convex analysis:

$$\partial f(\bar{x}) := \{\xi \,|\, f(x) - f(\bar{x}) \geq \xi \cdot (x - \bar{x}) \text{ for all } x \in R^n\}$$

The Fully Convex Case Cont.

Approach:

For a convex optimization problem, Fenchel Duality provides dual problem, conditions for ‘strong duality’, ‘multiplier’ interpretations of solutions to dual problem, etc.

- Compute subdifferentials of integral functionals involving extended valued integrands
- Check no duality gap and Fenchel Dual has solutions, which can identified with co-state arcs.

Hamiltonian Conditions

For $L(t,x,.)$ convex, the Hamiltonian:

$$H(t,x,p) = p \cdot v(t,x,p) - \tilde{L}(t,x,v(t,x,p)) \quad \text{(Legendre transformation)}$$

can expressed

$$H(t,x,p) = \sup\{p \cdot v - L(t,x,v) \,|\, v \in \mathbb{R}^n\}$$

- $L(t,.,.)$ convex $\implies$ $H(t,x,.)$ is convex, and $H(t,.,p)$ is concave

For $L(t,.,.)$ convex, Rockafellar showed under earlier conditions:

$$-\dot{p}(t) \in \tilde{\partial}_x H(t,\bar{x}(t),p(t)), \;\; \dot{\bar{x}}(t) \in \partial_p H(t,\bar{x}(t),p(t)).$$

'Hamiltonian inclusion'

$(\tilde{\partial}_x H := -\partial_x(-H)$ is 'concave' subdifferential)

Generalization to Non-convex Problems?

$$(P)\begin{cases}\text{Minimize } g(x(0),x(T))\\ \text{over } x(.)\in W^{1,1}([0,T];\mathbb{R}^n)\text{ satisfying}\\ \dot{x}(t)\in F(t,x(t))\\ (x(0),x(T))\in C\end{cases}$$

Major research theme (1973 -): **Derive necessary conditions for this non-convex problem**, e.g.

$$(\dot{p}(t),p(t))\in \text{`}\partial_{x,v}\text{'}\tilde{L}(t,\bar{x}(t),\dot{\bar{x}}(t))$$ Euler Lagrange inclusion

or $$(\tilde{L}(x,v)=\mathbb{I}_{\text{Graph}\,F(.)}(x,v))$$

$$(-\dot{p}(t),\dot{\bar{x}}(t))\in \text{`}\partial\text{'}\tilde{H}(t,\bar{x}(t),p(t))$$ Hamiltonian inclusion

- **How to interpret '∂'$\tilde{L}$ and '∂'H for non-convex functions?**

Concepts of Local Minimizer

'$x(.)$ *feasible*' F-trajectory $x(.)$ means '$(x(0), x(T)) \in C$'.

Definition. A feasible F-trajectory $\bar{x}(.)$ is a:

- L^∞-local minimizer if there exists $\epsilon > 0$ such that

$$g(\bar{x}(0), \bar{x}(T)) \le g(x(0), x(T))$$

  for all feasible $x(.)$'s s.t. $||x(.) - \bar{x}(.)||_{L^\infty} \le \epsilon$

- $W^{1,1}$-local minimizer if there exists $\epsilon > 0$ such that

$$g(\bar{x}(0), \bar{x}(T)) \le g(x(0), x(T))$$

  for all feasible $x(.)$'s s.t. $||x(.) - \bar{x}(.)||_{W^{1,1}} \le \epsilon$.

Choice of Subdifferentials, etc.

Take a close set $A \subset X$ (X top. vector space) and $\bar{x} \subset A$.

How to define tangent and normal cones 'T_A'$(\bar{x})$ and 'N_A'$(\bar{x})$?

Then define the corresponding subdifferential of l.s.c.

$f : X \to R \cup \{+\infty\}$ at $x \in \operatorname{dom} f(.)$:

$$\text{'}\partial f\text{'}(\bar{x}) \;:=\; \{\xi \,|\, (\xi, -1) \in \text{'}N_{\operatorname{epi} f(.)}\text{'}(\bar{x}, f(\bar{x}))\}\,.$$

Requirements:

- 'N_A'$(\bar{x})$ coincides with normal cone of convex analysis, etc, for convex sets/functions, etc.
- There is a suitably rich calculus ('$\partial(f_1 + f_2)(\bar{x}) \subset$ '∂'$f_1(\bar{x}) + f_2(\bar{x})$, . . .
- 'robustness'
- $f(\bar{x}) \le \inf\{f(x) \,|\, x \in X\} \implies 0 \in$ '∂'$f(\bar{x})$.

Nonsmooth Analysis: Francis Clarke

Clarke's Tangent Cone and Normal Cone

Key advance: Clarke's tangent cone and normal cone

$$T_A^C(\bar{x}) := \{\xi \,|\, \forall y_i \xrightarrow{A} \bar{x},\ \exists x_i \xrightarrow{A} \bar{x} \text{ and } \epsilon \downarrow 0 \text{ s.t. } \epsilon^{-1}(y_i - x_i) \rightarrow \xi\}$$

$$N_A^C(\bar{x}) = \{\eta \,|\, \eta \cdot \xi \leq 0 \text{ for all } \xi \in T_A^C(\bar{x})\}\,.$$

'polar cone' of Clarke tangent cone

$$\partial^C f(\bar{x}) := \{\xi \,|\, (\xi, -1) \in N^C_{\text{epi}\,(.)}(\bar{x}, f(\bar{x}))\}\,.$$

- **Satisfies main 'requirements'**
- **Suitable vehicle for derivation of necessary conditions in optimal control.**

The Hamiltonian Inclusion

$$(P) \begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(.) \in W([0,T] : \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

Standard Hypotheses (for given F-trajectory $\bar{x}(.)$): for some $c_F(.), k_F(.) \in L^1$ s.t.

(H1) : $F(t,x)$ is closed for each (t,x), $F(.,x)$ is meas., C is closed, $g(.,.)$ is locally LIpschitz

(H2): $F(t,x) \subset c_F(t)B$

for all $x \in \bar{x}(t) + \epsilon B$, a.e. $t \in [0,T]$.

(H3): $F(t,x) \subset F(t,x') + k_F(t)|x - x'|B$

for all $x, x' \in \bar{x}(t) + \epsilon B$, a.e. $t \in [0,T]$.

The Hamiltonian Inclusion, cont.

Theorem (Clarke 1973)

Take an L^∞-local minimizer $\bar{x}(.)$. Assume standard hypotheses and

(C): $F(t,x)$ is convex for all x.

Then there exists $p(.) \in W^{1,1}$ and $\lambda \geq 0$ such that

(i): $(p(.), \lambda) \neq (0,0)$

(ii): $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial^C_{x,p} H(t, \bar{x}(t), p(t))$ a.e.

(iii) $(p(0), -p(T)) \in \lambda \partial^C g(\bar{x}(0), \bar{x}(T)) + N^C_C(\bar{x}(0), \bar{x}(T))$

Nonconvex Normal Cones: A.Ya Kruger and B. Mordukhovich

Alexander Ya. Kruger

Boris Mordukhovich

Non-Convex Normal Cones and Subdifferentials

A non-convex normal cone (Kruger/Mordukhovich):

$$N_A(\bar{x}) \; := \; \limsup_{x' \to} \{\xi \,|\, \xi \cdot (x - x') \leq o(|x - x'|)\}$$

Hence, subdifferential $\partial f(\bar{x})$:

$$\partial f(\bar{x}) \; = \; \{\xi \,|\, (\xi, -1) \in N_{\text{epi}\, F(.)}\}(\bar{x}, f(\bar{x})).$$

Relation with Clarke normal cone and subdifferential:

$$N^C_A(\bar{x}) = \text{co}\, N^C_A(\bar{x}) \text{ and } \partial^C f(\bar{x}) = \text{co}\, \partial f(\bar{x})$$

(if $f(.)$ Lipschitz near $\bar{x}$).

Normal Cones and Subdifferentials (in this sense) have a rich calculus, and can be used to formulate refined necessary conditions.

Hamiltonian Inclusion, First Refinement

Theorem (First Refinement: improved transversality condition)

The assertions of the Hamiltonian Inclusion can be strengthened as follows:

There exists $p(.) \in W^{1,1}$ and $\lambda \geq 0$ such that

(i): $(p(.), \lambda) \neq (0, 0)$

(ii): $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co}\, \partial H(t, \bar{x}(t), p(t))$ a.e.

(iii) $(p(0), -p(T)) \in \lambda\, \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

Example: for $f(x) = -|x|$ and $\bar{x} = 0$,

$$\partial f(\bar{x}) = \{-1\} \cup \{1\} \overset{\text{strict}}{\subset} [-1, +1] = \partial^C f(\bar{x})$$

Two Further Refinements

1. Improvement of of Convexity Condition:
The Hamiltonian inclusion

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co}\, \partial_{x,p} H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

Note: convexification w.r.t. two variables can be replaced by

$$(-\dot{p}(.) \in \text{co}\{(\xi \,|\, (\xi, \dot{\bar{x}}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

which involves convexification w.r.t. only one variable

2. Strengthen Topology Defining 'Local Minimizer'

The Hamiltonian inclusion is valid for $W^{1,1}$-local minimizers (not just L^{∞}-local minimizers)

Theorem (Loewen Rockafellar, 1994)

Take an $W^{1,1}$-local minimizer $\bar{x}(.)$. Assume standard hypotheses and

(C): $F(x)$ is convex for all x.

Then there exists $p(.) \in AC$ and $\lambda \geq 0$ such that

(i): $(p(.), \lambda) \neq (0, 0)$

(ii): $(-\dot{p}(t) \in \text{co}\{(\xi \,|\, (\xi, \dot{\bar{x}}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t))$ a.e.

(iii) $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

Note: **(iii)** $\implies p(t) \cdot \dot{\bar{x}}(t) = \max\{p(t) \cdot v \,|\, v \in F(\bar{x}(t))\}$

(Weierstrass Condition)

The Generalized Euler-Lagrange Inclusion

Research direction since early 1990's: Find necessary conditions when $F(x)$ is non-convex.

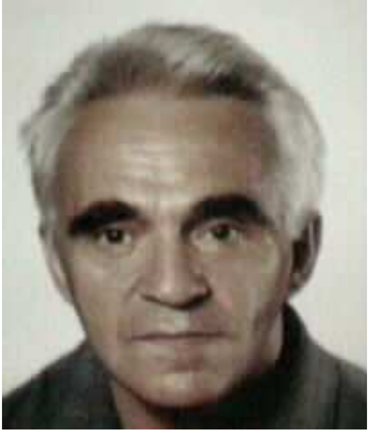
Generalizations of the classical Euler-Lagrange condition have been the most fruitful.

What should such conditions look like?

Here, generalizations of the classical Euler-Lagrange condition have been the most fruitful.

(Mordukhovich, Ioffe, Rockafellar, Vinter)

E-L inclusion: B. Mordukhovich and A.D. Ioffe



Generalized E-L Condition, Cont.

Differential inclusion problem formulated as extended Bolza problem:

Minimize $\{\int_0^T L(t, x(t), \dot{x}(t)) + g(x(0), x(1))$ over $x(.) \in AC\}$

where

$$L(x, v) \;=\; \mathbb{I}_{\text{Gr } F(t,.)}(x, v)$$

But

$$\partial \mathbb{I}_{\text{Gr } F(t,.)}(x, v) = N_{\text{Gr } F(t,.)}(x, v)$$

So natural generalization of Euler Lagrange condition is

$$(\dot{p}(t), p(t)) \in \text{co}\, N_{\text{Gr } F(.)}(x, v) \quad ?$$

Generalized E-L Condition, Cont.

Fully convexified E-L Inclusion is inadequate condition!

Example: $F(x) = \{|x|\}$, $\bar{x} = 0$ **and** $\bar{v} = 0$ **(Lipschitz data)**

$$\text{Gr}\, F(.) = \{(x, |x|) \,|\, x \in R\}$$

$$N_{\text{Gr}\, F(.)}(\bar{x}, \bar{v}) = \{\alpha(1,1) : \alpha \in R\} \cup \{\alpha(-1,1) : \alpha \in R\}$$

$$\text{co}\, N_{\text{Gr}\, F(.)}(\bar{x}, \bar{v}) = R \times R$$

So $(\dot{p}, p) \in \text{co}\, N_{\text{Gr}\, F(.)}(\bar{x}, \bar{v}) \implies (\dot{p}, p) \in R \times R$

But **no useful information**

$$\dot{p} \in \text{co}\{\xi \,|\, (\xi, p) \in N_{\text{Gr}\, F(.)}(0,0)\} \implies |\dot{p}| \le |p|$$

(partially convexification gives much more information)

'right condition'

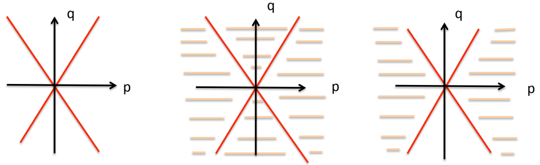
Normal cones to Gr $F(.)$

Take $F(x) = \{|x|\}$

(a): $(q, p) \in N_{\mathrm{Gr}\,F(.)}(0, 0)$

(b): $(q, p) \in \mathrm{co}\, N_{\mathrm{Gr}\,F(.)}(0, 0)$

(c): (q, p) s.t. $q \in \mathrm{co}\{\xi \,|\, (\xi, p)\, N_{\mathrm{Gr}\,F(.)}(0, 0)\}$



(a) (b) (c)

Nec. Condition for Non-Convex $F(x)$

Theorem (Ioffe 1997)

Take an $W^{1,1}$-local minimizer $\bar{x}(.)$. Assume standard hypotheses
Then there exists $p(.) \in AC$ and $\lambda \geq 0$ such that

(i): $(p(.), \lambda) \neq (0, 0)$

(ii): $(-\dot{p}(t) \in \text{co}\{(\xi \,|\, (\xi, p(t)) \in N_{\text{Gr}\, F(t,.)}(\bar{x}(t), \bar{x}(t))$ a.e.

(iii) $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

(iv): $p(t) \cdot \dot{\bar{x}}(t) = \max\{p(t) \cdot v \,|\, v \in F(t, \bar{x}(t))\}$ a.e.

(Earlier version without Weierstrass condition due to Morduckhovich 1995. Extensions allow unbounded $F(.)$, state constraints etc. Ioffe, Vinter, Clarke . .)

Notes: No convexity of $F(.)$. Must add Weierstrass Cond.

Duality

Different necessary conditions vary according to costate equation or inclusion.

Euler Lagrange inclusion subsumes earlier 'Hamiltonian inclusion' conditions

Theorem (Rockafellar 1996) Assume standard conditions

and

(C): $F(x)$ is convex for each x (convexity)

Then

$$\mathrm{co}\,\{q\,|\,(q,p(t))\in N_{\mathrm{Gr}\{F(t,.)\}}(\bar{x}(t),\dot{\bar{x}}(t))\}\;\subset$$
$$\mathrm{co}\,\{-q\,|\,(q,\bar{x}(t))\in\partial_{x,p}H(t,\bar{x}(t),p(t))\}\;.$$

Necessary Conditions: State of the Art 2005

Assume:

- Standard hypotheses
- Assume $\bar{x}(.)$ is a $W^{1,1}$ local minimizer

Then there exists (non-trivial) $(p(.) \in W^{1,1}$, $\lambda \geq 0$, satisfying transversality and Weierstrass conditions, s.t.

$$(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, p(t)) \in N_{\text{Gr}\, F(t,.)}(\bar{x}(t), \dot{\bar{x}}(t)) \text{ a.e.} \quad (1)$$

If also (Euler-Lagrange Inclusion)

- (C): $F(t, x)$ is convex for all x

Then (1) implies

$$(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, \dot{\bar{x}}(t)) \in \partial H(\bar{x}(t)' p()) \text{ a.e.}$$

(Partially Convexified Hamiltonian Inclusion)

Open Question, State of the Art 2005

Rockafellar Duality is only valid for convex $F(x)$'s.

Open the question: Are the conditions

$(-\dot{p}(.), \dot{\bar{x}}(t) \in \text{co}\, \partial_{x,p} H(t, \bar{x}(t), p(t))$ a.e.

or **(Fully Convexified Hamitonian Inclusion)**

$(-\dot{p}(t) \in \text{co}\{(q \,|\, (q, \dot{\bar{x}}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t))$ a.e.

(Partially Convexified Hamitonian Inclusion)

valid, when $F(x)$ is not convex?

This was partly resolved by Clarke in 2005.

Hamiltonian Inclusion for Non-Convex $F(x)$

Theorem (Clarke 2005)

Assume Standard Hypotheses and

- $\bar{x}(.)$ is L^∞-local minimizer.

Then there exists non-trivial $p(.) \in W^{1,1}$ and $\lambda \geq 0$ such that

(i): $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co}\, \partial_{x,p} H(t, \bar{x}(t), p(t))$ a.e.

(Fully Convexified Hamiltonian inclusion)

(ii) $(p(0), -p(T)) \in \lambda\, \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

Notes: L^∞-local minimizer and fully convexified Hamiltonian Inclusion

Remaining Open Questions

This work validates (for non-Convex $F(x)$'s)

The Fully Convexified Hamiltonian Inclusion for L^∞-minimizers.

BUT, It appeared to be an intermediate result to proving

The Partially Convexified Hamiltonian Inclusion for $W^{1,1}$-minimizers.

The status of these conditions is clarified by a recent counter-example.

Proposition. (Vinter 2014)

For some choice of data for (P) satisfying the Standard Hypotheses such that

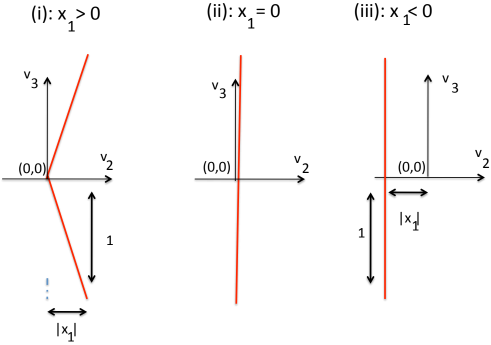
$F(x)$ **is not convex**,

and some F-trajectory $\bar{x}(.)$, we have

- $(\bar{x}(.)$ is a $W^{1,1}$ local minimizer,
- $(\bar{x}(.)$ is not a L^{∞} local minimizer,
- There exist non-trivial multipliers satisfying (the transversality condition and) the fully convexified Hamiltonian inclusion.
- There exist do not exist non-trivial multipliers satisfing t the fully convexified Hamiltonian inclusion.

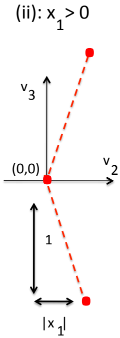
Example

$$(E)\begin{cases}\text{Minimize } \frac{1}{2}x_1(1) - x_2(1)\\ \text{over } x(.) = (x_1(.), x_2(.), x_3(.)) \text{ satisfying}\\ \dot{x}_1(t) = 0, (\dot{x}_2, \dot{x}_3) \in \tilde{F}(x_1)\\ x_1(0) \geq 0\end{cases}$$



(v2, v3) coordinates of points (v1, v2, v3) ε F(x)

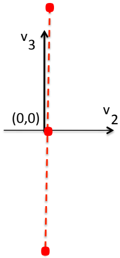
Differential Inclusion $\tilde{F}(x_1)$, $x_1 > 0$



(v_2, v_3) coordinates of points $(v_1, v_2, v_3) \in F(x)$

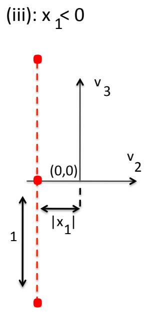
Differential Inclusion $\tilde{F}(x_1)$, $x_1 = 0$

(ii): $x_1 = 0$



(v_2, v_3) coordinates of points $(v_1, v_2, v_3) \in F(x)$

Differential Inclusion $\tilde{F}(x_1)$, $x_1 < 0$



(v 2, v 3) coordinates of points (v 1, v 2, v 3) ε F(x)

Example , continued

The example is very benign-seeming:

- Free right end-point problem.
- Can be re-formulated as a ‘finite Lagrangian problem’
- $F(x)$ is not convex for all x, BUT $F(\bar{x}(t))$ is convex

Note that

- **’Non-smooth’ of $x \to F(x)$ is essential feature of example**

Implications

Fully convexified Hamiltonian Inclusion is 2nd Class Citizen!

- The Hamilton Inclusion is valid, in general, only if $\bar{x}(.)$ is a L^∞ minimizer
- Gives new information in some cases:

  'Hamiltonian inclusion is not satisfied'

  $\Longrightarrow$ '$\bar{x}(.)$ is not a minimizer.'
- Degeneracy:Conditions can be given for the Hamiltonian inclusion to be non-degenerate when all other known conditions are degenerate.
- Relaxation: 'normality of Hamiltonian inclusion' $\Longrightarrow$ 'relaxation does not reduce minimum cost'

  (not known for other necessary conditions)

Some Concluding Remarks

- We have focused on necessary conditions for the most basic of differential inclusion problems. These are the building blocks for studying other problems - state constrained, impulse, hybrid., retarded . .
- The significance of ‘differential inclusion’ necessary conditions is to study structural properties of optimal controls.
- Pontryagin Maximum Principle is usually better for solving specific problems/ inspiring numerical methods
- The search for differential inclusions necessary conditions has had an enormous influence on development of techniques of nonlinear analysis.
- Open problem: validity of the partially convexified HI for non-convex $F(t, x)$,

Thank you

and

Happy holidays!