

Necessary Conditions in Dynamic Optimization

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Necessary Conditions in Dynamic Optimization

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Imperial College London

NetCo 2014

New Trends in Optimal Control

Tours, France

23-27 June 2014

NetCo (HJNet + SADCO)



Outline of the talk

- Early Necessary Conditions in Classical C. of V.
- Dynamic Constraints and The Maximum Principle
- **Differential Inclusion Problems**
- Clarke's Hamiltonian Inclusion for Convex Diff. Inclusions
- Euler Lagrange Inclusion for Non-convex Diff. Inclusions
- Return to the Hamiltonian Inclusion
- Uses of the Hamiltonian Inclusion and Open Questions

Some History

Calculus of Variations:

$$(Q) \begin{cases} \text{Minimize } \int_0^T L(t, x(t), \dot{x}(t)) dt \\ \text{over } x(\cdot) \in W^{1,1}[S, T]; \mathbb{R}^n \\ \text{satisfying} \\ x(0) = x_0 \text{ and } x(T) = x_1 . \end{cases}$$

($L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and points $x_0, x_1 \in \mathbb{R}^n$.)

$W^{1,1} \equiv$ **abs. continuous functions**

Take a minimizer $\bar{x}(\cdot)$.

Assume

- $L(\cdot, \cdot, \cdot)$ is continuous and $L(t, \cdot, \cdot)$ is C^1
- There exist $k(\cdot) \in L^1$ and $\epsilon > 0$ s.t.

$$|\nabla L(x, v)| \leq k(t)$$

for all $(x, v) \in (\bar{x}(t), \dot{\bar{x}}(t)) + \epsilon \mathbb{B}$ $\mathbb{B} \equiv$ 'closed unit ball'.

Necessary Conditions

Take a minimizer $\bar{x}(\cdot)$ for

$$\text{Minimize } \left\{ \int_0^T L(t, x(t), \dot{x}(t)) dt \mid x(\cdot) \in W^{1,1}, x(0) = x_0, x(T) = x_1 \right\}$$

There exists an absolutely continuous $p(\cdot) : [0, 1] \rightarrow \mathbb{R}^n$ such that

- $(\dot{p}(t), p(t)) = \nabla L(t, \bar{x}(t), \dot{\bar{x}}(t))$ a.e

(The Euler-Lagrange Condition)

- $p(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \max_{v \in \mathbb{R}^n} \{p(t) \cdot v - L(t, \bar{x}(t), v)\}$ a.e.

(The Weierstrass condition)

- If $L(t, x, v)$ is indep. of t ,

$$p(t) \cdot \dot{\bar{x}}(t) - L(t, \bar{x}(t), \dot{\bar{x}}(t)) = \text{const. a.e.}$$

(Constancy of the Hamiltonian)

Hamilton's Equations

Assume

- $L(t, \cdot, \cdot)$ is C^2
- $L_{vv}(t, x, v)$ is invertible.

Hamiltonian $H(t, x, p) : [0, T] \times R^n \times R^n \rightarrow R$:

$$H(t, x, p) = p \cdot v(t, x, p) - L(t, x, v(t, x, p)) \quad (\text{Legendre transformation})$$

$$(v(t, x, p) := \text{'solution function of } p \cdot v - L_v(t, x, v) = 0 \text{'})$$

Hamilton's equations:

$$(-\dot{p}(t), \dot{\bar{x}}(t)) = \nabla H(t, \bar{x}(t), p(t))$$

Permits study of minimizers via systems of differential equations.

The Optimal Control Problem with Dynamic Constraints

Aerospace applications ('optimal flight trajectories') motivated:

Pontryagin Formulation

$$\left\{ \begin{array}{l} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^n) \\ \quad \text{and measurable functions } u(\cdot) : [0, T] \rightarrow \mathbb{R}^m \text{ s.t.} \\ \dot{x}(t) \in f(t, x(t), u(t)) \quad \textit{a.e.} \\ u(t) \in U(t) \quad \textit{a.e.} \\ (x(0), x(T)) \in C \end{array} \right.$$

Data: $f(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $U(\cdot) : [0, T] \rightsquigarrow \mathbb{R}^m$,
 $C \subset \mathbb{R}^n \times \mathbb{R}^n$.



Dynamic constraint, cont.

Take minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$. Assume

- $f(\cdot, \cdot, \cdot)$ is continuous, $f(t, \cdot, u)$ is C^1 , C are closed
- $\text{Gr } U(\cdot)$ is a Borel set
- there exist $c(\cdot) \in L^1$ and $k_f(\cdot) \in L^1$ such that

$$|f(t, x, u) - f(t, x', u)| \leq k(t)|x - x'|,$$

$$|f(t, x, u)| \leq c(t)$$

for all $x, x' \in \bar{x}(t) + \epsilon B$, $u \in U$, a.e. $t \in [0, T]$.

Write $H(\cdot, \cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$

$$H(t, x, p, u) = p \cdot f(t, x, u).$$

(Un-maximized Hamiltonian)

Maximum Principle (L S Pontragin et al.) 1959

Take a minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$.

There exist $\lambda \geq 0$ and $p(\cdot) \in W^{1,1}$ such that

- $(\lambda, p(\cdot)) \neq (0, 0)$ (Nontriviality of multipliers)
- $-\dot{p}(t) = H_x(t, \bar{x}(t), \bar{u}(t))$ a.e. (Costate Equation)
- $H(t, \bar{x}(t), \bar{u}(t)) = \max\{H(t, \bar{x}(t), u) \mid u \in U\}$ a.e.

(Weierstrass Condition)

- $(p(0), -p(T)) = \lambda g_x(\bar{x}(0), \bar{x}(T)) + \xi$

where ξ is 'normal' to C at $(\bar{x}(0), \bar{x}(T))$.

(Transversality Condition)

- $H(t, \bar{x}(t), p(t), \bar{u}(t)) = \max\{H(t, \bar{x}(t), p(t), u) \mid u \in U(t)\}$ a.e.

(Constancy of Maximized Hamiltonian)

Problem Reformulation

The dynamic constraint

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ u(t) \in U(t) \end{cases}$$

is a constraint on possible velocities for a give state, and can be captured as a differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

where

$$F(t, x) := \{f(t, x, u) \mid u \in U\}.$$

The optimal control problem can be expressed directly in terms of $F(\cdot)$:

$$\begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T] : \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

Reformulation cont.

Since 1970's: interest in deriving necessary conditions based on a 'differential inclusion' formulation of the dynamic constraint:

Goal: Express both hypotheses and necessary conditions directly in terms of $F(\cdot)$ instead of ingredients of parameterization:

$$F(t, x) := f(t, x, U) .$$

Reasons:

- New kinds of necessary conditions
- Improvements on the Maximum Principle (validity under reduced hypotheses)
- Better tools for studying properties of optimal trajectories, value functions, etc.

First Ideas (R T Rockafellar)



First Ideas (R T Rockafellar)

Re-write the 'differential inclusion' problem

$$\left\{ \begin{array}{l} \text{Minimize } \int_0^T L(t, x(t), \dot{x}(t)) dt + g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T], \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{array} \right.$$

as a **Generalized Problem of Bolza**:

$$\left\{ \begin{array}{l} \text{Minimize } \tilde{g}(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T], \mathbb{R}^n) \end{array} \right.$$

in which

('hard penalization')

$$L(t, x, v) := \begin{cases} 0 & \text{if } v \in F(t, x) \\ +\infty & \text{otherwise} \end{cases}, \quad \tilde{g}(x_0, x_1) = \begin{cases} g(x_0, x_1) & \text{if } (x_0, x_1) \in C \\ +\infty & \text{otherwise} \end{cases}$$

('Generalized' because it involves extended valued functions)

Optimality Conditions for Fully Convex Problems

Theorem (Rockafellar 1970, Fully Convex Case)

Let $\bar{x}(\cdot)$ be a solution to the **Generalized Bolza Problem**. Assume

- $\tilde{L}(t, \cdot, \cdot)$ and $\tilde{g}(\cdot, \cdot)$ are proper convex functions. ('full convexity')
- $\tilde{L}(t, x, \cdot)$ has superlinear growth, $g(\cdot, \cdot)$ linearly bounded below.
- $\text{rel interior } \{\tilde{\mathcal{C}}\} \cap \text{rel interior } \{\mathcal{R}\} \neq \emptyset$ ('controllability')

$$(\tilde{\mathcal{C}} = \{(x_0, x_1) \mid \tilde{g}(x_0, x_1) < \infty\} \text{ and } \mathcal{R} = \text{'reachable set'}).$$

Then there exist $p(\cdot) \in AC$ such that

- $(\dot{p}(t), p(t)) \in \partial L(t, \bar{x}(t), \dot{\bar{x}}(t))$ a.e. (Euler-Lagrange Condition)
- $(p(0), -p(T)) \in \partial \tilde{g}(\bar{x}(0), \bar{x}(T))$

$\partial f(\bar{x})$ is subdifferential of convex analysis:

$$\partial f(\bar{x}) := \{\xi \mid f(x) - f(\bar{x}) \geq \xi \cdot (x - \bar{x}) \text{ for all } x \in R^n\}$$

The Fully Convex Case Cont.

Approach:

For a convex optimization problem, **Fenchel Duality** provides dual problem, conditions for 'strong duality', 'multiplier' interpretations of solutions to dual problem, etc.

- Compute subdifferentials of integral functionals involving extended valued integrands
- Check no duality gap and Fenchel Dual has solutions, which can be identified with co-state arcs.

Hamiltonian Conditions

For $L(t, x, \cdot)$ convex, the Hamiltonian:

$$H(t, x, p) = p \cdot v(t, x, p) - \tilde{L}(t, x, v(t, x, p)) \quad (\text{Legendre transformation})$$

can be expressed

$$H(t, x, p) = \sup\{p \cdot v - L(t, x, v) \mid v \in \mathbb{R}^n\}$$

- $L(t, \cdot, \cdot)$ convex $\implies H(t, x, \cdot)$ is concave, and $H(t, \cdot, p)$ is convex

For $L(t, \cdot, \cdot)$ convex, Rockafellar showed under earlier conditions:

$$-\dot{p}(t) \in \tilde{\partial}_x H(t, \bar{x}(t), p(t)), \quad \dot{\bar{x}}(t) \in \partial_p H(t, \bar{x}(t), p(t)).$$

‘Hamiltonian inclusion’

$(\tilde{\partial}_x H := -\partial_x(-H))$ is ‘concave’ subdifferential

Generalization to Non-convex Problems?

$$(P) \begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

Major research theme (1973 -): **Derive necessary conditions for this non-convex problem**, e.g.

$$(\dot{p}(t), p(t)) \in \partial_{x,v} \tilde{L}(t, \bar{x}(t), \dot{\bar{x}}(t)) \quad \text{Euler Lagrange inclusion}$$

or
$$(\tilde{L}(x, v) = \mathbb{I}_{\text{Graph } F(\cdot)}(x, v))$$

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial \tilde{H}(t, \bar{x}(t), p(t)) \quad \text{Hamiltonian inclusion}$$

- **How to interpret $\partial \tilde{L}$ and ∂H for non-convex functions?**

Concepts of Local Minimizer

' $x(\cdot)$ feasible' F -trajectory $x(\cdot)$ means ' $(x(0), x(T)) \in C$ '.

Definition. A feasible F -trajectory $\bar{x}(\cdot)$ is a:

- L^∞ -local minimizer if there exists $\epsilon > 0$ such that

$$g(\bar{x}(0), \bar{x}(T)) \leq g(x(0), x(T))$$

for all feasible $x(\cdot)$'s s.t. $\|x(\cdot) - \bar{x}(\cdot)\|_{L^\infty} \leq \epsilon$

- $W^{1,1}$ -local minimizer if there exists $\epsilon > 0$ such that

$$g(\bar{x}(0), \bar{x}(T)) \leq g(x(0), x(T))$$

for all feasible $x(\cdot)$'s s.t. $\|x(\cdot) - \bar{x}(\cdot)\|_{W^{1,1}} \leq \epsilon$.

Choice of Subdifferentials, etc.

Take a close set $A \subset X$ (X top. vector space) and $\bar{x} \in A$.

How to define tangent and normal cones ' T_A '(\bar{x}) and ' N_A '(\bar{x}) ?

Then define the corresponding subdifferential of l.s.c.

$$f : X \rightarrow R \cup \{+\infty\} \text{ at } x \in \text{dom } f(.):$$

$$' \partial f(\bar{x}) := \{ \xi \mid (\xi, -1) \in 'N_{\text{epi } f(.)}'(\bar{x}, f(\bar{x})) \}.$$

Requirements:

- ' N_A '(\bar{x}) coincides with normal cone of convex analysis, etc, for convex sets/functions, etc.
- There is a suitably rich calculus
($' \partial(f_1 + f_2)(\bar{x}) \subset ' \partial f_1(\bar{x}) + f_2(\bar{x}), \dots$)
- 'robustness'
- $f(\bar{x}) \leq \inf\{f(x) \mid x \in X\} \implies 0 \in ' \partial f(\bar{x})$.

Nonsmooth Analysis: Francis Clarke



Clarke's Tangent Cone and Normal Cone

Key advance: **Clarke's tangent cone and normal cone**

$$T_A^C(\bar{x}) := \{ \xi \mid \forall y_i \xrightarrow{A} \bar{x}, \exists x_i \xrightarrow{A} \bar{x} \text{ and } \epsilon \downarrow 0 \text{ s.t. } \epsilon^{-1}(y_i - x_i) \rightarrow \xi \}$$

$$N_A^C(\bar{x}) = \{ \eta \mid \eta \cdot \xi \leq 0 \text{ for all } \xi \in T_A^C(\bar{x}) \}.$$

'polar cone' of Clarke tangent cone

$$\partial^C f(\bar{x}) := \{ \xi \mid (\xi, -1) \in N_{\text{epi}(\cdot)}^C(\bar{x}, f(\bar{x})) \}.$$

- **Satisfies main 'requirements'**
- **Suitable vehicle for derivation of necessary conditions in optimal control.**

The Hamiltonian Inclusion

$$(P) \begin{cases} \text{Minimize } g(x(0), x(T)) \\ \text{over } x(\cdot) \in W([0, T] : \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(t) \in F(t, x(t)) \\ (x(0), x(T)) \in C \end{cases}$$

Standard Hypotheses (for given F -trajectory $\bar{x}(\cdot)$): for some $c_F(\cdot), k_F(\cdot) \in L^1$ s.t.

(H1) : $F(t, x)$ is closed for each (t, x) , $F(\cdot, x)$ is meas.,
 C is closed, $g(\cdot, \cdot)$ is locally Lipschitz

(H2): $F(t, x) \subset c_F(t)B$

for all $x \in \bar{x}(t) + \epsilon B$, a.e. $t \in [0, T]$.

(H3): $F(t, x) \subset F(t, x') + k_F(t)|x - x'|B$

for all $x, x' \in \bar{x}(t) + \epsilon B$, a.e. $t \in [0, T]$.

The Hamiltonian Inclusion, cont.

Theorem (Clarke 1973)

Take an L^∞ -local minimizer $\bar{x}(\cdot)$. Assume standard hypotheses and

(C): $F(t, x)$ is convex for all x .

Then there exists $p(\cdot) \in W^{1,1}$ and $\lambda \geq 0$ such that

(i): $(p(\cdot), \lambda) \neq (0, 0)$

(ii): $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial_{x,p}^{\mathcal{C}} H(t, \bar{x}(t), p(t))$ a.e.

(iii) $(p(0), -p(T)) \in \lambda \partial^{\mathcal{C}} g(\bar{x}(0), \bar{x}(T)) + N_{\mathcal{C}}^{\mathcal{C}}(\bar{x}(0), \bar{x}(T))$

Nonconvex Normal Cones: A.Ya Kruger and B. Mordukhovich



Alexander Ya. Kruger



Boris Mordukhovich

Non-Convex Normal Cones and Subdifferentials

A non-convex normal cone (Kruger/Mordukhovich):

$$N_A(\bar{x}) := \limsup_{x' \rightarrow \bar{x}} \{\xi \mid \xi \cdot (x - x') \leq o(|x - x'|)\}$$

Hence, subdifferential $\partial f(\bar{x})$:

$$\partial f(\bar{x}) = \{\xi \mid (\xi, -1) \in N_{\text{epi } F(\cdot)}(\bar{x}, f(\bar{x}))\}.$$

Relation with Clarke normal cone and subdifferential:

$$N_A^C(\bar{x}) = \text{co } N_A(\bar{x}) \text{ and } \partial^C f(\bar{x}) = \text{co } \partial f(\bar{x})$$

(if $f(\cdot)$ Lipschitz near \bar{x}).

Normal Cones and Subdifferentials (in this sense) have a rich calculus, and can be used to formulate refined necessary conditions.

Hamiltonian Inclusion, First Refinement

Theorem (First Refinement: **improved transversality condition**)

The assertions of the Hamiltonian Inclusion can be strengthened as follows:

There exists $p(\cdot) \in W^{1,1}$ and $\lambda \geq 0$ such that

- (i): $(p(\cdot), \lambda) \neq (0, 0)$
- (ii): $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co } \partial H(t, \bar{x}(t), p(t))$ a.e.
- (iii) $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

Example: for $f(x) = -|x|$ and $\bar{x} = 0$,

$$\partial f(\bar{x}) = \{-1\} \cup \{1\} \stackrel{\text{strict}}{\subset} [-1, +1] = \partial^C f(\bar{x})$$

Two Further Refinements

1. Improvement of of Convexity Condition:

The Hamiltonian inclusion

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co } \partial_{x,p} H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

Note: **convexification w.r.t. two variables** can be replaced by

$$(-\dot{p}(\cdot) \in \text{co}\{(\xi \mid (\xi, \dot{\bar{x}}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

which involves **convexification w.r.t. only one variable**

2. Strengthen Topology Defining ‘Local Minimizer’

The Hamiltonian inclusion is valid for $W^{1,1}$ -local minimizers (not just L^∞ -local minimizers)

Refined Hamiltonian Inclusion

Theorem (Loewen Rockafellar, 1994)

Take an $W^{1,1}$ -local minimizer $\bar{x}(\cdot)$. Assume standard hypotheses and

(C): $F(x)$ is convex for all x .

Then there exists $p(\cdot) \in AC$ and $\lambda \geq 0$ such that

(i): $(p(\cdot), \lambda) \neq (0, 0)$

(ii): $(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, \dot{\bar{x}}(t)) \in \partial_{x,p}H(t, \bar{x}(t), p(t)) \text{ a.e.}$

(iii) $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

Note: **(iii)** $\implies p(t) \cdot \dot{\bar{x}}(t) = \max\{p(t) \cdot v \mid v \in F(\bar{x}(t))\}$

(Weierstrass Condition)

The Generalized Euler-Lagrange Inclusion

Research direction since early 1990's: **Find necessary conditions when $F(x)$ is non-convex.**

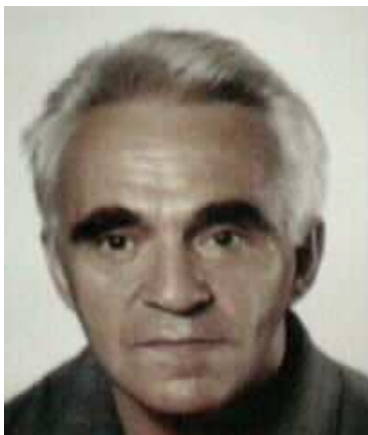
Generalizations of the classical Euler-Lagrange condition have been the most fruitful.

What should such conditions look like?

Here, generalizations of the classical Euler-Lagrange condition have been the most fruitful.

(Mordukhovich, Ioffe, Rockafellar, Vinter)

E-L inclusion: B. Mordukhovich and A.D. Ioffe



Generalized E-L Condition, Cont.

Differential inclusion problem formulated as extended Bolza problem:

Minimize $\{\int_0^T L(t, x(t), \dot{x}(t)) + g(x(0), x(1))\}$ over $x(\cdot) \in AC$

where

$$L(x, v) = \mathbb{I}_{\text{Gr } F(t, \cdot)}(x, v)$$

But

$$\partial \mathbb{I}_{\text{Gr } F(t, \cdot)}(x, v) = N_{\text{Gr } F(t, \cdot)}(x, v)$$

So natural generalization of Euler Lagrange condition is

$$(\dot{p}(t), p(t)) \in \text{co } N_{\text{Gr } F(\cdot)}(x, v) \quad ?$$

Generalized E-L Condition, Cont.

Fully convexified E-L Inclusion is inadequate condition!

Example: $F(x) = \{|x|\}$, $\bar{x} = 0$ and $\bar{v} = 0$ (Lipschitz data)

$$\text{Gr } F(\cdot) = \{(x, |x|) \mid x \in \mathbb{R}\}$$

$$N_{\text{Gr } F(\cdot)}(\bar{x}, \bar{v}) = \{\alpha(1, 1) : \alpha \in \mathbb{R}\} \cup \{\alpha(-1, 1) : \alpha \in \mathbb{R}\}$$

$$\text{co } N_{\text{Gr } F(\cdot)}(\bar{x}, \bar{v}) = \mathbb{R} \times \mathbb{R}$$

$$\text{So } (\dot{p}, p) \in \text{co } N_{\text{Gr } F(\cdot)}(\bar{x}, \bar{v}) \implies (\dot{p}, p) \in \mathbb{R} \times \mathbb{R}$$

But **no useful information**

$$\dot{p} \in \text{co}\{\xi \mid (\xi, p) \in N_{\text{Gr } F(\cdot)}(0, 0)\} \implies |\dot{p}| \leq |p|$$

(partially convexification gives much more information)

'right condition'

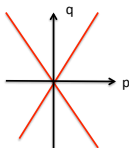
Normal cones to $\text{Gr } F(\cdot)$

Take $F(x) = \{|x|\}$

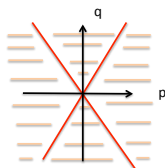
(a): $(q, p) \in N_{\text{Gr } F(\cdot)}(0, 0)$

(b): $(q, p) \in \text{co } N_{\text{Gr } F(\cdot)}(0, 0)$

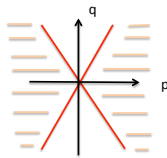
(c): (q, p) s.t. $q \in \text{co}\{\xi \mid (\xi, p) \in N_{\text{Gr } F(\cdot)}(0, 0)\}$



(a)



(b)



(c)

Nec. Condition for Non-Convex $F(x)$

Theorem (loffe 1997)

Take an $W^{1,1}$ -local minimizer $\bar{x}(\cdot)$. Assume standard hypotheses

Then there exists $p(\cdot) \in AC$ and $\lambda \geq 0$ such that

- (i): $(p(\cdot), \lambda) \neq (0, 0)$
- (ii): $(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, p(t)) \in N_{\text{Gr } F(t, \cdot)}(\bar{x}(t), \bar{x}(t)) \text{ a.e.}$
- (iii) $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$
- (iv): $p(t) \cdot \dot{\bar{x}}(t) = \max\{p(t) \cdot v \mid v \in F(t, \bar{x}(t))\} \text{ a.e.}$

(Earlier version without Weierstrass condition due to Morduckhovich 1995. Extensions allow unbounded $F(\cdot)$, state constraints etc. loffe, Vinter, Clarke . . .)

Notes: No convexity of $F(\cdot)$. Must add Weierstrass Cond.

Different necessary conditions vary according to costate equation or inclusion.

Euler Lagrange inclusion subsumes earlier 'Hamiltonian inclusion' conditions

Theorem (Rockafellar 1996) Assume standard conditions and

(C): $F(x)$ is convex for each x (convexity)

Then

$$\text{co} \{q \mid (q, p(t)) \in N_{\text{Gr}\{F(t, \cdot)\}}(\bar{x}(t), \dot{\bar{x}}(t))\} \subset \\ \text{co} \{-q \mid (q, \bar{x}(t)) \in \partial_{x,p} H(t, \bar{x}(t), p(t))\} .$$

Necessary Conditions: State of the Art 2005

Assume:

- Standard hypotheses
- Assume $\bar{x}(\cdot)$ is a $W^{1,1}$ local minimizer

Then there exists (non-trivial) $(p(\cdot) \in W^{1,1}, \lambda \geq 0$, satisfying transversality and Weierstrass conditions, s.t.

$$(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, p(t)) \in N_{\text{Gr } F(t, \cdot)}(\bar{x}(t), \dot{\bar{x}}(t)) \text{ a.e. (1)}$$

If also (Euler-Lagrange Inclusion)

- (C): $F(t, x)$ is convex for all x

Then (1) implies

$$(-\dot{p}(t) \in \text{co}\{(\xi \mid (\xi, \dot{\bar{x}}(t)) \in \partial H(\bar{x}(t)'p()) \text{ a.e.}$$

(Partially Convexified Hamiltonian Inclusion)

Rockafellar Duality is only valid for convex $F(x)$'s.

Open the question: Are the conditions

$$(-\dot{p}(\cdot), \dot{\bar{x}}(t) \in \text{co } \partial_{x,p}H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

or **(Fully Convexified Hamiltonian Inclusion)**

$$(-\dot{p}(t) \in \text{co}\{(q \mid (q, \dot{\bar{x}}(t)) \in \partial_{x,p}H(t, \bar{x}(t), p(t)) \text{ a.e.}$$

(Partially Convexified Hamiltonian Inclusion)

valid, when $F(x)$ is not convex?

This was partly resolved by Clarke in 2005.

Hamiltonian Inclusion for Non-Convex $F(x)$

Theorem (Clarke 2005)

Assume Standard Hypotheses and

- $\bar{x}(\cdot)$ is L^∞ -local minimizer.

Then there exists non-trivial $p(\cdot) \in W^{1,1}$ and $\lambda \geq 0$ such that

(i): $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co } \partial_{x,p} H(t, \bar{x}(t), p(t)) \quad \text{a.e.}$

(Fully Convexified Hamiltonian inclusion)

(ii) $(p(0), -p(T)) \in \lambda \partial g(\bar{x}(0), \bar{x}(T)) + N_C(\bar{x}(0), \bar{x}(T))$

Notes: L^∞ -local minimizer and fully convexified Hamiltonian Inclusion

Remaining Open Questions

This work validates (for non-Convex $F(x)$'s)

The Fully Convexified Hamiltonian Inclusion for L^∞ -minimizers.

BUT, It appeared to be an intermediate result to proving

The Partially Convexified Hamiltonian Inclusion for $W^{1,1}$ -minimizers.

The status of these conditions is clarified by a recent counter-example.

Proposition. (Vinter 2014)

For some choice of data for (P) satisfying the Standard Hypotheses such that

$F(x)$ is not convex,

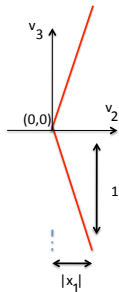
and some F -trajectory $\bar{x}(\cdot)$, we have

- $(\bar{x}(\cdot))$ is a $W^{1,1}$ local minimizer,
- $(\bar{x}(\cdot))$ is not a L^∞ local minimizer,
- There exist non-trivial multipliers satisfying (the transversality condition and) the fully convexified Hamiltonian inclusion.
- There exist do not exist non-trivial multipliers satisfying t the fully convexified Hamiltonian inclusion.

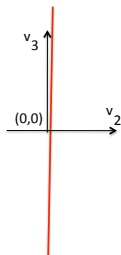
Example

$$(E) \begin{cases} \text{Minimize } \frac{1}{2}x_1(1) - x_2(1) \\ \text{over } x(\cdot) = (x_1(\cdot), x_2(\cdot), x_3(\cdot)) \text{ satisfying} \\ \dot{x}_1(t) = 0, (\dot{x}_2, \dot{x}_3) \in \tilde{F}(x_1) \\ x_1(0) \geq 0 \end{cases}$$

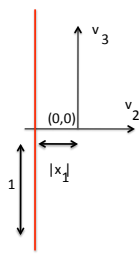
(i): $x_1 > 0$



(ii): $x_1 = 0$

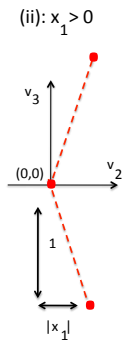


(iii): $x_1 < 0$



(v_2, v_3) coordinates of points $(v_1, v_2, v_3) \in F(x)$

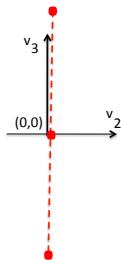
Differential Inclusion $\tilde{F}(x_1), x_1 > 0$



(v_2, v_3) coordinates of points $(v_1, v_2, v_3) \in F(x)$

Differential Inclusion $\tilde{F}(x_1), x_1 = 0$

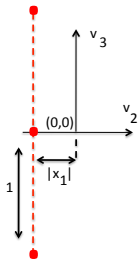
(ii): $x_1 = 0$



(v_2, v_3) coordinates of points $(v_1, v_2, v_3) \in F(x)$

Differential Inclusion $\tilde{F}(x_1)$, $x_1 < 0$

(iii): $x_1 < 0$



(v_2, v_3) coordinates of points $(v_1, v_2, v_3) \in F(x)$

Example , continued

The example is very benign-seeming:

- Free right end-point problem.
- Can be re-formulated as a 'finite Lagrangian problem'
- $F(x)$ is not convex for all x , **BUT $F(\bar{x}(t))$ is convex**

Note that

- **'Non-smooth' of $x \rightarrow F(x)$ is essential feature of example**

Fully convexified Hamiltonian Inclusion is 2nd Class Citizen!

- The Hamilton Inclusion is valid, in general, only if $\bar{x}(\cdot)$ is a L^∞ minimizer
- Gives new information in some cases:
'Hamiltonian inclusion is not satisfied'
 \implies ' $\bar{x}(\cdot)$ is not a minimizer.'
- **Degeneracy:** Conditions can be given for the Hamiltonian inclusion to be non-degenerate when all other known conditions are degenerate.
- **Relaxation:** 'normality of Hamiltonian inclusion' \implies 'relaxation does not reduce minimum cost'

(not known for other necessary conditions)

Some Concluding Remarks

- We have focused on necessary conditions for the most basic of differential inclusion problems. These are the building blocks for studying other problems - state constrained, impulse, hybrid., retarded . .
- The significance of 'differential inclusion' necessary conditions is to study structural properties of optimal controls.
- Pontryagin Maximum Principle is usually better for solving specific problems/ inspiring numerical methods
- The search for differential inclusions necessary conditions has had an enormous influence on development of techniques of nonlinear analysis.
- **Open problem: validity of the partially convexified HI for non-convex $F(t, x)$,**

Thank you

and

Happy holidays!