Degenerate second order mean field games systems

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We consider the following second order mean field games (MFG) system with degenerate diffusion and a local coupling

$$\begin{cases} -\partial_t \phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t)) \\ \partial_t m - \partial_{ij}(A_{ij} m) - \mathrm{div}(m D_p H(x, D\phi)) = 0 \\ m(0) = m_0, \ \phi(x,T) = \phi_T(x) \end{cases}$$

where

- $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is symmetric and nonnegative
- the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is convex in the second variable
- the coupling $f : \mathbb{R}^d \times [0, +\infty) \to [0, +\infty)$ is increasing with respect to the second variable
- m_0 is a probability density
- $\phi_T : \mathbb{R}^d \to \mathbb{R}$ is a given function

Introduced by Lasry and Lions and by Huang, Caines and Malhamé to describe Nash equilibria in differential games with infinitely many players

The optimal control problem of a typical small player

$\phi(t,x)$ is the value function

Dynamics :

$$dX_s = \alpha_s ds + \Sigma(s, X_s) dB_s,$$

where (α_s) is the control, (B_s) is a $d-$dimensional Brownian motion and $\Sigma\Sigma^T = A$

Cost :

$$\mathbb{E}\left[\int_0^T H^*(s, X_s, -\alpha_s) + f(X_s, m(s, X_s))\ ds + \phi_T(X_T)\right]$$

$\forall t \in [0, T]$ $m(t,x)$ denotes the density of population of small players at position x

the optimal control is formally given by the feedback $(t,x) \to -D_pH(x, D\phi(t,x))$

the second equation in (MFG) is the Kolmogorov equation of the process (X_s) when the small player plays in an optimal way

In the MFG systems with uniformly parabolic diffusions (typically $A_{ij}\partial_{ij}\phi = \Delta\phi$) the solutions are smooth, (at least if the coupling is nonlocal and regularizing or if it has a “small growth")

Analysis by PDE methods : see Cardaliaguet, Lasry, Lions and Porretta (2012), Lasry and Lions (2006, 2007), Gomes, Pimentel and Sánchez-Morgado (2013), Lions (2011), Porretta (2013)

Analysis by stochastic techniques : see Carmona and Delarue (2013), Huang, Malhamé and Caines (2006)

The case of couplings with an arbitrary growth has been discussed in Cardaliaguet, Lasry, Lions and Porretta (2012) (for quadratic hamiltonians and smooth solutions) and in Porretta (2013) (for more general hamiltonians but weak solutions)

We consider degenerate parabolic equations : lack of regularity $\Rightarrow$ break down of the uniformly parabolic techniques

IDEA : use convex optimization methods from optimal transport problems (see Benamou and Brenier (2000), Cardaliaguet, Carlier and Nazaret (2012))

These techniques were already used to study first order MFG systems (i.e., $A \equiv 0$): see Cardaliaguet (2013), Cardaliaguet and Graber (2013), Graber (2013)

We show the existence and uniqueness of a weak solution for the degenerate MFG system as well as the stability of solutions with respect to perturbation of the data

Assumptions

(H1) $f : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ is continuous in both variables, increasing wrt the second variable m, and $\exists q > 1, C_1$ s.t.

$$\frac{1}{C_1}|m|^{q-1} - C_1 \le f(x,m) \le C_1|m|^{q-1} + C_1 \qquad \forall m \ge 0$$

Moreover $f(x,0) = 0 \qquad \forall x \in \mathbb{T}^d$

(H2) $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in both variables, convex and differentiable in the second variable, with D_pH continuous in both variable, and $\exists r > 1, C_2 > 0$ s.t.

$$\frac{1}{rC_2}|\xi|^r - C_2 \le H(x,\xi) \le \frac{C_2}{r}|\xi|^r + C_2 \qquad \forall (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d$$

Assumptions

(H3) $A : \mathbb{T}^d \to \mathbb{R}^{d\times d}$ is Lipschitz continuous with symmetric nonnegative values: $\exists C_3 > 0$ s.t.

$$|A(x) - A(y)| \leq C_3|x-y| \qquad \forall x, y \in \mathbb{T}^d,\ \xi \in \mathbb{R}^d$$

Moreover, either $r \geq p$ or $A \equiv 0$, where $\frac{1}{p} + \frac{1}{q} = 1$

(H4) $\phi_T : \mathbb{T}^d \to \mathbb{R}$ is of class $\mathcal{C}^2$, while $m_0 : \mathbb{T}^d \to \mathbb{R}$ is a C^1 positive density

The Fenchel conjugate H^* of H wrt the second variable is continuous and

$$\frac{1}{r' C_2}|\xi|^{r'} - C_2 \leq H^*(x,\xi) \leq \frac{C_2}{r'}|\xi|^{r'} + C_2 \qquad \forall (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d\ ,$$

where r' is s.t. : $\dfrac{1}{r} + \dfrac{1}{r'} = 1$

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where r' is s.t. : $\dfrac{1}{r} + \dfrac{1}{r'} = 1$

Let
$$F(x,m) = \begin{cases} \displaystyle\int_0^m f(x,\tau)d\tau & \text{if } \ m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Then F is continuous on $\mathbb{T}^d \times (0, +\infty)$, derivable and strictly convex in m and

$$\frac{1}{qC_1}|m|^q - C_1 \leq F(x,m) \leq \frac{C_1}{q}|m|^q + C_1 \qquad \forall m \geq 0$$

Let F^* be the Fenchel conjugate of F wrt the second variable then $F^*(x,a) = 0$ for $a \leq 0$, moreover,

$$\frac{1}{pC_1}|a|^p - C_1 \leq F^*(x,a) \leq \frac{C_1}{p}|a|^p + C_1 \qquad \forall a \geq 0\,,$$

where p is s.t. : $1/p + 1/q = 1$

Key point : integral estimates

Assume that (H2) and (H3) hold true and let ϕ satisfy

$$\begin{cases} -\partial_t \phi - A_{ij}(x)\partial_{ij}\phi + H(x, D\phi) \leq \alpha(t,x) \\ \phi(x,T) \leq \phi_T(x) \end{cases}$$

in the sense of distribution, with $\alpha \in L^p((0,T)\times\mathbb{T}^d)$, $\phi_T \in L^\infty(\mathbb{T}^d)$

i.e. for any nonnegative test function $\zeta \in C_c^\infty((0,T]\times\mathbb{T}^d)$,

$$-\int_{\mathbb{T}^d} \zeta(T)\phi_T + \int_0^T \int_{\mathbb{T}^d} \phi\partial_t\zeta + \langle D\zeta, AD\phi\rangle + \zeta(\partial_i A_{ij}\partial_j\phi + H(x,D\phi)) \leq \int_0^T \int_{\mathbb{T}^d} \alpha\zeta$$

Key point : integral estimates

Theorem

Then, if ϕ is bounded below, we have

$$\|\phi\|_{L^\infty((0,T),L^m(\mathbb{T}^d))}+\|\phi\|_{L^\gamma((0,T)\times\mathbb{T}^d)} \leq C\left(\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)} + \|\phi_T\|_{L^m(\mathbb{T}^d)}\right)$$

where $m = \frac{d(r(p-1)+1)}{d-r(p-1)}$ *and* $\gamma = \frac{rp(1+d)}{d-r(p-1)}$ *if* $p < 1 + \frac{d}{r}$ *and* $m = \gamma = +\infty$ *if* $p > 1 + \frac{d}{r}$,

with a constant C *depending on* T, p, d, r, C_2, C_3 *and on* $\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)}, \|\phi_T\|_{L^m(\mathbb{T}^d)}$ *and* $\|\phi_-\|_\infty$.

When $r > 2$ and $p > 1 + d/r$, Cardaliaguet and Silvestre 2012 proved that the solution ϕ is locally Hölder continuous, with Hölder modulus depending only on the growth condition of H, the L^∞ bound on u and the L^p norm of α

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where $m=\frac{d(r(p-1)+1)}{d-r(p-1)}$ *and* $\gamma=\frac{rp(1+d)}{d-r(p-1)}$ *if* $p<1+\frac{d}{r}$
and $m=\gamma=+\infty$ *if* $p>1+\frac{d}{r}$,

with a constant C *depending on* T, p, d, r, C_2, C_3 *and on* $\|\alpha\|_{L^p((0,T)\times\mathbb{T}^d)}, \|\phi_T\|_{L^m(\mathbb{T}^d)}$ *and* $\|\phi_-\|_\infty$.

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Optimal control problems in duality

An optimal control problem for a backward HJ equation :

the state variable ϕ is controlled by a distributed control $\alpha : (0, T) \times \mathbb{T}^d \to \mathbb{R}$ in order to minimize the criterium

$$\int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha(t,x)) \; dxdt - \int_{\mathbb{T}^d} \phi(0,x) dm_0(x)$$

when the state ϕ is driven by

$$\begin{cases} -\partial_t \phi - A_{ij}(x)\partial_{ij}\phi + H(x, D\phi) = \alpha(t,x) \\ \phi(x, T) = \phi_T(x) \end{cases}$$

Precisely : $$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$$

where $\mathcal{K}_0 := \{\phi \in \mathcal{C}^2([0,T] \times \mathbb{T}^d) | \; \phi(T,x) = \phi_T(x)\}$ and

$$\mathcal{A}(\phi) = \int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t\phi(t,x) - A_{ij}\partial_{ij}\phi + H(x, D\phi(t,x))) \; dxdt - \int_{\mathbb{T}^d} \phi(0,x) dm_0(x)$$

The second optimal control problem is an optimal control problem for a Fokker-Plank equation :

we control the state variable m through a vector field $v : (0, T) \times \mathbb{T}^d \to \mathbb{R}^d$ in order to minimize

$$\int_0^T \int_{\mathbb{T}^d} m(t,x)H^*(x, -v(t,x)) + F(x, m(t,x))\, dxdt + \int_{\mathbb{T}^d} \phi_T(x)m(T,x)dx$$

when m solves the Fokker-Plank equation

$$\partial_t m - \partial_{ij}(A_{ij}(x)m) + \mathrm{div}(mv) = 0$$

Precisely : $$\inf_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m,w)$$

where

$$\mathcal{K}_1 := \Big\{ (m,w) \in L^1((0,T)\times\mathbb{T}^d) \times L^1((0,T)\times\mathbb{T}^d,\mathbb{R}^d) \Big|$$

$$m(t,x) \geq 0 \text{ a.e.}, \int_{\mathbb{T}^d} m(t,x)dx = 1 \text{ for a.e. } t\in(0,T), \text{ and}$$

$$\partial_t m - \partial_{ij}(A_{ij}(x)m) + \mathrm{div}(w) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \qquad m(0) = m_0$$

$$\text{in the sense of distribution}\Big\}$$

$$\mathcal{B}(m,w) = \int_0^T \int_{\mathbb{T}^d} m(t,x) H^*\left(x, -\frac{w(t,x)}{m(t,x)}\right) + F(x,m(t,x))\, dxdt + \int_{\mathbb{T}^d} \phi_T(x) m(T,x)dx$$

for $m(t,x) = 0$, $m(t,x)H^*\left(x, -\frac{w(t,x)}{m(t,x)}\right) = \begin{cases} +\infty & \text{if } w(t,x) \neq 0 \\ 0 & \text{if } w(t,x) = 0 \end{cases}$

Lemma

We have

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

Moreover, the minimum in the right-hand side is achieved by a unique pair $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ *satisfying*

$(\bar{m}, \bar{w}) \in L^q((0,T) \times \mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}((0,T) \times \mathbb{T}^d)$

IDEA : Use the Fenchel-Rockafellar duality Theorem

Relaxation

Let

$$\mathcal{K} := \Big\{ (\phi, \alpha) \in L^{\gamma}((0,T)\times\mathbb{T}^d) \times L^{p}((0,T)\times\mathbb{T}^d) \Big| \; D\phi \in L^{r}((0,T)\times\mathbb{T}^d)$$

and which satisfies in the sense of distribution

$$-\partial_t\phi - A_{ij}(x)\partial_{ij}\phi + H(x, D\phi) \le \alpha, \qquad \phi(T,\cdot) \le \phi_T \Big\}$$

Due to the presence of second order derivatives we do not expect the function ϕ to be BV (as in the first order case)

What about trace properties ? ϕ has a trace in a weak sense

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$. Then, for any Lipschitz continuous map $\zeta : \mathbb{T}^d \to \mathbb{R}$, the map $t \to \int_{\mathbb{T}^d} \zeta(x)\phi(t,x)dx$ has a BV representative on $[0, T]$.

Moreover, if we denote by $\int_{\mathbb{T}^d} \zeta(x)\phi(t^+,x)dx$ its right limit at $t \in [0, T)$, then the map $\zeta \to \int_{\mathbb{T}^d} \zeta(x)\phi(t^+,x)dx$ is continuous in $L^{m'}(\mathbb{T}^d)$.

$\Rightarrow$ for any nonnegative C^1 map $\zeta : [0, T] \times \mathbb{T}^d \to \mathbb{R}$, one can write the integration by part formula: for any $0 \le t_1 \le t_2 \le T$,

$$-\left[\int_{\mathbb{T}^d} \zeta\phi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{T}^d} \phi\partial_t\zeta + \langle D\zeta, AD\phi\rangle + \zeta(\partial_i A_{ij}\partial_j\phi + H(x, D\phi)) \le \int_{t_1}^{t_2}\int_{\mathbb{T}^d} \alpha\zeta$$

Lemma

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$$-\left[\int_{\mathbb{T}^d} \zeta\phi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{T}^d} \phi\partial_t\zeta + \langle D\zeta, AD\phi\rangle + \zeta(\partial_i A_{ij}\partial_j\phi + H(x, D\phi)) \leq \int_{t_1}^{t_2}\int_{\mathbb{T}^d} \alpha\zeta$$

Proof :

For any Lipschitz continuous, nonnegative map ζ,

$$-\frac{d}{dt}\int_{\mathbb{T}^d}\zeta\phi(t)+\int_{\mathbb{T}^d}\langle D\zeta, AD\phi(t)\rangle+\zeta(\partial_i A_{ij}\partial_j\phi+H(x,D\phi)-\alpha)\leq 0,$$

in the sense of distribution

As the second integral is in $L^1((0,T))$, the map $t\to\int_{\mathbb{T}^d}\zeta\phi(t)$ is BV

If now ζ is Lipschitz continuous and changes sign, one can write $\zeta=\zeta^+-\zeta^-$ and the map $t\to\int_{\mathbb{T}^d}\zeta\phi(t)=\int_{\mathbb{T}^d}\zeta^+\phi(t)-\int_{\mathbb{T}^d}\zeta^-\phi(t)$ is still BV

The continuity with respect to ζ comes from the $L^\infty((0,T),L^m(\mathbb{T}^d))$ estimate on ϕ

We extend the functional $\mathcal{A}$ to $\mathcal{K}$

$$\mathcal{A}(\phi,\alpha)=\int_0^T\int_{\mathbb{T}^d}F^*(x,\alpha(x,t))\,dxdt-\int_{\mathbb{T}^d}\phi(x,0)m_0(x)\,dx\ \ \forall(\phi,\alpha)\in\mathcal{K}$$

Proposition

We have

$$\inf_{\phi\in\mathcal{K}_0}\mathcal{A}(\phi)=\inf_{(\phi,\alpha)\in\mathcal{K}}\mathcal{A}(\phi,\alpha)$$

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$ *and* $(m, w) \in \mathcal{K}_1$.
Assume that $mH^*(\cdot, -w/m) \in L^1((0,T) \times \mathbb{T}^d)$ *and* $m \in L^q$.

Then, for a.e. $t \in [0,T]$, $m(t)\phi(t)$ *is integrable and*

$$\left[\int_{\mathbb{T}^d} m\phi\right]_t^T + \int_t^T \int_{\mathbb{T}^d} m\left(\alpha + H^*(x, -\frac{w}{m})\right) \geq 0$$

and

$$\left[\int_{\mathbb{T}^d} m\phi\right]_0^t + \int_0^t \int_{\mathbb{T}^d} m\left(\alpha + H^*(x, -\frac{w}{m})\right) \geq 0.$$

Moreover, if equality holds in the first inequality for $t = 0$, *then* $w = -mD_pH(x, D\phi)$ *a.e.*

Proof :

Inequality $\inf_{\phi\in\mathcal{K}_0} \mathcal{A}(\phi) \geq \inf_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi,\alpha)$ holds obviously

Let $(\phi,\alpha)\in\mathcal{K}$, then $\forall(m,w)\in\mathcal{K}_1$ with $mH^*(\cdot,-w/m)\in L^1$ we have,

$$\begin{aligned}
\mathcal{A}(\phi,\alpha) &= \int_0^T\int_{\mathbb{T}^d} F^*(\alpha) - \int_{\mathbb{T}^d}\phi(0)m_0 \\
&\geq \int_0^T\int_{\mathbb{T}^d} \alpha m - F(m) - \int_{\mathbb{T}^d} m_0\phi(0) \\
&\geq \int_0^T\int_{\mathbb{T}^d} -mH^*(x,-\frac{w}{m}) - F(m) - \int_{\mathbb{T}^d} m(T)\phi_T = -\mathcal{B}(m,w)
\end{aligned}$$

Taking the sup with respect to (m,w) in the right-hand side

$$\mathcal{A}(\phi,\alpha) \geq -\inf_{(m,w)\in\mathcal{K}_1}\mathcal{B}(m,w) = \inf_{\phi\in\mathcal{K}_0}\mathcal{A}(\phi)$$

Proposition

The relaxed problem has at least one solution $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ *which is bounded below by a constant depending on* $\|\phi_T\|_{C^2}$*, on* $\|A_{ij}\|_{C^0}$ *and on* $\|H(\cdot, D\phi_T)\|_\infty$.

Therefore, thanks to the equivalence, we have a infimum for $\mathcal{A}$ over $\mathcal{K}_0$ and a minimum for $\mathcal{B}$ over $\mathcal{K}_1$

$$\min_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi,\alpha) = \inf_{\phi\in\mathcal{K}_0} \mathcal{A}(\phi) = -\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m,w)$$

Existence and uniqueness

We say that a pair $(\phi, m) \in L^{\gamma}((0,T) \times \mathbb{T}^d) \times L^q((0,T) \times \mathbb{T}^d)$ is a weak solution to (MFG) if

(i) $D\phi \in L^r,\ mH^*(\cdot, D_pH(\cdot, D\phi)) \in L^1 \quad \text{and} \quad mD_pH(\cdot, D\phi)) \in L^1$

(ii) The following inequality holds in the sense of distribution

$$-\partial_t\phi - \partial_i(A_{ij}(x)\partial_j\phi) + (\partial_i A_{ij})\partial_j\phi + H(x, D\phi) \le f(x,m) \quad \text{in } (0,T)\times\mathbb{T}^d,$$

with $\phi(T,\cdot) \le \phi_T$

(iii) The following equation holds in the sense of distribution

$$\partial_t m - \partial_{ij}(A_{ij}(x)m) - \mathrm{div}(mD_pH(x, D\phi))) = 0 \text{ in } (0,T)\times\mathbb{T}^d, m(0) = m_0$$

(iv) The following equality holds :

$$\int_0^T \int_{\mathbb{T}^d} m(t,x)\left(f(x,m(t,x)) + H^*(x, D_pH(x, D\phi)(t,x))\right) dxdt + \int_{\mathbb{T}^d} m(T,x)\phi_T(x) - m_0(x)\phi(0,x)dx = 0$$

Theorem

There exists a weak solution $(\bar\phi, \bar m)$ *to (MFG).*

Moreover this solution is unique in the following sense: if $(\bar\phi, \bar m)$ *and* $(\bar\phi', \bar m')$ *are two solutions, then* $\bar m = \bar m'$ *a.e. and* $\bar\phi = \bar\phi'$ *in* $\{\bar m > 0\}$.

Finally, there exists a solution which is bounded below by a constant depending on $\|\phi_T\|_{C^2}$, *on* $\|A_{ij}\|_{C^0}$ *and on* $\|H(\cdot, D\phi_T)\|_\infty$.

Proof :

Let $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ be a minimizer of $\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m, w)$
and $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ be a minimizer of $\min_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi, \alpha)$

We have $\quad -\mathcal{B}(\bar{m}, \bar{w}) = \mathcal{A}(\bar{\phi}, \bar{\alpha})$

$$\int_0^T \int_{\mathbb{T}^d} F^*(x, \bar{\alpha}) + F(x, \bar{m}) + \bar{m} H^*\left(x, -\frac{\bar{w}}{\bar{m}}\right) dxdt + \int_{\mathbb{T}^d} \phi_T \bar{m}(T) - \bar{\phi}(0) m_0 dx = 0$$

By convexity of F,

$$F^*(x, \bar{\alpha}) + F(x, \bar{m}) - \bar{\alpha}\bar{m} \geq 0,$$

hence

$$\int_0^T \int_{\mathbb{T}^d} \bar{\alpha}\bar{m} + \bar{m} H^*\left(x, -\frac{\bar{w}}{\bar{m}}\right) dxdt + \int_{\mathbb{T}^d} \phi_T \bar{m}(T) - \bar{\phi}(0) m_0 dx \leq 0$$

Therefore $\bar{w} = -\bar{m} D_p H(\cdot, D\bar{\phi})$ a.e. and $\bar{\alpha}(t, x) = f(x, \bar{m}(t, x))$ a.e.

and $(\bar{\phi}, \bar{m})$ is a weak solution

Recall that :

Lemma

Let $(\phi, \alpha) \in \mathcal{K}$ and $(m, w) \in \mathcal{K}_1$. Assume that $mH^*(\cdot, -w/m) \in L^1((0,T) \times \mathbb{T}^d)$ and $m \in L^q$.

Then, for a.e. $t \in [0,T]$, $m(t)\phi(t)$ is integrable and

$$\left[\int_{\mathbb{T}^d} m\phi\right]_t^T + \int_t^T \int_{\mathbb{T}^d} m\left(\alpha + H^*(x, -\frac{w}{m})\right) \geq 0$$

and

$$\left[\int_{\mathbb{T}^d} m\phi\right]_0^t + \int_0^t \int_{\mathbb{T}^d} m\left(\alpha + H^*(x, -\frac{w}{m})\right) \geq 0.$$

Moreover, if equality holds in the first inequality for $t = 0$, then $w = -mD_pH(x, D\phi)$ a.e.

Conversely, using the convexity of F and the Lemma, we can show that any weak solution $(\bar{\phi}, \bar{m})$ of (MFG) is such that :

the pair $(\bar{m}, -\bar{m}D_pH(\cdot, D\bar{\phi}))$ is the minimizer of $\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m, w)$

while $(\bar{\phi}, f(\cdot, \bar{m}))$ is a minimizer of $\min_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi, \alpha)$

Uniqueness : Since the solution of $\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m, w)$ is unique $\bar{m}$ is unique

The proof of the uniqueness of $\bar{\phi}$ is more technical

Stability

Assume that (A^n), (H^n), (f^n) m_0^n and ϕ_T^n satisfy conditions (H1)...(H4) uniformly with respect to n and converge to A, H, f, m_0 and ϕ_T locally uniformly.

Theorem

Let (ϕ^n, m^n) be a weak solution of (MFG) associated with A^n, H^n, f^n and with the initial and terminal conditions m_0^n and ϕ_T^n.
Assume also that the sequence ϕ^n is uniformly bounded below.

Then (m^n) converges strongly to m in L^q while ϕ^n converges weakly and up to a subsequence to a map ϕ in L^γ, where the pair (ϕ, m) is a weak solution to (MFG).

In particular the weak solution is stable with respect to viscous approximation

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