Regularization of chattering phenomena via bounded variation controls

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Outline

1. Motivation
   - what is chattering?
   - how often does it occur?
2. Convergence results
   - convergence for the perturbed problem
   - rate of convergence and switching times
3. Open problems
   - rate of convergence for the perturbed problem
   - generic rate of convergence

Fuller's Problem [Fuller 1961]

Control System

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \qquad |u| \leq 1$$

Cost

$$\underset{\substack{x(0)=\bar{x} \\ x(t_f)=0}}{\text{Minimize}} \int_0^{t_f} x_1^2(s)\, ds$$

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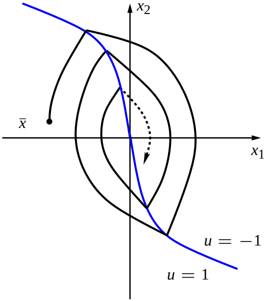


- optimal controls are bang-bang with *infinitely many* switchings
- switchings take place on the curve $\{x_1 + C|x_2|x_2 = 0\}, C \sim 0.44$
- time intervals between consecutive switches decrease in a *geometric progression*

Examples of chattering

Control System

$$\dot{x} = f(x,u) \quad u \in \mathcal{U}$$

Cost

$$\underset{\substack{x(0)=\bar{x} \\ x(t_f)=0}}{\text{Minimize}} \int_0^{t_f} L(s,x,u)\,ds$$

Examples of chattering: state constraints

Control System

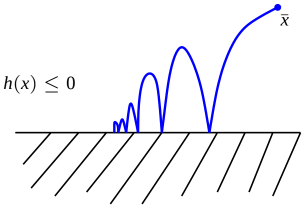
$$\dot{x} = f(x,u) \quad u \in \mathcal{U}$$

Cost

$$\underset{\substack{x(0)=\bar{x} \\ x(t_f)=0 \\ h(x(s))\leq 0}}{\text{Minimize}} \int_0^{t_f} L(s,x,u)\, ds$$

state constraints $\Longrightarrow$

accumulation of contact points with the constraint's boundary



Examples of chattering

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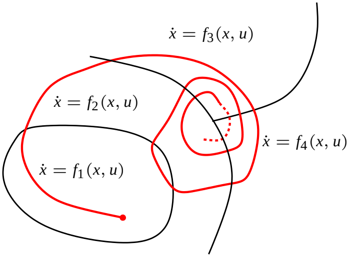
Examples of chattering: switching systems

switching systems



Cost

$$\underset{\substack{x(0)=\bar{x} \\ x(t_f)=0}}{\text{Minimize}} \int_0^{t_f} L(s, x, u)\, ds$$

accumulation of switchings between different locations

Occurring of chattering

Optimal control problem

$$\dot{x} = f_0(x) + u f_1(x) \quad u \in \mathcal{U} \qquad \underset{\substack{x(0)=\bar{x} \\ x(t_f)=0}}{\text{Minimize}} \int_0^{t_f} L_0(x) + uL_1(x)\, ds$$

- $f_0, f_1 \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$
- $\mathcal{U} = \{v(\cdot) \text{ measurable } v(t) \in [-1, 1]\}$
- $L_0, L_1 \in \mathcal{C}^\infty(\mathbb{R}^n)$

Ubiquity of Fuller's phenomenon [Kupka 1990]

In sufficiently high state dimension, for single-input control affine problems with control affine cost solutions are *generically* chattering.

Regularization method

OCP

$$\begin{cases} \text{Minimize}_{\substack{x(0)=\bar{x} \\ x(t_f)=0}} \int L(s,x,u)\, ds \\ \dot{x} = f(x,u), \quad u \in \mathcal{U} \end{cases}$$

OCP$_\varepsilon$

$$\begin{cases} \text{Minimize}_{\substack{x(0)=\bar{x} \\ x(t_f)=0}} \int L(s,x,u) ds + \varepsilon\, TV(u) \\ \dot{x} = f(x,u), \quad u \in \mathcal{U} \end{cases}$$

- $\varepsilon > 0$
- $TV(u) = \sup \sum_{i=1}^{n} |u(t_{i+1}) - u(t_i)|$

- problem OCP$_\varepsilon$ admits a solution u_ε which is "regular" ($TV(u_\varepsilon) < \infty$)
- convergence $\int L(s, x_\varepsilon, u_\varepsilon) \longrightarrow \int L(s, x^*, u^*)$ as $\varepsilon \to 0$, where u^* is a chattering solution of OCP

Regularization method

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General setting and assumptions I

Optimal control problem

$$\dot{x} = f(x,u), \quad u \in \mathcal{U} \qquad\qquad \underset{\substack{x(0)=\bar{x}\\ x(t_f)=0}}{\text{Minimize}} \int_0^{t_f} L(s,x(s),u(s))\,ds$$

where

- $f \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, $f(0,0) = 0$
- $\mathcal{U} = \{u(\cdot) \text{ measurable}, u(t) \in \mathbf{U}\}$, for a given subset $\mathbf{U} \subset \mathbb{R}^m$, $0 \in \mathring{\mathbf{U}}$
- $L \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m)$

General setting and assumptions II

Assumptions

Controllability

1. $\mathrm{Lie}_0\{f(u,\cdot) \mid u \in \mathbf{U}\} = \mathbb{R}^n$
2. 0 is small time locally controllable

Weak convexity

3. for every $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ the set

$$\{(f(x,u), L(t,x,u)+\gamma) \mid u \in \mathbf{U}, \gamma \geq 0\} \quad \text{is convex}$$

Compactness

4. $\mathbf{U}$ is compact
5. trajectories associated with admissible controls are uniformly bounded

Convergence

Theorem

Assume conditions 1-5 and assume that OCP has a solution (u^*, x^*)*. Then, for every* $\varepsilon > 0$*, OCP*$_\varepsilon$ *has a solution* $(u_\varepsilon, x_\varepsilon)$ *and*

$$\lim_{\varepsilon \to 0} \int_0^{t_\varepsilon} L(s, x_\varepsilon(s), u_\varepsilon(s))ds = \int_0^{t^*} L(s, x^*(s), u^*(s))ds.$$

Moreover,

- $t_\varepsilon \to t^*$,
- $x_\varepsilon \to x^*$ *uniformly.*

Idea of the proof

Step 1.

$$\left.\begin{array}{l}\text{controllability}\\ \text{compactness of } \{u \in \mathcal{U} \mid TV(u) \leq 1\}\\ \text{lower s.c. of } TV(\cdot)\end{array}\right\} \Rightarrow \text{existence for OCP}_\varepsilon$$

Step 2.

$$\left.\begin{array}{l}\text{convexity}\\ \text{compactness}\end{array}\right\} \Rightarrow \begin{array}{l}\text{existence of } w \in \mathcal{U} \text{ and } \gamma \in L^1, \gamma \geq 0 \text{ such that}\\ (f(x_\varepsilon, u_\varepsilon), L(\cdot, x_\varepsilon, u_\varepsilon)) \underset{w^* - L^\infty}{\longrightarrow} (f(x_w, w), L(\cdot, x_w, w) + \gamma)\end{array}$$

Step 3.

$$\left.\begin{array}{l}u_\varepsilon \text{ is optimal for OCP}_\varepsilon\\ \text{density of bv controls}\end{array}\right\} \Rightarrow \begin{array}{l}w \text{ is optimal for OCP,}\\ \text{i.e., } \int L(s, x^*, u^*) = \int L(s, x_w, w)\end{array}$$

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lower s.c. of $TV(\cdot)$
$\Rightarrow$ existence for OCP$_\varepsilon$

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$\Rightarrow$ existence of $w \in \mathcal{U}$ and $\gamma \in L^1, \gamma \geq 0$ such that
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Rate of convergence

Proposition

Assume that

- *conditions 1, 2 (controllability), and 4,5 (compactness) are satisfied;*
- *OCP has a solution* (u^*, x^*) *and there exists* $t_n \uparrow T^*$ *such that* $TV(u^*|_{[0,t_n]}) < \infty$*;*
- *the time-optimal map is* $\mathcal{C}^{0,\alpha}$ *for some* $\alpha \in (0,1]$ *on a neighborhood of* 0*.*

Then there exist a sequence $v_n : [0, T_n] \to \mathbf{U}$ *of admissible controls such that*

- $TV(v_n) < \infty$,
- $v_n \to u^*$ *strongly in* L^1*, and*

$$\int_0^{T_n} L(t, x_n, v_n)\,dt - \int_0^{T^*} L(t, x^*, u^*)\,dt \leq C(T^* - t_n)^\alpha.$$

Remarks

Theorem	Proposition
▶ general setting ▶ computable controls u_ε ▶ convergence only	▶ general setting ▶ controls v_n given by truncations of u^* ▶ convergence + rate in terms of switching times

[Zelikin 1999] - [Manita 1996]

- specific OCP (small perturbation of Fuller's dynamics in higher dimension with Fuller's cost
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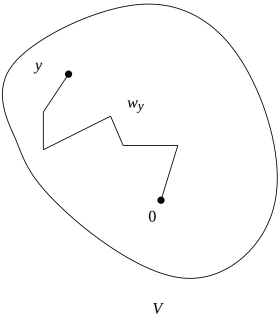
Open problems I

1. Rate of convergence for the regularized problem OCP$_\varepsilon$

$\Rightarrow$ uniform small time normal reachability property



Controllability assumptions $\Rightarrow$

- w_y is piecewise constant with at most q switchings,
- the time τ_y to steer y to 0 satisfies $\tau_y \leq C|y|$,

where q, C do not depend on y ?

Open problems II

2. Generic rate of convergence

- Fuller's problem: switching times converge *geometrically*
- Kupka 1990 $\Rightarrow$ exponential rate of convergence is "stable"

Can we expect lower rates of convergence? What rates are stable?