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# Regularization of chattering phenomena via bounded variation controls

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joint work with  
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# Outline

- 1 Motivation
  - what is chattering?
  - how often does it occur?
- 2 Convergence results
  - convergence for the perturbed problem
  - rate of convergence and switching times
- 3 Open problems
  - rate of convergence for the perturbed problem
  - generic rate of convergence

# Fuller's Problem [Fuller 1961]

## Control System

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \quad |u| \leq 1$$

## Cost

$$\begin{aligned} &\text{Minimize} && \int_0^{t_f} x_1^2(s) ds \\ &x(0) = \bar{x} \\ &x(t_f) = 0 \end{aligned}$$

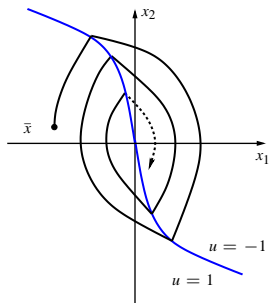
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- optimal controls are bang-bang with *infinitely many* switchings
- switchings take place on the curve  $\{x_1 + C|x_2|x_2 = 0\}$ ,  $C \sim 0.44$
- time intervals between consecutive switches decrease in a *geometric progression*

# Examples of chattering

## Control System

$$\dot{x} = f(x, u) \quad u \in \mathcal{U}$$

## Cost

$$\begin{array}{l} \text{Minimize} \\ x(0) = \bar{x} \\ x(t_f) = 0 \end{array} \int_0^{t_f} L(s, x, u) ds$$

# Examples of chattering: state constraints

## Control System

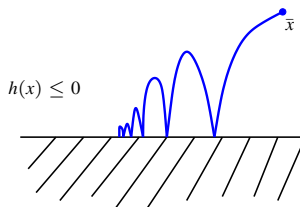
$$\dot{x} = f(x, u) \quad u \in \mathcal{U}$$

state constraints  $\implies$

## Cost

$$\begin{aligned} &\text{Minimize} && \int_0^{t_f} L(s, x, u) ds \\ &x(0) = \bar{x} \\ &x(t_f) = 0 \\ &h(x(s)) \leq 0 \end{aligned}$$

accumulation of contact points with the constraint's boundary



# Examples of chattering

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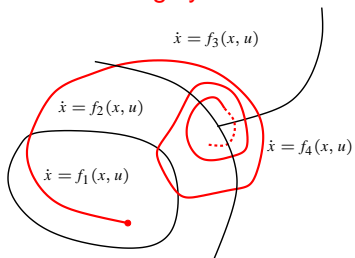
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# Examples of chattering: switching systems

switching systems



Cost

$$\text{Minimize} \int_0^{t_f} L(s, x, u) ds$$

$$x(0) = \bar{x}$$

$$x(t_f) = 0$$

accumulation of switchings between different locations

# Occurring of chattering

## Optimal control problem

$$\dot{x} = f_0(x) + u f_1(x) \quad u \in \mathcal{U} \quad \text{Minimize} \quad \int_0^{t_f} L_0(x) + u L_1(x) ds$$

$$x(0) = \bar{x} \quad x(t_f) = 0$$

- $f_0, f_1 \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$
- $\mathcal{U} = \{v(\cdot) \text{ measurable } v(t) \in [-1, 1]\}$
- $L_0, L_1 \in \mathcal{C}^\infty(\mathbb{R}^n)$

## Ubiquity of Fuller's phenomenon [Kupka 1990]

In sufficiently high state dimension, for single-input control affine problems with control affine cost solutions are *generically* chattering.

# Regularization method

## OCP

$$\left\{ \begin{array}{l} \text{Minimize } \int L(s, x, u) ds \\ x(0) = \bar{x} \\ x(t_f) = 0 \\ \dot{x} = f(x, u), \quad u \in \mathcal{U} \end{array} \right.$$

## OCP<sub>ε</sub>

$$\left\{ \begin{array}{l} \text{Minimize } \int L(s, x, u) ds + \varepsilon TV(u) \\ x(0) = \bar{x} \\ x(t_f) = 0 \\ \dot{x} = f(x, u), \quad u \in \mathcal{U} \end{array} \right.$$

$$- \varepsilon > 0$$

$$- TV(u) = \sup \sum_{i=1}^n |u(t_{i+1}) - u(t_i)|$$

- problem OCP<sub>ε</sub> admits a solution  $u_\varepsilon$  which is “regular” ( $TV(u_\varepsilon) < \infty$ )
- convergence  $\int L(s, x_\varepsilon, u_\varepsilon) \rightarrow \int L(s, x^*, u^*)$  as  $\varepsilon \rightarrow 0$ , where  $u^*$  is a chattering solution of OCP

# Regularization method

OCP

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# General setting and assumptions I

## Optimal control problem

$$\dot{x} = f(x, u), \quad u \in \mathcal{U}$$

$$\begin{aligned} \text{Minimize} \quad & \int_0^{t_f} L(s, x(s), u(s)) ds \\ & x(0) = \bar{x} \\ & x(t_f) = 0 \end{aligned}$$

where

- $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ ,  $f(0, 0) = 0$
- $\mathcal{U} = \{u(\cdot) \text{ measurable}, u(t) \in \mathbf{U}\}$ , for a given subset  $\mathbf{U} \subset \mathbb{R}^m$ ,  $0 \in \overset{\circ}{\mathbf{U}}$
- $L \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m)$

# General setting and assumptions II

## Assumptions

### Controllability

1.  $\text{Lie}_0\{f(u, \cdot) \mid u \in \mathbf{U}\} = \mathbb{R}^n$
2. 0 is small time locally controllable

### Weak convexity

3. for every  $(t, x) \in \mathbb{R} \times \mathbb{R}^n$  the set

$$\{(f(x, u), L(t, x, u) + \gamma) \mid u \in \mathbf{U}, \gamma \geq 0\} \text{ is convex}$$

### Compactness

4.  $\mathbf{U}$  is compact
5. trajectories associated with admissible controls are uniformly bounded

# Convergence

## Theorem

Assume conditions 1-5 and assume that OCP has a solution  $(u^*, x^*)$ . Then, for every  $\varepsilon > 0$ ,  $OCP_\varepsilon$  has a solution  $(u_\varepsilon, x_\varepsilon)$  and

$$\lim_{\varepsilon \rightarrow 0} \int_0^{t_\varepsilon} L(s, x_\varepsilon(s), u_\varepsilon(s)) ds = \int_0^{t^*} L(s, x^*(s), u^*(s)) ds.$$

Moreover,

- $t_\varepsilon \rightarrow t^*$ ,
- $x_\varepsilon \rightarrow x^*$  uniformly.



# Idea of the proof

## Step 1.

controllability  
 compactness of  $\{u \in \mathcal{U} \mid TV(u) \leq 1\}$   
 lower s.c. of  $TV(\cdot)$

$$\left. \vphantom{\begin{array}{l} \text{controllability} \\ \text{compactness of } \{u \in \mathcal{U} \mid TV(u) \leq 1\} \\ \text{lower s.c. of } TV(\cdot) \end{array}} \right\} \Rightarrow \text{existence for OCP}_\varepsilon$$

## Step 2.

convexity  
 compactness

$$\left. \vphantom{\begin{array}{l} \text{convexity} \\ \text{compactness} \end{array}} \right\} \Rightarrow \text{existence of } w \in \mathcal{U} \text{ and } \gamma \in L^1, \gamma \geq 0 \text{ such that}$$

$$(f(x_\varepsilon, u_\varepsilon), L(\cdot, x_\varepsilon, u_\varepsilon)) \xrightarrow{w^* - L^\infty} (f(x_w, w), L(\cdot, x_w, w) + \gamma)$$

## Step 3.

$u_\varepsilon$  is optimal for  $\text{OCP}_\varepsilon$   
 density of bv controls

$$\left. \vphantom{\begin{array}{l} u_\varepsilon \text{ is optimal for } \text{OCP}_\varepsilon \\ \text{density of bv controls} \end{array}} \right\} \Rightarrow \begin{array}{l} w \text{ is optimal for OCP,} \\ \text{i.e., } \int L(s, x^*, u^*) = \int L(s, x_w, w) \end{array}$$

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$$\left. \vphantom{\begin{array}{l} w \text{ is optimal for OCP} \\ \text{density of bv controls} \end{array}} \right\} \Rightarrow \begin{array}{l} \text{convergence} \\ \int L(s, x_\varepsilon, u_\varepsilon) \text{ to } \int L(s, x^*, u^*) \end{array}$$

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# Rate of convergence

## Proposition

Assume that

- conditions 1, 2 (controllability), and 4,5 (compactness) are satisfied;
- OCP has a solution  $(u^*, x^*)$  and there exists  $t_n \uparrow T^*$  such that  $TV(u^*|_{[0, t_n]}) < \infty$ ;
- the time-optimal map is  $C^{0, \alpha}$  for some  $\alpha \in (0, 1]$  on a neighborhood of 0.

Then there exist a sequence  $v_n : [0, T_n] \rightarrow \mathbf{U}$  of admissible controls such that

- $TV(v_n) < \infty$ ,
- $v_n \rightarrow u^*$  strongly in  $L^1$ , and

$$\int_0^{T_n} L(t, x_n, v_n) dt - \int_0^{T^*} L(t, x^*, u^*) dt \leq C(T^* - t_n)^\alpha.$$

# Remarks

Theorem	Proposition
<ul style="list-style-type: none"><li>▶ general setting</li><li>▶ computable controls <math>u_\varepsilon</math></li><li>▶ convergence only</li></ul>	<ul style="list-style-type: none"><li>▶ general setting</li><li>▶ controls <math>v_n</math> given by truncations of <math>u^*</math></li><li>▶ convergence + rate in terms of switching times</li></ul>

[Zelikin 1999] - [Manita 1996]

- specific OCP (small perturbation of Fuller's dynamics in higher dimension with Fuller's cost)
- controls  $v_n$  given by truncations of  $u^*$
- convergence + rate in terms of switching times (exponential in this case)



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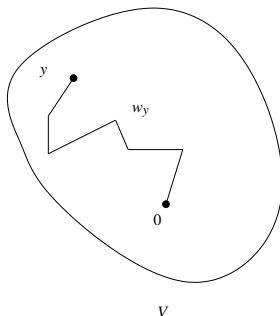
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# Open problems I

## 1. Rate of convergence for the regularized problem $\text{OCP}_\varepsilon$

$\Rightarrow$  uniform small time normal reachability property



Controllability assumptions  $\Rightarrow$

- $w_y$  is piecewise constant with **at most  $q$  switchings**,
- the time  $\tau_y$  to steer  $y$  to  $0$  satisfies  **$\tau_y \leq C|y|$** ,

where  **$q, C$  do not depend on  $y$** ?

# Open problems II

## 2. Generic rate of convergence

- Fuller's problem: switching times converge *geometrically*
- Kupka 1990  $\Rightarrow$  exponential rate of convergence is "stable"

Can we expect lower rates of convergence? What rates are stable?