



# Mean field games on graphs

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# Mean field games on graphs

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# Outline

- 1 Introduction
- 2 MFG on graphs: setup
- 3 Existence result
- 4 Uniqueness result
- 5 Potential games and Master equation

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# Introduction

- Mean field games: introduced in 2005 by Pierre-Louis Lions and Jean-Michel Lasry
- Stochastic games when the number of (identical) players  $\rightarrow \infty$ .
- Mean field approach to model interactions between players/agents: each player considers the distribution of others instead of individual states.

## Usual framework

- Each individual solves a (stochastic) control problem. The objective function depends on the distribution of other players.  
→ HJB equation, backward in time.
- The distribution of players evolves according to players' optimal controls.  
→ transport equation, forward in time.

### MFG PDEs (continuous state space - usual framework)

$$(HJB) \quad \partial_t u + \frac{\sigma^2}{2} \Delta u + H(\nabla u, x, m) = -f(x, m)$$

$$(K) \quad \partial_t m - \frac{\sigma^2}{2} \Delta m + \nabla \cdot (m \partial_p H(\nabla u, x, m)) = 0$$

$$u(T, x) = g(x, m(T, \cdot)), \quad m(0, x) = m_0(x)$$

## Literature

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- Mean field game theory:
  - Existence and uniqueness results (Lasry-Lions, ...)
  - Special cases, mainly linear-quadratic MFGs
  - Long time behavior
  - Limit of  $N$ -player games
  - Probabilistic approach (Carmona and Delarue)



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- Mean field game theory:
  - Existence and uniqueness results (Lasry-Lions, ...)
  - Special cases, mainly linear-quadratic MFGs
  - Long time behavior
  - Limit of  $N$ -player games
  - Probabilistic approach (Carmona and Delarue)
- Numerical approximations of MFG PDEs:
  - General finite difference methods (see e.g. Achdou et al.)
  - Specific schemes with (unusual) monotonicity properties in the case of quadratic Hamiltonians
  - Gradient methods in the case of potential games
  - Talks today

# Literature

- Applications:
  - Many toy models (G., Lasry, Lions)
  - Growth models (e.g. Lucas and Moll)  
*Remark: Interesting paper “PDE Models in Macroeconomics”*
  - Competition between oil producers (Chan, Sircar)
  - Population dynamics (Achdou et al.)
  - Maybe applications to machine learning in the future...

Our goal here is to prove existence and (very general) uniqueness results for mean field games on a graph (discrete state space). In particular, we will consider congestion effects.

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# Framework

- Mean Field Games: Continuum of anonymous and identical (indistinguishable) players
- Graph  $\mathcal{G}$ . Nodes indexed by integers from 1 to  $N$ .
- $\forall i \in \mathcal{N} = \{1, \dots, N\}$ :
  - $\mathcal{V}(i) \subset \mathcal{N} \setminus \{i\}$  the set of nodes  $j$  for which a directed edge exists from  $i$  to  $j$  (cardinal:  $d_i$ ).
  - $\mathcal{V}^{-1}(i) \subset \mathcal{N} \setminus \{i\}$  the set of nodes  $j$  for which a directed edge exists from  $j$  to  $i$ .

## Framework (continued)

- Each player's position: Markov chain  $(X_t)_t$  with values in  $\mathcal{G}$
- Distribution of players:  $t \mapsto m(t) = (m(1, t), \dots, m(N, t))$
- Instantaneous transition probabilities at time  $t$ :

$$\lambda_t(i, \cdot) : \mathcal{V}(i) \rightarrow \mathbb{R}_+ \quad (\forall i \in \mathcal{N})$$

- Running payoff:
  - Running costs:  $\mathcal{L}(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}, m(t))$  to set the value of  $\lambda_t(i, j)$  to  $\lambda_{i,j}$ .
  - Running gain:  $f(i, m(t))$ , where  $i$  is the current state.
- Terminal payoff:  $g(i, m(T))$ .

# Assumptions

- Continuity:  $\forall i \in \mathcal{N}$ , the functions  $\mathcal{L}(i, \cdot, \cdot)$ ,  $f(i, \cdot)$  and  $g(i, \cdot)$  are continuous.
- Convexity of the cost functions:  $\forall i \in \mathcal{N}, \forall \mu \in \mathbb{R}_+^N$ ,  $\mathcal{L}(i, \cdot, \mu)$  is a strictly convex function.
- Asymptotic super-linearity of the cost functions:

$$\forall i \in \mathcal{N}, \forall K > 0, \quad \lim_{\lambda \in \mathbb{R}_+^{d_i}, |\lambda| \rightarrow +\infty} \inf_{\mu \in [0, K]^N} \frac{\mathcal{L}(i, \lambda, \mu)}{|\lambda|} = +\infty.$$

We define the Hamiltonian functions  $\mathcal{H}(i, \cdot, \cdot)$  by:

$$\forall i \in \mathcal{N}, (p, \mu) \in \mathbb{R}^{d_i} \times \mathbb{R}_+^N \mapsto \mathcal{H}(i, p, \mu) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i, \lambda, \mu).$$

# Nash-MFG equilibrium

- We denote by  $\mathcal{A}$  the set of admissible controls  $\lambda$  ( $L^\infty$  and deterministic).
- For a given differentiable function  $m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$  (total mass=1) we define:

$$J_m(i, t, \lambda) = \mathbb{E} \left[ \int_t^T (-\mathcal{L}(X_s, \lambda_s(X_s, \cdot), m(s)) + f(X_s, m(s))) ds + g(X_T, m(T)) \right]$$

for  $(X_s)_{s \in [t, T]}$  a Markov chain on  $\mathcal{G}$ , starting from  $i$  at time  $t$ , with instantaneous transition probabilities given by  $(\lambda_s)_{s \in [t, T]}$ .

# Nash-MFG equilibrium

## Nash-MFG equilibrium

A differentiable function

$m : t \in [0, T] \mapsto (m(1, t), \dots, m(N, t)) \in \mathcal{P}_N$  is said to be a Nash-MFG equilibrium, if there exists an admissible control  $\lambda \in \mathcal{A}$  such that:

$$\forall \tilde{\lambda} \in \mathcal{A}, \forall i \in \mathcal{N}, J_m(i, 0, \lambda) \geq J_m(i, 0, \tilde{\lambda}),$$

and

$$\forall i \in \mathcal{N}, \frac{d}{dt} m(i, t) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_t(j, i) m(j, t) - \sum_{j \in \mathcal{V}(i)} \lambda_t(i, j) m(i, t)$$



# $\mathcal{G}$ -MFG equations

The  $\mathcal{G}$ -MFG equations consist in a system of  $2N$  equations:

## $\mathcal{G}$ -MFG equations

$N$  Hamilton-Jacobi equations:

$$\frac{d}{dt}u(i, t) + \mathcal{H}\left(i, (u(j, t) - u(i, t))_{j \in \mathcal{V}(i)}, m(1, t), \dots, m(N, t)\right) \\ + f(i, m(1, t), \dots, m(N, t)) = 0$$

with  $u(i, T) = g(i, m(1, T), \dots, m(N, T))$ .

and...

## $\mathcal{G}$ -MFG equations

### $\mathcal{G}$ -MFG equations (continued)

...  $N$  forward transport equations:

$$\begin{aligned} \frac{d}{dt} m(i, t) = & \sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial \mathcal{H}(j, \cdot, m(t))}{\partial p_i} \left( (u(k, t) - u(j, t))_{k \in \mathcal{V}(j)} \right) m(j, t) \\ & - \sum_{j \in \mathcal{V}(i)} \frac{\partial \mathcal{H}(i, \cdot, m(t))}{\partial p_j} \left( (u(k, t) - u(i, t))_{k \in \mathcal{V}(i)} \right) m(i, t) \end{aligned}$$

with  $(m(1, 0), \dots, m(N, 0)) = m^0$ .

*Remark: in the literature on MFG, people usually do not care about verification (why?). On graphs, there is no issue as we shall find  $C^1$  solutions.*

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# Existence

## Theorem (Existence)

*Under the assumptions made above, there exists a  $C^1$  solution  $(u, m)$  of the  $\mathcal{G}$ -MFG equations.*

To prove this result we need a Lemma. This lemma is a comparison principle.

## Comparison principle

### Lemma (Comparison principle)

Let  $m : [0, T] \rightarrow \mathcal{P}_N$  be a continuous function.

Let  $u : [0, T] \rightarrow \mathbb{R}^n$  be a  $C^1$  function that verifies:

$$\forall i, -\frac{d}{dt}u(i, t) - \mathcal{H}\left(i, (u(j, t) - u(i, t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \leq 0$$

with  $u(i, T) \leq g(i, m(T))$ .

Let  $v : [0, T] \rightarrow \mathbb{R}^n$  be a  $C^1$  function that verifies:

$$\forall i, -\frac{d}{dt}v(i, t) - \mathcal{H}\left(i, (v(j, t) - v(i, t))_{j \in \mathcal{V}(i)}, m(t)\right) - f(i, m(t)) \geq 0$$

with  $v(i, T) \geq g(i, m(T))$ .

Then,  $\forall i \in \mathcal{N}, \forall t \in [0, T], v(i, t) \geq u(i, t)$ .

## Sketch of proof for the Theorem

- $N$  HJ equations defines a function  $B : m \mapsto u$
- Comparison principle leads a priori bounds on  $u$

$$\sup_{i \in \mathcal{N}} \|u(i, \cdot)\|_{\infty} \leq \sup_{i \in \mathcal{N}} \|g(i, \cdot)\|_{\infty} \\ + T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |\mathcal{H}(i, 0, \mu)| + T \sup_{i \in \mathcal{N}, \mu \in \mathcal{P}_N} |f(i, \mu)|.$$

- $N$  transport equations defines a function  $F : u \mapsto m$
- Bounds on  $u$  lead to bounds on  $\frac{dm}{dt}$
- Ascoli + Schauder permit to obtain a fixed point to  $m \mapsto F(B(m))$ .

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## Classical uniqueness result

It is possible to derive a uniqueness result equivalent to the one presented in Lasry-Lions' initial paper:

### Theorem (Classical Uniqueness)

Assume that  $g$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (g(i, \nu) - g(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

Assume that  $f$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (f(i, \nu) - f(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu$$



## Classical uniqueness result

### Theorem (Classical Uniqueness (continued))

Assume that  $\mathcal{H}$  **does not** depend on  $m$ .

Then, if  $(\hat{u}, \hat{m})$  and  $(\tilde{u}, \tilde{m})$  are two  $C^1$  solutions of the  $\mathcal{G}$ -MFG equations, we have  $\hat{m} = \tilde{m}$  and  $\hat{u} = \tilde{u}$ .

In words, this says that if people **do not** want to live together and if the cost to move **does not** depend on the number of people at specific nodes, then uniqueness is ensured.

How can this be extended? By adding  $m$  in the the Hamiltonian functions!

→ Congestion models for instance.

# General uniqueness result

## Theorem (Uniqueness)

Assume that  $g$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (g(i, \nu) - g(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu.$$

Assume that  $f$  is such that:

$$\forall (\nu, \mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^N (f(i, \nu) - f(i, \mu))(\nu_i - \mu_i) \geq 0 \implies \nu = \mu$$

Assume that  $\mathcal{H}$  is such that  $\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \frac{\partial \mathcal{H}}{\partial p_j}(i, \cdot, \cdot)$  is a  $C^1$  function on  $\mathbb{R}^{d_i} \times \mathbb{R}_+^N$ .

# General uniqueness result

## Theorem (Uniqueness (continued))

Let us define

$A : (q_1, \dots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\alpha_{ij}(q_i, \mu))_{i,j} \in \mathcal{M}_N$  by:

$$\alpha_{ij}(q_i, \mu) = -\frac{\partial \mathcal{H}}{\partial \mu_j}(i, q_i, \mu).$$

Let us also define

$\forall i \in \mathcal{N}, B^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\beta_{jk}^i(q_i, \mu))_{j,k} \in \mathcal{M}_{N,d_i}$  by:

$$\beta_{jk}^i(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_j \partial q_{ik}}(i, q_i, \mu).$$

# General uniqueness result

## Theorem (Uniqueness (continued))

Let us also define,  $\forall i \in \mathcal{N}$ ,

$C^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left( \gamma_{jk}^i(q_i, \mu) \right)_{j,k} \in \mathcal{M}_{d_i, N}$  defined by:

$$\gamma_{jk}^i(q_i, \mu) = \frac{\mu_i}{2} \frac{\partial^2 \mathcal{H}}{\partial \mu_k \partial q_{ij}}(i, q_i, \mu).$$

Let us finally define,  $\forall i \in \mathcal{N}$ ,

$D^i : (q_i, \mu) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left( \delta_{jk}^i(q_i, \mu) \right)_{j,k} \in \mathcal{M}_{d_i}$  defined by:

$$\delta_{jk}^i(q_i, \mu) = \mu_i \frac{\partial^2 \mathcal{H}}{\partial q_{ij} \partial q_{ik}}(i, q_i, \mu).$$

# General uniqueness result

## Theorem (Uniqueness (continued))

Assume that  $\forall (q_1, \dots, q_N, \mu) \in \prod_{i=1}^N \mathbb{R}^{d_i} \times \mathcal{P}_N$ :

$$\begin{pmatrix} A(q_1, \dots, q_N, \mu) & B^1(q_1, \mu) & \cdots & \cdots & \cdots & B^N(q_N, \mu) \\ C^1(q_1, \mu) & D^1(q_1, \mu) & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ C^N(q_N, \mu) & 0 & \cdots & \cdots & 0 & D^N(q_N, \mu) \end{pmatrix} \geq 0.$$

Then, if  $(\hat{u}, \hat{m})$  and  $(\tilde{u}, \tilde{m})$  are two  $C^1$  solutions of the  $\mathcal{G}$ -MFG equations, we have  $\hat{m} = \tilde{m}$  and  $\hat{u} = \tilde{u}$ .

# General uniqueness result – Proof

## Sketch of proof:

Classical idea of computing the value of

$$I = \sum_{i=1}^N \int_0^T \frac{d}{dt} ((\hat{u}(i, t) - \tilde{u}(i, t))(\hat{m}(i, t) - \tilde{m}(i, t))) dt$$

in two different ways.

We first know directly that

$$I = \sum_{i=1}^N (g(i, \hat{m}(T)) - g(i, \tilde{m}(T)))(\hat{m}(i, T) - \tilde{m}(i, T)).$$

# General uniqueness result – Proof

We introduce:

- $M(q_1, \dots, q_N, \mu)$  the matrix of the Theorem
- Convex combinations of solutions:  $u^\theta = \theta \hat{u} + (1 - \theta) \tilde{u}$  and  $m^\theta = \theta \hat{m} + (1 - \theta) \tilde{m}$
- A vector  $V$ :

$$V(t) = (\hat{m}(t) - \tilde{m}(t), ((\hat{u}(k, t) - \tilde{u}(k, t)) - (\hat{u}(1, t) - \tilde{u}(1, t))))_{k \in \mathcal{V}(1)}, \dots, ((\hat{u}(k, t) - \tilde{u}(k, t)) - (\hat{u}(N, t) - \tilde{u}(N, t))))_{k \in \mathcal{V}(N)}).$$

## General uniqueness result – Proof

If we differentiate the expression in  $I$  and rearrange the terms (not obvious), we have:

$$I = - \int_0^T \sum_{i=1}^N (f(i, \hat{m}(t)) - f(i, \tilde{m}(t))) (\hat{m}(i, t) - \tilde{m}(i, t)) dt + J,$$

where

$$J = \int_0^T \int_0^1 V(t) M((u^\theta(k, t) - u^\theta(1, t))_{k \in \mathcal{V}(1)}, \dots, (u^\theta(k, t) - u^\theta(N, t))_{k \in \mathcal{V}(N)}, m^\theta(t)) V(t)' d\theta dt.$$

Uniqueness then comes from  $M \geq 0$  and  $f$  and  $g$  decreasing.



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# Potential games

We consider the special case where:

- The Hamiltonian functions (or equivalently the cost functions) do not depend on  $m$ .
- There are two functions  $F$  and  $G$  such that:  
$$\frac{\partial F}{\partial m_i} = f(i, \cdot) \text{ and } \frac{\partial G}{\partial m_i} = g(i, \cdot)$$

# Potential games

We consider the problem of **one** planner:

$$\sup_{\lambda \in \mathcal{A}} \mathcal{J}(0, m^0, \lambda)$$

where  $\mathcal{J}(t, m^t, \lambda)$

$$= \int_t^T \left( F(m(s)) - \sum_{i=1}^N L(i, (\lambda_s(i, j))_{j \in \mathcal{V}(i)}) m(s, i) \right) ds + G(m(T))$$

where  $\forall i \in \mathcal{N}, m(t, i) = m_i^t$  and  $\forall i \in \mathcal{N}, \forall s \in [t, T]$ ,

$$\frac{d}{ds} m(s, i) = \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_s(j, i) m(s, j) - \sum_{j \in \mathcal{V}(i)} \lambda_s(i, j) m(s, i).$$

## Potential games - HJ equation

The HJ equation associated to this problem is:

$$\frac{\partial \Phi}{\partial t}(t, m_1, \dots, m_N) + \mathcal{H}(m_1, \dots, m_N, \nabla \Phi) + F(m_1, \dots, m_N) = 0$$

Terminal conditions:  $\Phi(T, m_1, \dots, m_N) = G(m_1, \dots, m_N)$ .

The Hamiltonian function is  $\mathcal{H}(m_1, \dots, m_N, p) =$

$$\begin{aligned} \sup_{(\lambda_{i,j})_{i \in \mathcal{N}, j}} \sum_{i=1}^N \left[ \left( \sum_{j \in \mathcal{V}^{-1}(i)} \lambda_{j,i} m_j - \sum_{j \in \mathcal{V}(i)} \lambda_{i,j} m_i \right) p_i - L(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}) m_i \right] \\ = \sum_{i=1}^N m_i H \left( i, (p_j - p_i)_{j \in \mathcal{V}(i)} \right) \end{aligned}$$

# The Master equation

Differentiating formally we obtain for  $U = \nabla\Phi$  the following equations:

$$0 = \frac{\partial U_i}{\partial t} + H\left(i, (U_j - U_i)_{j \in \mathcal{V}(i)}\right) \\ + \sum_{j=1}^N m_j \sum_{k \in \mathcal{V}(j)} \left( \frac{\partial U_i}{\partial m_k} - \frac{\partial U_i}{\partial m_j} \right) \frac{\partial H(j, \cdot)}{\partial p_k} \left( (U_l - U_j)_{l \in \mathcal{V}(j)} \right) + f(i, \cdot) \\ U_i(T, m) = g(i, m)$$

This would be the HJ equation of the initial MFG equation, had we considered  $m$  a state variable. This is the **Master equation!**

# Conclusion

With potential games:

- 1  $HJ$  equations
- $N$  equations (Master equation)
- $2N$  equations for the underlying MFG problem. A solution is:

$$m(0) = m^0, \quad \frac{d}{dt}m(t, i) =$$

$$\sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial H(j, \cdot)}{\partial p_i} \left( (U_k(t, m(t)) - U_j(t, m(t)))_{k \in \mathcal{V}(j)} \right) m(t, j)$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial H(i, \cdot)}{\partial p_j} \left( (U_k(t, m(t)) - U_i(t, m(t)))_{k \in \mathcal{V}(i)} \right) m(t, i)$$

$$\forall i \in \mathcal{N}, u(t, i) = U_i(t, m(t, 1), \dots, m(t, N))$$

Questions?