

# Asymptotic models for Hamilton-Jacobi-Bellman equations

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# Asymptotic models for Hamilton–Jacobi–Bellman equations

**A. Siconolfi**

Università di Roma *La Sapienza*

partially in collaboration with Nguyen Thuong

Tours, June 2014

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However [Arstein–Gaitsgory](#) (2000) have studied a similar model with other techniques where periodicity is in a sense replaced by [coercivity](#) conditions.

[Our aim](#) is to recover their results adapting Alvarez–Bardi techniques to the nonperiodic case.

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- $f, g : \mathbb{R}^N \times \mathbb{R}^N \times A$  satisfy the usual conditions in order to have, for any initial datum  $(x_0, y_0)$  and control  $\alpha$  unique solution  $(x(t; x_0, y_0, \alpha), y(t; x_0, y_0, \alpha))$  to  $(\text{CD}_\varepsilon)$

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$$\begin{aligned} x(\cdot) &= x(\cdot, x_0, y_0, \alpha) \\ y(\cdot) &= y(\cdot, x_0, y_0, \alpha) \end{aligned}$$

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- study the convergence of the value functions of  $(\text{Min}_\varepsilon)$  to the value function of the limit problem.

The form of the [limit problem](#) is well understood thanks to a series of papers of [Arstein](#) and [Gaitsgory](#) (1998–2005) where they adapt, to describe it, the [Krylov–Bogolyubov](#) theory of invariant measures for flows.

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We define a probability measure on  $\mathbb{R}^N \times A$  via

$$\frac{1}{t} \int_0^t \mathbb{I}_E(y(s), \alpha(s)) ds$$

where  $E$  is any Borel subset of  $\mathbb{R}^N \times A$  and  $\mathbb{I}_E$  the corresponding indicator function.

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The **controls of the limit system** (depending on the state) are given by the closure  $M(x_0)$  with respect to the **weak convergence** of the previous measures.

These are called **limit occupational** or **holonomic** measures.

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# Convergence of the value functions

We write the Hamiltonian

$$H(x, y, p, q) = \max_{a \in A} \{-p \cdot f(x, y, a) - q \cdot g(x, y, a) - \ell(x, y, a)\}$$

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we write the family of Hamilton–Jacobi–Bellman equations

$$\begin{cases} v_t^\varepsilon + H\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon}\right) & = 0 \\ v^\varepsilon(x, y, 0) & = u_0(x) \end{cases} \quad (\text{Hj}_\varepsilon)$$



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Under our assumptions on  $\ell$ ,  $u_0$ , the value functions  $v^\varepsilon$  of  $(\text{Min}_\varepsilon)$  uniquely solve  $(\text{Hj}_\varepsilon)$ .

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We aim at studying the asymptotic behavior  $v^\varepsilon$  in relation to the limit problem  $(\bar{H}j)$ .

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plus a [controllability](#) condition :

- Given  $x_0$ , any pair  $y_1, y_2$  of values the fast variable can be linked (up to a vector of  $\mathbb{Z}^N$ ) by a trajectory of a fast dynamics, with  $x_0$  fixed, in a time  $t \leq T(x_0)$ .

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## Theorem

*If the system satisfies the periodicity and controllability conditions, and, in addition, the limit equation satisfies a comparison principle, then the value functions  $v^\varepsilon(x, y, t)$  [locally uniformly converge](#) to the value function  $v(x, t)$  of the limit problem  $(\overline{H_j})$ .*



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If the comparison principle does not hold for the limit equation then a weaker convergence result is obtained through [semilimits](#).

The theorem is based on the analysis of the family of the so-called [cell problems](#), which are stationary Hamilton–Jacobi–Bellman equations in the fast variable.

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The controllability condition can be viewed as a weak form of coercivity, which is needed for homogenization. There are examples of periodic **noncoercive** Hamiltonians for which **homogenization does not take place**.



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## Proposition

*Under periodicity and controllability assumptions, there is, for any  $(x_0, p_0)$ , an unique value  $c_0 = c_0(x_0, p_0)$  such that the corresponding cell equation  $(\text{Cell}(x_0, p_0)_c)$ , with  $c = c_0$ , admits an usc periodic subsolution and a lsc periodic supersolution problem. In this case*

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Periodic sub/super solutions of the cell problem play the role of **correctors** allowing to adapt Evans perturbed test function method to this setting.

Our contribution is to extend this technique to the noncompact (non periodic) case, replacing periodicity by a coercivity condition on the cost:

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We also assume a local version of controllability

- Given  $x_0$ , the previous controllability condition holds in any compact subset, say  $K$  of the fast variable and the time to link two points of  $K$ , say  $y_1$  and  $y_2$ , depends on  $x_0$  and  $K$ .

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We obtain the following result:

## Theorem

*If the system satisfies the coercivity and local controllability conditions, and, in addition, the limit equation satisfies a comparison principle, then we get the same assertion of Alvarez–Bardi Theorem.*

In this new setting there is **no hope to have periodic or even bounded sub/supersolutions** of the cell problem  $(\text{Cell}(x_0, p_0)_c)$  equation for some distinguished  $c$ .



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Natural questions are:

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- If yes, are the sub/supersolution so selected suitable for adapted perturbed test function method ?

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Answer to the other questions are positive, and this is actually the way of proving our convergence theorem.

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- Use representation formulas for solutions to  $H_k = c_0^k$  via Aubry set to show that  $c_0^k$  is the unique value for which there are coercive solutions
- Exploit local controllability to show that there is a sequence of locally equibounded coercive solutions  $u_k$  to  $H_k = c_0^k$

The argument of ergodicity result is based on the following steps:

- **Approximate**, in the local uniform convergence, the Hamiltonian  $(y, p) \mapsto H(x_0, y, p_0, p)$  by a sequence of coercive Hamiltonians  $H_k(y, p)$ , decreasing in  $k$
- Define  $c_0^k = \min\{c \mid H_k = c \text{ admits subsolutions in } \mathbb{R}^N\}$
- Exploit coercivity condition to prove that there is a corresponding nonempty **Aubry sets**  $\mathcal{A}_k$
- Use **representation formulas** for solutions to  $H_k = c_0^k$  via **Aubry set** to show that  $c_0^k$  is the unique value for which there are **coercive solutions**
- Exploit local controllability to show that there is a **sequence of locally equibounded coercive solutions**  $u_k$  to  $H_k = c_0^k$
- **pass to the semilimits**, as  $k \rightarrow +\infty$  to get sub/supersolution to

$$H(x_0, y, p_0, Du) = c_0(x_0, p_0) =: \lim_k c_k$$

with the desired properties.