Asymptotic models for Hamilton-Jacobi-Bellman equations

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Tours, June 2014

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Our aim is to recover their results adapting Alvarez–Bardi techniques to the nonperiodic case.

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- α is measurable bounded control taking values in a compact subset A of $\mathbb{R}^M$
- $f, g : \mathbb{R}^N \times \mathbb{R}^N \times A$ satisfy the usual conditions in order to have, for any initial datum (x_0, y_0) and control α unique solution $(x(t; x_0, y_0, \alpha), y(t; x_0, y_0, \alpha))$ to $(\mathrm{CD}_\varepsilon)$

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and the related optimal control problem

$$\min_{\alpha} \int_0^t \ell(x(s), y(s), \alpha(s))\, ds + u_0(x(t)) \qquad (\mathrm{Min}_\varepsilon)$$

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$$\begin{aligned} x(\cdot) &= x(\cdot, x_0, y_0, \alpha) \\ y(\cdot) &= y(\cdot, x_0, y_0, \alpha) \end{aligned}$$

solutions to $(\mathrm{CD}_\varepsilon)$ with initial data (x_0, y_0).

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The form of the limit problem is well understood thanks to a series of papers of Arstein and Gaitsgory (1998–2005) where they adapt, to describe it, the Krylov–Bogolyubov theory of invariant measures for flows.

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We define a probability measure on $\mathbb{R}^N \times A$ via

$$\frac{1}{t} \int_0^t \mathbb{I}_E(y(s), \alpha(s))\, ds$$

where E is any Borel subset of $\mathbb{R}^N \times A$ and $\mathbb{I}_E$ the corresponding indicator function.

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The controls of the limit system (depending on the state) are given by the closure $M(x_0)$ wit respect to the weak convergence of the previous measures.

These are called limit occupational or holonomic measures.

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and the limit cost by

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Convergence of the value functions

We write the Hamiltonian

$$H(x, y, p, q) = \max_{a \in A}\{-p \cdot f(x, y, a) - q \cdot g(x, y, a) - \ell(x, y, a)\}$$

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we write the family of Hamilton–Jacobi–Bellman equations

$$\begin{cases} v_t^\varepsilon + H\left(x, y, D_x v^\varepsilon, \frac{D_y v^\varepsilon}{\varepsilon}\right) & = 0 \\ v^\varepsilon(x, y, 0) & = u_0(x) \end{cases} \qquad (\mathrm{Hj}_\varepsilon)$$

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Under our assumptions on ℓ, u_0, the value functions v^ε of $(\mathrm{Min}_\varepsilon)$ uniquely solve $(\mathrm{Hj}_\varepsilon)$.

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We aim at studying the asymptotic behavior v^ε in relation to the limit problem $(\overline{Hj})$.

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plus a controllability condition :

- Given x_0, any pair y_1, y_2 of values the fast variable can be linked (up to a vector of $\mathbb{Z}^N$) by a trajectory of a fast dynamics, with x_0 fixed, in a time $t \leq T(x_0)$.

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Theorem

If the system satisfies the periodicity and controllability conditions, and, in addition, the limit equation satisfies a comparison principle, then the value functions $v^\varepsilon(x, y, t)$ locally uniformly converge to the value function $v(x, t)$ of the limit problem $(\overline{Hj})$.

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If the comparison principle does not hold for the limit equation then a weaker convergence result is obtained through semilimits.

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The controllability condition can be viewed as a weak form of coercivity, which is needed for homogenization. There are examples of periodic noncoercive Hamiltonians for which homogenization does not take place.

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Proposition

Under periodicity and controllability assumptions, there is, for any (x_0, p_0)*, an unique value* $c_0 = c_0(x_0, p_0)$ *such that the corresponding cell equation* $(\mathrm{Cell}(x_0, p_0)_c)$*, with* $c = c_0$*, admits an usc periodic subsolution and a lsc periodic supersolution problem. In this case*

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Periodic sub/super solutions of the cell problem play the role of correctors allowing to adapt Evans perturbed test function method to this setting.

Our contribution is to extend this technique to the noncompact (non periodic) case, replacing periodicity by a coercivity condition on the cost:

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We also assume a local version of controllability

- Given x_0 , the previous controllability condition holds in any compact subset, say K of the fast variable and the time to link two points of K, say y_1 and y_2, depends on x_0 and K.

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We obtain the following result:

Theorem

If the system satisfies the coercivity and local controllability conditions, and, in addition, the limit equation satisfies a comparison principle, then we get the same assertion of Alvarez–Bardi Theorem.

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Natural questions are:

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- If yes, are the sub/supersolution so selected suitable for adapted perturbed test function method ?

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Answer to the other questions are positive, and this is actually the way of proving our convergence theorem.

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- Exploit local controllability to show that there is a sequence of locally equibounded coercive solutions u_k to $H_k = c_0^k$
- pass to the semilimits, as $k \to +\infty$ to get sub/supersolution to

$$H(x_0, y, p_0, Du) = c_0(x_0, p_0) =: \lim_k c_k$$

with the desired properties.