



# Optimal control problems on stratifiable state constraints sets.

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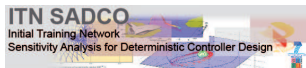
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# Optimal control problems on stratifiable state constraints sets.

Cristopher HERMOSILLA  
Joint work with Hasnaa ZIDANI

Commands, INRIA Saclay Île-de-France  
UMA, ENSTA ParisTech

NetCo Conference 2014



# Introduction

We consider an **infinite horizon** problem with **state constraints**  $\mathcal{K}$  :

$$(P) \quad \inf \left\{ \int_0^{\infty} e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \mid \begin{array}{l} u : [0, +\infty) \rightarrow \mathcal{A} \text{ measurable} \\ y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0 \end{array} \right\}.$$

where  $\lambda > 0$  is fixed and  $y_{x,u}(\cdot)$  is a trajectory of the control system

$$\begin{cases} \dot{y} = f(y, u) & \text{a.e. } t \geq 0 \\ y(0) = x \in \mathcal{K} \end{cases}$$

We are mainly concerned with a **characterization** of the value function of (P) as the **bilateral solution** to a Hamilton-Jacobi-Bellman equation.

# Standing Hypothesis (SH)

- $\mathcal{K}$  is **closed** and  $\mathcal{A} \subseteq \mathbb{R}^m$  is nonempty and **compact**.
- $\ell : \mathbb{R}^N \times \mathcal{A} \rightarrow [0, +\infty)$  and  $f : \mathbb{R}^N \times \mathcal{A} \rightarrow \mathbb{R}^N$  are **continuous and Lipschitz** w.r.t. the state :

$$\exists L > 0 \text{ such that } \left. \begin{array}{l} |f(x, u) - f(y, u)| \\ |\ell(x, u) - \ell(y, u)| \end{array} \right\} \leq L|x - y| \quad \forall u \in \mathcal{A}.$$

- We assume **convexity** of the augmented dynamics :

$$\left\{ \left( \begin{array}{c} f(x, u) \\ e^{-\lambda t} \ell(x, u) + r \end{array} \right) \mid \begin{array}{l} u \in \mathcal{A} \\ r \geq 0 \end{array} \right\}, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

# The value function

## Basic properties

$$v(x) := \inf_{u \in \mathbb{A}(x)} \int_0^{\infty} e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt, \quad \forall x \in \mathcal{K},$$

where

$$\mathbb{A}(x) = \{u : [0, +\infty) \rightarrow \mathcal{A} \text{ measurable} \mid y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0\}.$$

## Proposition

Suppose that (SH) holds. Then,  $\exists \lambda_0 = \lambda_0(f, \ell) > 0$  so that if  $\lambda > \lambda_0$

- if  $v(x) \in \mathbb{R}$  then there exists  $u \in \mathbb{A}(x)$  an *optimal control*.
- $v : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  is *lower semicontinuous*.
- $v$  has *linear growth* on its domain :

$$\exists c_v > 0 \quad |v(x)| \leq c_v(1 + |x|) \quad \forall x \in \text{dom } \mathbb{A}.$$

# An example

$$(\mathcal{P}) \quad \min \int_0^\infty e^{-\lambda t} u(t)^2 dt,$$

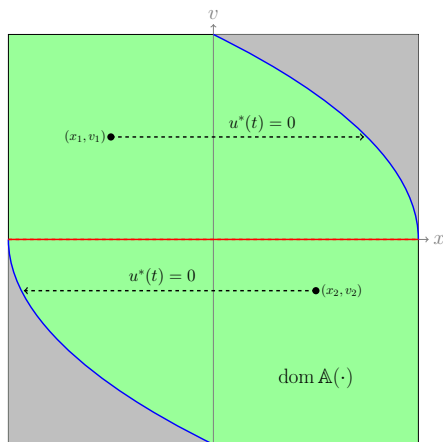
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

$$y_1(t), y_2(t) \in [-r, r]$$



# Dynamic Programming Principle

## Proposition

The value function *satisfies the DPP* : For any  $T > 0$

$$v(x) = \inf_{u \in \mathbb{A}(x)} \left\{ \int_0^T e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt + e^{-\lambda T} v(y_{x,u}(T)) \right\}.$$

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## Definition

Let  $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  be l.s.c.

i)  $\varphi$  is *weakly decreasing* provided  $\forall x \in \text{dom } \varphi, \exists u \in \mathbb{A}(x)$  such that

$$e^{-\lambda t} \varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s)) ds \leq \varphi(x) \quad \forall t \geq 0.$$

ii)  $\varphi$  is *strongly increasing* provided  $\text{dom } \mathbb{A} \subseteq \text{dom } \varphi$  and  $\forall x \in \mathcal{K}, \forall u \in \mathbb{A}(x)$

$$e^{-\lambda t} \varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s)) ds \geq \varphi(x) \quad \forall t \geq 0.$$



# The value function

## Comparison principle

### Proposition

Let  $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  be l.s.c. with *linear growth*.

- If  $\varphi$  is weakly decreasing, then  $v(x) \leq \varphi(x)$  for all  $x \in \mathcal{K}$ .
- If  $\varphi$  is strongly increasing, then  $v(x) \geq \varphi(x)$  for all  $x \in \mathcal{K}$ .

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### Corollary

Suppose that (SH) holds with  $\lambda > \lambda_0$ . Then  $v(\cdot)$  is the *unique l.s.c. function with linear growth* defined on  $\mathcal{K}$  which is *weakly decreasing* and *strongly increasing* at the same time.

# Weakly Decreasing Principle

## Characterization of supersolutions

### Definition (Subdifferentials)

Let  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$  be a l.s.c. function. A vector  $\zeta \in \mathbb{R}^N$  is called a **viscosity subgradient** of  $\varphi$  at  $x \in \text{dom } \varphi$  if and only if :

$\exists g \in \mathcal{C}^1(\mathbb{R}^N)$  s.t.  $\nabla g(x) = \zeta$  and  $\varphi - g$  attains a local minimum at  $x$ .

Furthermore,  $\zeta$  is called a **proximal subgradient** of  $\varphi$  at  $x$  if for some  $\sigma > 0$ ,

$$g(y) := \langle \zeta, y - x \rangle - \sigma |y - x|^2.$$

The set of all proximal subgradients at  $x$  is denoted by  $\partial_P \varphi(x)$ .

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### Proposition

Suppose that (SH) holds and let  $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a l.s.c. function. Then  $\varphi$  is **weakly decreasing** if and only if

$$\lambda \varphi(x) + H(x, \zeta) \geq 0 \quad \forall x \in \mathcal{K}, \quad \forall \zeta \in \partial_P \varphi(x).$$

# Strongly Increasing Principle

## Characterization of subsolutions

### Proposition

Suppose that (SH) holds and let  $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a *l.s.c. function*. If  $\varphi$  is *strongly increasing* then

$$(1) \quad \lambda\varphi(x) + H(x, \zeta) \leq 0 \quad \forall x \in \text{int}(\mathcal{K}), \forall \zeta \in \partial_P \varphi(x).$$

# Strongly Increasing Principle

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### Remark

- The converse **does not hold** without additional hypothesis.
- (1) provides information only for trajectories that **touch** the boundary a **finite number** of times.
- There is **no information** of what happens on the boundary.

# Feasible Neighboring Trajectories Approach

Soner, Frankowska-Vinter, Clarke-Stern, among many others

When does (1) become sufficient??

- $v(\cdot)$  is **continuous** up to the boundary.
- **Interior approximation** of trajectories.
- Some **monotonicity properties** of the solutions to (1).

# Feasible Neighboring Trajectories Approach

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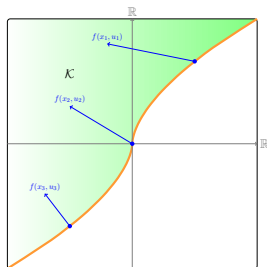
When does (1) become sufficient??

- $v(\cdot)$  is **continuous** up to the boundary.
- **Interior approximation** of trajectories.
- Some **monotonicity properties** of the solutions to (1).

What do we need to achieve one of these??

↪  $\mathcal{K} = \overline{\text{int}(\mathcal{K})}$  and tameness properties.

↪ Pointing Conditions  
(Inward or Outward).





# Our example

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

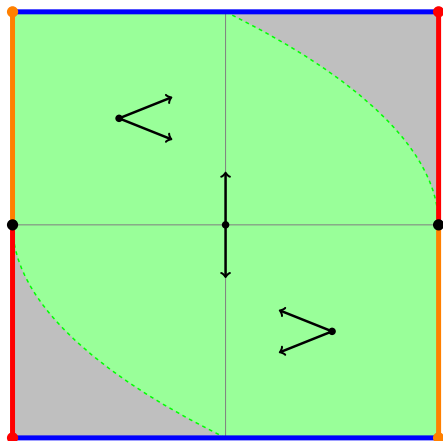
$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

$$y_1(t), y_2(t) \in [-r, r]$$

Note that :

$$\left\langle \begin{pmatrix} 0 \\ u \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 0$$

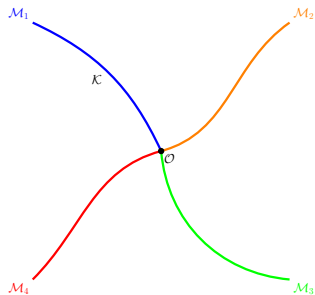
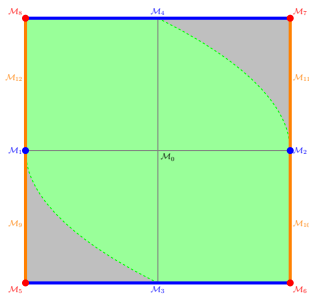


# Stratifiable sets

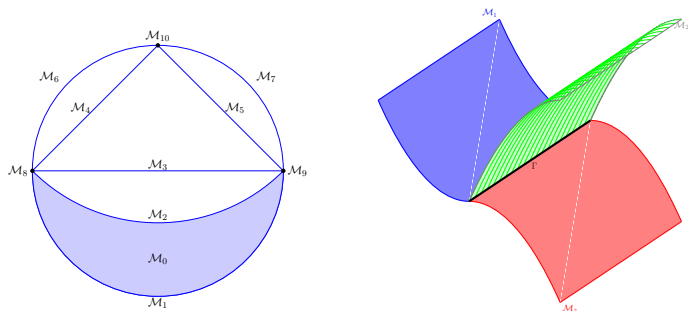
## Definition

A closed set  $\mathcal{K} \subseteq \mathbb{R}^N$  is said to be **stratifiable** if there exists a locally finite collection  $\{\mathcal{M}_i : i \in \mathcal{I}\}$  of embedded manifolds of  $\mathbb{R}^N$  such that :

- $\mathcal{K} = \bigcup_{i \in \mathcal{I}} \mathcal{M}_i$  and  $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$  whenever  $i \neq j$ .
- $\mathcal{M}_i \cap \overline{\mathcal{M}_j} \neq \emptyset$ , then  $\mathcal{M}_i \subseteq \overline{\mathcal{M}_j}$  and  $\dim(\mathcal{M}_i) < \dim(\mathcal{M}_j)$ .



# Stratifiable sets



The class of stratifiable sets on  $\mathbb{R}^N$  is wide, it includes :

- Semilinear sets  $\rightarrow$  finite union of open polyhedra.
- Semialgebraic sets  $\rightarrow$  finite union of polynomial manifolds.
- Subanalytic sets  $\rightarrow$  locally finite union of analytic manifolds.

# Characterization of subsolutions

## Some basic definitions and notation

Assume that  $\mathcal{K}$  is **stratifiable** and let  $\{\mathcal{M}_i\}$  be its strata. Then, for each stratum  $\mathcal{M}_i$  we define

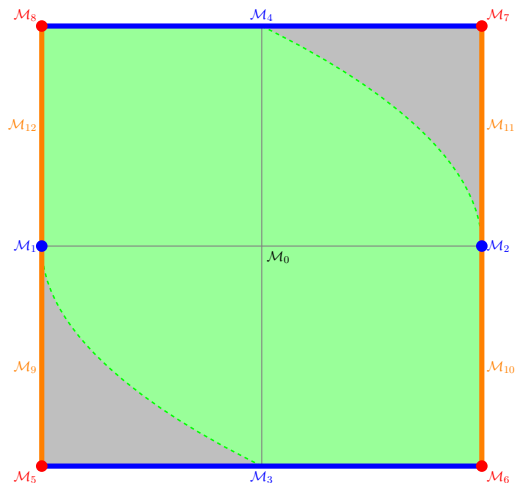
- the **set of tangent controls** as the map  $\mathcal{A}_i : \mathcal{M}_i \rightrightarrows \mathcal{A}$  given by

$$\mathcal{A}_i(x) := \{u \in \mathcal{A} \mid f(x, u) \in \mathcal{T}_{\mathcal{M}_i}(x)\}.$$

- the **tangential Hamiltonian** as the map  $H_i : \mathcal{M}_i \times \mathbb{R}^N \rightarrow \mathbb{R}$  given by

$$H_i(x, \zeta) = \max_{u \in \mathcal{A}_i(x)} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\}.$$

# Tangent controls



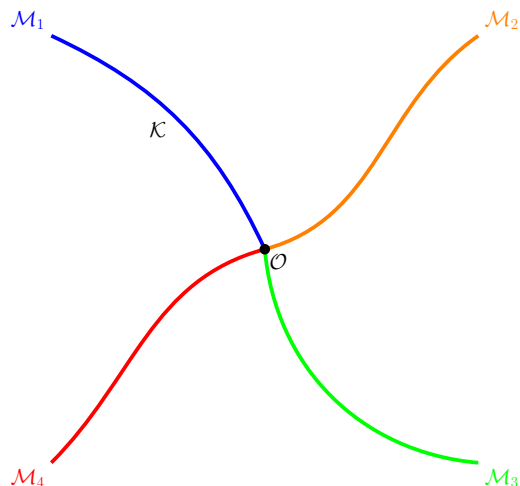
$$f(x_1, x_2, u) = (x_2, u) \\ u \in [-1, 1]$$

$$\mathcal{A}_0(x) = \mathcal{A}$$

$$\mathcal{A}_i(x) = \{0\} \text{ for } i = 1, \dots, 4$$

$$\mathcal{A}_j(x) = \emptyset \text{ for } j = 5, \dots, 12$$

# Tangent controls



$\exists \mathcal{A}_i \subseteq \mathcal{A}$  such that

$$f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i).$$

$$\mathcal{A}_i(x) = \mathcal{A}_i, \quad \forall i = 1, \dots, 4$$

Let  $M_0 = \{\mathcal{O}\}$  :

$$\mathcal{A}_0(\mathcal{O}) = \{u \in \mathcal{A} \mid f(\mathcal{O}, u) = 0\}.$$

# Characterization of subsolutions

## Proposition (CH - Zidani)

Suppose that (SH) holds with  $\mathcal{K}$  is *stratifiable* and let  $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a *l.s.c. function*. Assume that

(H<sub>0</sub>)  $\mathcal{A}_i$  is a *Lipschitz* set-valued map or has *empty images* on  $\mathcal{M}_i$ .

If  $\varphi$  is *strongly increasing* then for each  $i \in \mathcal{I}$

$$(\star) \quad \lambda\varphi(x) + H_i(x, \zeta) \leq 0 \quad \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P \varphi_i(x),$$

where  $\varphi_i(x) = \varphi(x)$  if  $x \in \overline{\mathcal{M}}_i$  and  $+\infty$  otherwise.

## Remark

( $\star$ ) is equivalent to say :  $\forall i \in \mathcal{I}, \forall x \in \mathcal{M}_i$  and  $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$  such that  $\varphi - g$  attains a local minimum at  $x$  *relative to*  $\mathcal{M}_i$ ;

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

# The converse ?

Strong Invariance Principle on  $\text{int}(\mathcal{K})$

$$\lambda\varphi(x) + \max_{u \in \mathcal{A}} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\} \leq 0 \quad \forall x \in \text{int}(\mathcal{K}), \forall \zeta \in \partial_P \varphi(x).$$



$y_{x,u}(s) \in \text{int}(\mathcal{K})$  for every  $s \in (a, b)$ , where  $0 \leq a < b < +\infty$  then

$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)}\varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds.$$



Any admissible trajectory  $y_{x,u}$  defined on  $[0, T]$  with  $y_{x,u}(s) \in \text{int}(\mathcal{K})$  for every  $s \in (0, T)$  satisfies the **Strong Increasing Inequality**.



# The converse ?

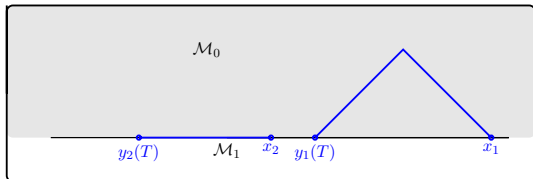
Strong Invariance Principle on each stratum

$$\lambda\varphi(x) + \max_{u \in \mathcal{A}_i(x)} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\} \leq 0 \quad \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P \varphi_i(x).$$

$\Downarrow$

$y_{x,u}(s) \in \mathcal{M}_i$  for every  $s \in (a, b)$ , where  $0 \leq a < b < +\infty$  then

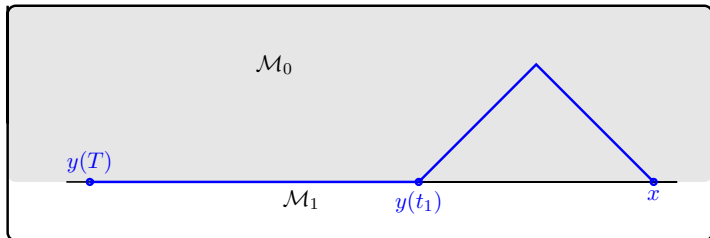
$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)}\varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds.$$



Suppose that  $y_{x,u}(s) \in \mathcal{K}$ ,  $\forall s \in [0, T]$  and **there exists a partition** of  $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T\}$$

so that  $\forall l \in \{0, \dots, n\}, \exists \mathcal{M}_l$  with  $y_{x,u}(s) \in \mathcal{M}_l, \forall s \in (t_l, t_{l+1})$ .



Whence,  $\forall l \in \{0, \dots, n\}$  we have

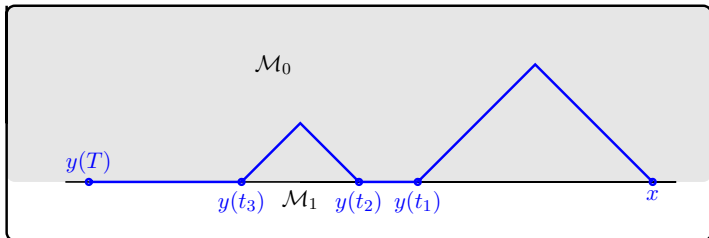
$$\varphi(y_{x,u}(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)} \varphi(y_{x,u}(t_{l+1})) + e^{\lambda t_l} \int_{t_l}^{t_{l+1}} e^{-\lambda s} \ell(y_{x,u}, u) ds.$$

Therefore  $y_{x,u}$  satisfies the **Strong Increasing Inequality !!**

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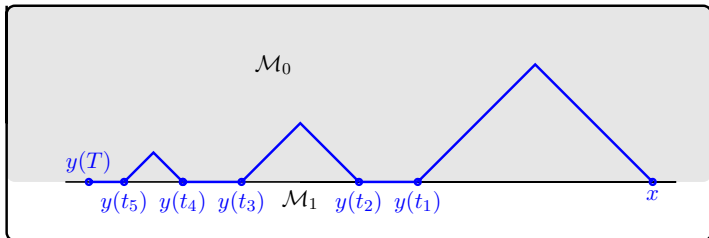
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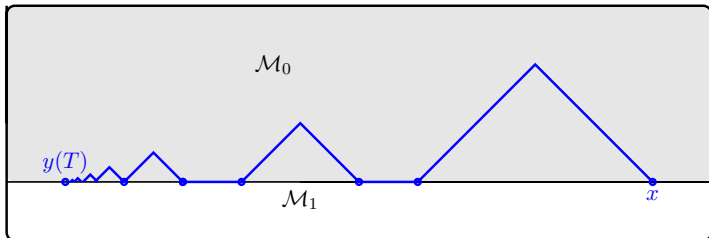
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Therefore  $y_{x,u}$  satisfies the **Strong Increasing Inequality !!**

Suppose that  $y_{x,u}(s) \in \mathcal{K}$ ,  $\forall s \in [0, T]$  and there is NO partition of  $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T\}$$

so that  $\forall l \in \{0, \dots, n\}, \exists \mathcal{M}_l$  with  $y_{x,u}(s) \in \mathcal{M}_l, \forall s \in (t_l, t_{l+1})$ .



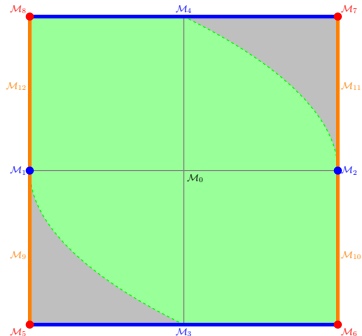
What if the trajectory "chatters" between two or more strata ???

# Controllability assumption

$$(H_1) \quad \begin{cases} \forall i \in \mathcal{I} \text{ with } \text{dom } \mathcal{A}_i \neq \emptyset, \exists \delta_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}^{(t)}(x) \cap \overline{\mathcal{M}}_i \subseteq \bigcup_{s \in [0, \Delta_i t]} \mathcal{R}_i^{(s)}(x), \quad \forall x \in \mathcal{M}_i, \forall t \in [0, \delta_i]. \end{cases}$$

$\mathcal{R}^{(t)}(\cdot)$  : reachable set at time  $t$  of  $x \mapsto f(x, \mathcal{A})$ .

$\mathcal{R}_i^{(t)}(\cdot)$  : reachable set at time  $t$  of  $x \mapsto f(x, \mathcal{A}_i(x))$ .



$$f(x_1, x_2, u) = (x_2, u) \\ u \in [-1, 1]$$

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## Controllability assumption

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$\mathcal{R}_i^{(t)}(\cdot)$  : reachable set at time  $t$  of  $x \mapsto f(x, \mathcal{A}_i(x))$ .

### Lemma

Suppose (SH) with  $\mathcal{K}$  stratifiable. Assume that  $(H_0)$  and  $(H_1)$  hold.

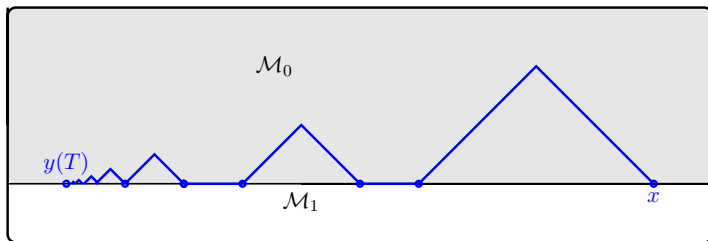
For any  $x \in \mathcal{K}$  and  $T > 0$  there exists  $L > 0$  such that :  $\forall u \in \mathbb{A}(x), \forall \varepsilon > 0$  if  $y_{x,u}(b), y_{x,u}(a) \in \mathcal{M}_i$  with  $0 \leq a < b \leq T$  for some  $i \in \mathcal{I}$  then,

$$\varphi(y_{x,u}(a)) \leq e^{\lambda \varepsilon} \left( e^{-\lambda(b-a)} \varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds \right) + L\varepsilon.$$

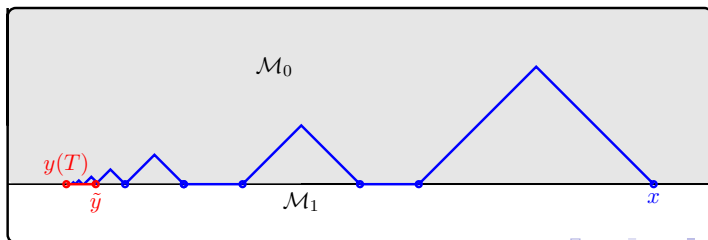
provided  $\text{meas}(\{t \in [a, b] \mid y_{x,u}(t) \notin \mathcal{M}_i\}) < \varepsilon$ .

# Controllability assumption :

Chattering trajectory :



Approximated trajectory :





# Characterization of the strong increasing principle

## Proposition (CH - Zidani)

Suppose that (SH) with  $\mathcal{K}$  stratifiable. Assume that  $(H_0)$  and  $(H_1)$  hold. Let  $\varphi : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a l.s.c. function with  $\text{dom } \mathbb{A} \subseteq \text{dom } \varphi$  that satisfies

$$(\star) \quad \lambda\varphi(x) + H_i(x, \zeta) \leq 0 \quad \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P \varphi_i(x), \forall i \in \mathcal{I}.$$

Then  $\varphi$  is *strongly increasing*.

## Recall

$(\star)$  is equivalent to say :  $\forall i \in \mathcal{I}, \forall x \in \mathcal{M}_i$  and  $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$  such that  $\varphi - g$  attains a local minimum at  $x$  *relative to*  $\mathcal{M}_i$

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

# Characterization of the value function

## Theorem (CH - Zidani)

Suppose (SH) with  $\mathcal{K}$  **stratifiable** and  $\lambda > \lambda_0$ . Assume  $(H_0)$  and  $(H_1)$ . Then the value function

$$v(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \mid u \in \mathbb{A}(x) \right\}.$$

is the **unique l.s.c.** function with **linear growth** which is  $+\infty$  on  $\mathbb{R}^N \setminus \mathcal{K}$  and that satisfies :

$$\lambda v(x) + H(x, \zeta) \geq 0 \quad \forall x \in \mathcal{K}, \forall \zeta \in \partial_P v(x),$$

$$\lambda v(x) + H_i(x, \zeta) \leq 0 \quad \forall x \in \mathcal{M}_i, \forall \zeta \in \partial_P v_i(x), \forall i \in \mathcal{I},$$

$$\text{where } v_i(x) = \begin{cases} v(x) & \text{if } x \in \overline{\mathcal{M}}_i \\ +\infty & \text{otherwise.} \end{cases}$$

# Characterization of the value function : $\mathcal{M}_0 = \text{int}(\mathcal{K})$ .

## Theorem (CH - Zidani)

Suppose (SH) with  $\mathcal{K}$  stratifiable with  $\text{int}(\mathcal{K}) \neq \emptyset$  and  $\lambda > \lambda_0$ . Assume  $(H_0)$  and  $(H_1)$ . Then the value function

$$v(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \mid u \in \mathbb{A}(x) \right\}.$$

is the *unique l.s.c.* function with *linear growth* which is  $+\infty$  on  $\mathbb{R}^N \setminus \mathcal{K}$  and that satisfies :

$$\lambda v(x) + H(x, \zeta) \geq 0 \quad \forall x \in \mathcal{K}, \quad \forall \zeta \in \partial_P v(x),$$

$$\lambda v(x) + H(x, \zeta) \leq 0 \quad \forall x \in \text{int}(\mathcal{K}), \quad \forall \zeta \in \partial_P v(x),$$

$$\lambda v(x) + H_i(x, \zeta) \leq 0 \quad \forall x \in \mathcal{M}_i, \quad \forall \zeta \in \partial_P v_i(x), \quad \forall i \in \mathcal{I} \setminus \{0\},$$

$$\text{where } v_i(x) = \begin{cases} v(x) & \text{if } x \in \overline{\mathcal{M}}_i \\ +\infty & \text{otherwise.} \end{cases}$$

# Applications to Networks

Suppose  $\mathcal{K}$  is a network with one junction  $\mathcal{O}$  and let  $\mathcal{M}_1, \dots, \mathcal{M}_p$  be its branches.

Assume that for each  $i \in \{1, \dots, p\}$ ,  $\exists \mathcal{A}_i \subseteq \mathcal{A}$  s.t.

$$(H_3) \quad f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i), \quad \forall x \in \mathcal{M}_i.$$

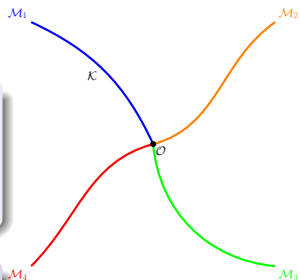
For instance, for some  $v_i \in \mathbb{R}^N \setminus \{0\}$

$$\mathcal{M}_i = (0, +\infty)v_i \text{ and } f(x, u) = f_i(x, u)v_i, \quad \forall x \in \mathcal{M}_i$$

with and  $f_i$  real-valued.

## Claim

The hypothesis  $(H_0)$  and  $(H_1)$  are satisfied.



# Applications to Networks

## Theorem (CH - Zidani)

Suppose that (SH) with  $\mathcal{K}$  a *network* as before and  $\lambda > \lambda_0$ . Assume that  $(H_3)$  holds and let

$$\mathcal{A}_0 = \{u \in \mathcal{A} \mid f(\mathcal{O}, u) = 0\}.$$

Then the value function is the *unique l.s.c.* function with *linear growth* which is  $+\infty$  on  $\mathbb{R}^N \setminus \mathcal{K}$  and that satisfies :

$$\lambda v(x) + \max_{u \in \mathcal{A}_i} \{-\langle \zeta, f(x, u) \rangle - \ell(x, u)\} = 0 \quad \forall x \in \mathcal{M}_i, \quad \forall \zeta \in \partial_P v(x),$$

$$\lambda v(\mathcal{O}) + H(\mathcal{O}, \zeta) \geq 0 \quad \forall \zeta \in \partial_P v(\mathcal{O}),$$

$$\lambda v(\mathcal{O}) - \min_{u \in \mathcal{A}_0} \ell(\mathcal{O}, u) \leq 0.$$

## Final remarks

- The interior of  $\mathcal{K}$  can always be taken as a stratum and so, the constrained Hamilton-Jacobi equation proposed by Soner is included in the set of equations proposed in the main theorem.
- The characterization of the value function neither requires its continuity nor that the state constraint set has empty interior.
- Under the continuity of the value function (on its domain) the controllability assumption can be dropped.
- The characterization for networks can be extended to a suitable notion of generalized network where the junction is replaced by a manifold.

Thanks for your attention !