Optimal control problems on stratifiable state constraints sets.

Cristopher Hermosilla, Hasnaa Zidani

Optimal control problems on stratifiable state constraints sets.

Cristopher HERMOSILLA
Joint work with Hasnaa ZIDANI

Commands, INRIA Saclay Île-de-France
UMA, ENSTA ParisTech

NetCo Conference 2014

Introduction

We consider an infinite horizon problem with state constraints $\mathcal{K}$:

$$(P) \quad \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \;\middle|\; \begin{array}{c} u : [0, +\infty) \to \mathcal{A} \text{ measurable} \\ y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0 \end{array} \right\}.$$

where $\lambda > 0$ is fixed and $y_{x,u}(\cdot)$ is a trajectory of the control system

$$\begin{cases} \dot{y} = f(y, u) & \text{a.e. } t \geq 0 \\ y(0) = x \in \mathcal{K} \end{cases}$$

We are mainly concerned with a characterization of the value function of (P) as the bilateral solution to a Hamilton-Jacobi-Bellman equation.

Standing Hypothesis (SH)

- $\mathcal{K}$ is closed and $\mathcal{A} \subseteq \mathbb{R}^m$ is nonempty and compact.
- $\ell : \mathbb{R}^N \times \mathcal{A} \to [0, +\infty)$ and $f : \mathbb{R}^N \times \mathcal{A} \to \mathbb{R}^N$ are continuous and Lipschitz w.r.t. the state :

$$\exists L > 0 \text{ such that } \quad \left.\begin{array}{c} |f(x,u) - f(y,u)| \\ |\ell(x,u) - \ell(y,u)| \end{array}\right\} \leq L|x-y| \qquad \forall u \in \mathcal{A}.$$

- We assume convexity of the augmented dynamics :

$$\left\{ \begin{pmatrix} f(x,u) \\ e^{-\lambda t}\ell(x,u) + r \end{pmatrix} \;\middle|\; \begin{array}{c} u \in \mathcal{A} \\ r \geq 0 \end{array} \right\}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

The value function

Basic properties

$$v(x) := \inf_{u \in \mathbb{A}(x)} \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt, \qquad \forall x \in \mathcal{K},$$

where

$$\mathbb{A}(x) = \{u : [0, +\infty) \to \mathcal{A} \text{ measurable } \mid y_{x,u}(t) \in \mathcal{K} \quad \forall t \geq 0\}.$$

Proposition

Suppose that (SH) holds. Then, $\exists \lambda_0 = \lambda_0(f, \ell) > 0$ *so that if* $\lambda > \lambda_0$

- *if* $v(x) \in \mathbb{R}$ *then there exists* $u \in \mathbb{A}(x)$ *an optimal control.*
- $v : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous.*
- v *has linear growth on its domain :*

$$\exists c_v > 0 \qquad |v(x)| \leq c_v(1 + |x|) \qquad \forall x \in \operatorname{dom} \mathbb{A}.$$

An example

$$(\mathcal{P}) \qquad \min \int_0^\infty e^{-\lambda t} u(t)^2 dt,$$

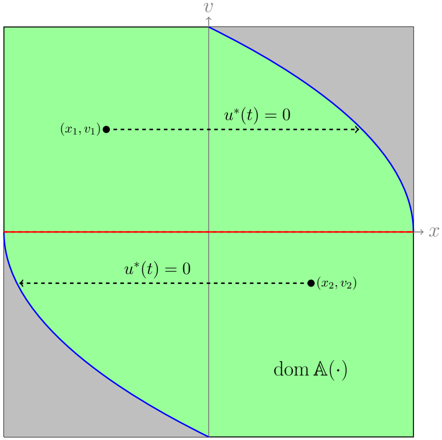
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

$$y_1(t), y_2(t) \in [-r, r]$$



Dynamic Programming Principle

Proposition

The value function satisfies the DPP : For any $T > 0$

$$v(x) = \inf_{u \in \mathbb{A}(x)} \left\{ \int_0^T e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt + e^{-\lambda T} v(y_{x,u}(T)) \right\}.$$

Dynamic Programming Principle

Proposition

The value function *satisfies the DPP* *: For any* $T > 0$

$$v(x) = \inf_{u \in \mathbb{A}(x)} \left\{ \int_0^T e^{-\lambda t} \ell(y_{x,u}(t), u(t))dt + e^{-\lambda T} v(y_{x,u}(T)) \right\}.$$

Definition

Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ *be l.s.c.*

i) φ *is* *weakly decreasing* *provided* $\forall x \in \operatorname{dom} \varphi$, $\exists u \in \mathbb{A}(x)$ *such that*

$$e^{-\lambda t} \varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s))ds \leq \varphi(x) \quad \forall t \geq 0.$$

ii) φ *is* *strongly increasing* *provided* $\operatorname{dom} \mathbb{A} \subseteq \operatorname{dom} \varphi$ *and* $\forall x \in \mathcal{K}$, $\forall u \in \mathbb{A}(x)$

$$e^{-\lambda t} \varphi(y_{x,u}(t)) + \int_0^t e^{-\lambda s} \ell(y_{x,u}(s), u(s))ds \geq \varphi(x) \quad \forall t \geq 0.$$

The value function

Comparison principle

Proposition

Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ *be l.s.c. with linear growth.*

- *If* φ *is weakly decreasing, then* $\upsilon(x) \leq \varphi(x)$ *for all* $x \in \mathcal{K}$*.*
- *If* φ *is strongly increasing, then* $\upsilon(x) \geq \varphi(x)$ *for all* $x \in \mathcal{K}$*.*

Proposition

Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be l.s.c. with linear growth.

- *If φ is weakly decreasing, then $\upsilon(x) \leq \varphi(x)$ for all $x \in \mathcal{K}$.*
- *If φ is strongly increasing, then $\upsilon(x) \geq \varphi(x)$ for all $x \in \mathcal{K}$.*

Corollary

Suppose that (SH) holds with $\lambda > \lambda_0$. Then $\upsilon(\cdot)$ is the unique l.s.c. function with linear growth defined on $\mathcal{K}$ which is weakly decreasing and strongly increasing at the same time.

Weakly Decreasing Principle

Characterization of supersolutions

Definition (Subdifferentials)

Let $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. A vector* $\zeta \in \mathbb{R}^N$ *is called a* ***viscosity subgradient*** *of* φ *at* $x \in \operatorname{dom}\varphi$ *if and only if :*

$$\exists g \in \mathcal{C}^1(\mathbb{R}^N) \text{ s.t. } \nabla g(x) = \zeta \text{ and } \varphi - g \text{ attains a local minimum at } x.$$

Furthermore, ζ *is called a* ***proximal subgradient*** *of* φ *at* x *if for some* $\sigma > 0$,

$$g(y) := \langle \zeta, y - x\rangle - \sigma|y - x|^2.$$

The set of all proximal subgradients at x *is denoted by* $\partial_P\varphi(x)$.

Weakly Decreasing Principle

Characterization of supersolutions

Definition (Subdifferentials)

Let $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. A vector $\zeta \in \mathbb{R}^N$ is called a viscosity subgradient of φ at $x \in \operatorname{dom}\varphi$ if and only if :

$$\exists g \in \mathcal{C}^1(\mathbb{R}^N) \text{ s.t. } \nabla g(x) = \zeta \text{ and } \varphi - g \text{ attains a local minimum at } x.$$

Furthermore, ζ is called a proximal subgradient of φ at x if for some $\sigma > 0$,

$$g(y) := \langle \zeta, y - x \rangle - \sigma |y - x|^2.$$

The set of all proximal subgradients at x is denoted by $\partial_P \varphi(x)$.

Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Then φ is weakly decreasing if and only if

$$\lambda\varphi(x) + H(x, \zeta) \geq 0 \qquad \forall x \in \mathcal{K},\ \forall \zeta \in \partial_P \varphi(x).$$

Strongly Increasing Principle

Characterization of subsolutions

Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. If* φ *is strongly increasing then*

$$(1) \qquad \lambda\varphi(x) + H(x,\zeta) \leq 0 \qquad \forall x \in \text{int}(\mathcal{K}),\ \forall \zeta \in \partial_P \varphi(x).$$

Strongly Increasing Principle

Characterization of subsolutions

Proposition

Suppose that (SH) holds and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. If φ is strongly increasing then

$$(1) \qquad \lambda\varphi(x) + H(x,\zeta) \leq 0 \qquad \forall x \in \mathrm{int}(\mathcal{K}),\ \forall \zeta \in \partial_P \varphi(x).$$

Remark

- The converse does not hold without additional hypothesis.
- (1) provides information only for trajectories that touch the boundary a finite number of times.
- There is no information of what happens on the boundary.

Feasible Neighboring Trajectories Approach

Soner, Frankowska-Vinter, Clarke-Stern, among many others

When does (1) become sufficient ? ?

- $v(\cdot)$ is continuous up to the boundary.
- Interior approximation of trajectories.
- Some monotonicity properties of the solutions to (1).

Feasible Neighboring Trajectories Approach

Soner, Frankowska-Vinter, Clarke-Stern, among many others

When does (1) become sufficient ? ?

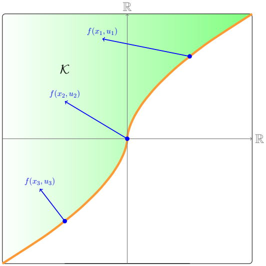
- $v(\cdot)$ is continuous up to the boundary.
- Interior approximation of trajectories.
- Some monotonicity properties of the solutions to (1).

What do we need to achieve one of these ? ?

↬ $\mathcal{K} = \overline{\text{int}(\mathcal{K})}$ and tameness properties.

↬ Pointing Conditions

(Inward or Outward).



Our example

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ u \end{pmatrix}$$

$$y_1(0) = x,$$

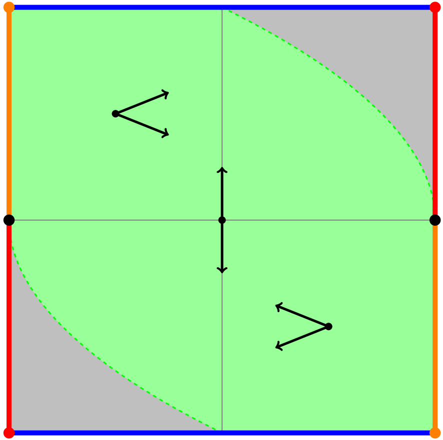
$$y_2(0) = v,$$

$$u \in \mathcal{A} := [-1, 1]$$

$$y_1(t), y_2(t) \in [-r, r]$$

Note that :

$$\left\langle \begin{pmatrix} 0 \\ u \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 0$$

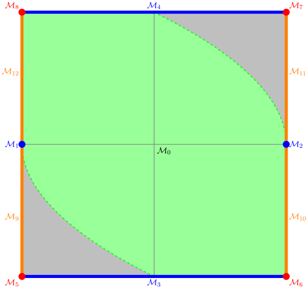


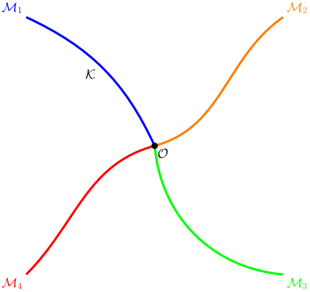
Stratifiable sets

Definition

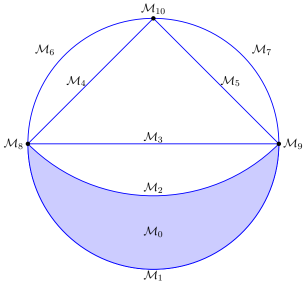
A closed set $\mathcal{K} \subseteq \mathbb{R}^N$ is said to be stratifiable *if there exists a locally finite collection $\{\mathcal{M}_i : i \in \mathcal{I}\}$ of embedded manifolds of $\mathbb{R}^N$ such that :*

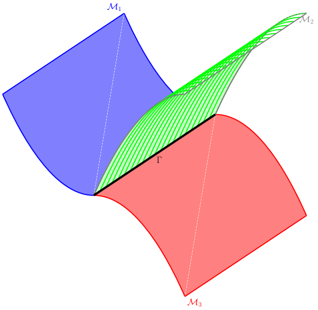
- $\mathcal{K} = \bigcup_{i\in\mathcal{I}} \mathcal{M}_i$ *and* $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ *whenever* $i \neq j$.
- $\mathcal{M}_i \cap \overline{\mathcal{M}_j} \neq \emptyset$, *then* $\mathcal{M}_i \subseteq \overline{\mathcal{M}_j}$ *and* $\dim(\mathcal{M}_i) < \dim(\mathcal{M}_j)$.





Stratifiable sets





The class of stratifiable sets on $\mathbb{R}^N$ is wide, it includes :

- Semilinear sets $\rightarrow$ finite union of open polyhedra.
- Semialgebraic sets $\rightarrow$ finite union of polynomial manifolds.
- Subanalytic sets $\rightarrow$ locally finite union of analytic manifolds.

Characterization of subsolutions

Some basic definitions and notation

Assume that $\mathcal{K}$ is stratifiable and let $\{\mathcal{M}_i\}$ be its strata. Then, for each stratum $\mathcal{M}_i$ we define

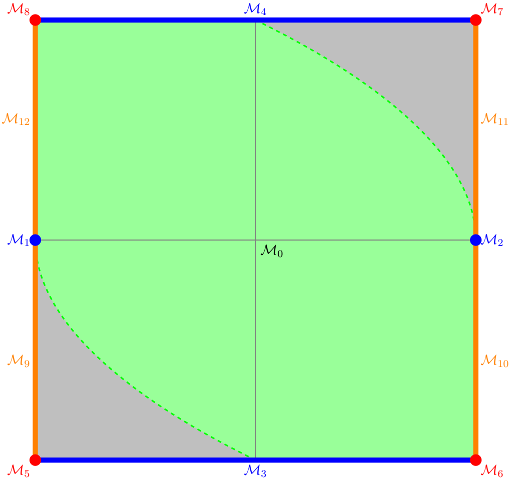
- the set of tangent controls as the map $\mathcal{A}_i : \mathcal{M}_i \rightrightarrows \mathcal{A}$ given by

$$\mathcal{A}_i(x) := \{u \in \mathcal{A} \mid f(x,u) \in \mathcal{T}_{\mathcal{M}_i}(x)\}.$$

- the tangential Hamiltonian as the map $H_i : \mathcal{M}_i \times \mathbb{R}^N \to \mathbb{R}$ given by

$$H_i(x,\zeta) = \max_{u \in \mathcal{A}_i(x)} \{-\langle \zeta, f(x,u)\rangle - \ell(x,u)\}.$$

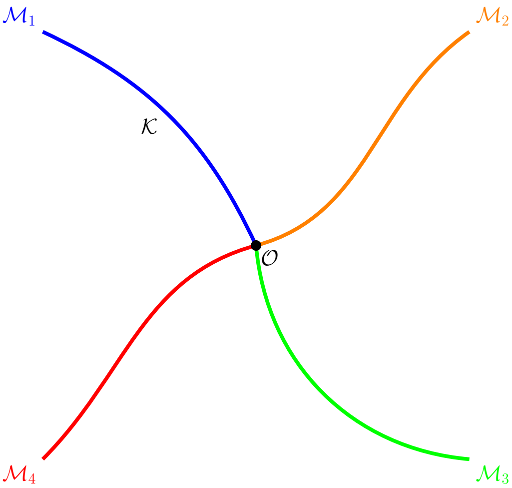
Tangent controls



$$f(x_1, x_2, u) = (x_2, u)$$
$$u \in [-1, 1]$$

$$\mathcal{A}_0(x) = \mathcal{A}$$
$$\mathcal{A}_i(x) = \{0\} \text{ for } i = 1, \ldots, 4$$
$$\mathcal{A}_j(x) = \emptyset \text{ for } j = 5, \ldots, 12$$

Tangent controls



$\exists A_i \subseteq \mathcal{A}$ such that

$$f(x,\mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x,\mathcal{A}_i).$$
$$\mathcal{A}_i(x) = \mathcal{A}_i, \quad \forall i = 1,\dots,4$$

Let $M_0 = \{\mathcal{O}\}$:

$$\mathcal{A}_0(\mathcal{O}) = \{u \in \mathcal{A} \mid f(\mathcal{O},u) = 0\}.$$

Characterization of subsolutions

Proposition (CH - Zidani)

Suppose that (SH) holds with $\mathcal{K}$ is stratifiable and let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Assume that

(H_0) $\quad$ *$\mathcal{A}_i$ is a Lipschitz set-valued map or has empty images on $\mathcal{M}_i$.*

If φ is strongly increasing then for each $i \in \mathcal{I}$

$$(\bigstar) \qquad \lambda\varphi(x) + H_i(x, \zeta) \leq 0 \qquad \forall x \in \mathcal{M}_i, \ \forall \zeta \in \partial_P \varphi_i(x),$$

where $\varphi_i(x) = \varphi(x)$ if $x \in \overline{\mathcal{M}}_i$ and $+\infty$ otherwise.

Remark

$(\bigstar)$ is equivalent to say : $\forall i \in \mathcal{I}$, $\forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x relative to $\mathcal{M}_i$

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

The converse ?

Strong Invariance Principle on $\mathrm{int}(\mathcal{K})$

$$\lambda\varphi(x) + \max_{u\in\mathcal{A}}\{-\langle\zeta, f(x,u)\rangle - \ell(x,u)\} \leq 0 \qquad \forall x \in \mathrm{int}(\mathcal{K}),\ \forall\zeta \in \partial_P\varphi(x).$$

$$\Downarrow$$

$y_{x,u}(s) \in \mathrm{int}(\mathcal{K})$ for every $s \in (a,b)$, where $0 \leq a < b < +\infty$ then

$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)}\varphi(y_{x,u}(b)) + e^{\lambda a}\int_a^b e^{-\lambda s}\ell(y_{x,u},u)ds.$$

$$\Downarrow$$

Any admissible trajectory $y_{x,u}$ defined on $[0,T]$ with $y_{x,u}(s) \in \mathrm{int}(\mathcal{K})$ for every $s \in (0,T)$ satisfies the Strong Increasing Inequality.

The converse ?

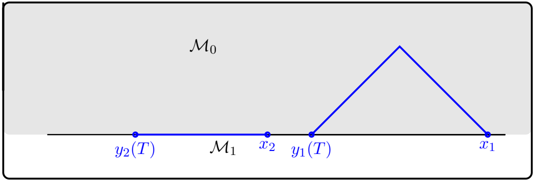
Strong Invariance Principle on each stratum

$$\lambda\varphi(x) + \max_{u\in\mathcal{A}_i(x)} \{-\langle\zeta, f(x,u)\rangle - \ell(x,u)\} \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall\zeta \in \partial_P\varphi_i(x).$$

$$\Downarrow$$

$y_{x,u}(s) \in \mathcal{M}_i$ for every $s \in (a,b)$, where $0 \leq a < b < +\infty$ then

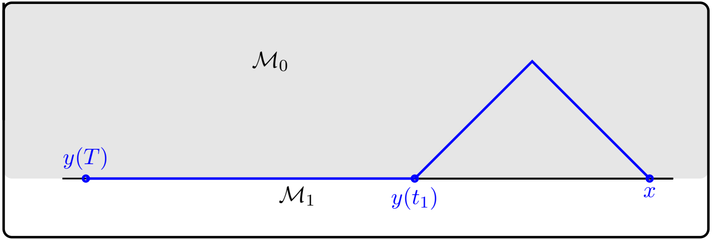
$$\varphi(y_{x,u}(a)) \leq e^{-\lambda(b-a)}\varphi(y_{x,u}(b)) + e^{\lambda a}\int_a^b e^{-\lambda s}\ell(y_{x,u},u)ds.$$



Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there exists a partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}, \exists \mathcal{M}_i$ with $y_{x,u}(s) \in \mathcal{M}_i, \forall s \in (t_l, t_{l+1})$.



Whence, $\forall l \in \{0, \ldots, n\}$ we have

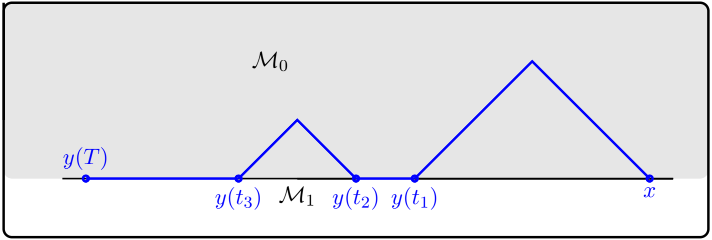
$$\varphi(y_{x,u}(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)}\varphi(y_{x,u}(t_{l+1})) + e^{\lambda t_l} \int_{t_l}^{t_{l+1}} e^{-\lambda s}\ell(y_{x,u}, u)ds.$$

Therefore $y_{x,u}$ satisfies the Strong Increasing Inequality ! !

Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there exists a partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}, \exists \mathcal{M}_i$ with $y_{x,u}(s) \in \mathcal{M}_i, \forall s \in (t_l, t_{l+1})$.



Whence, $\forall l \in \{0, \ldots, n\}$ we have

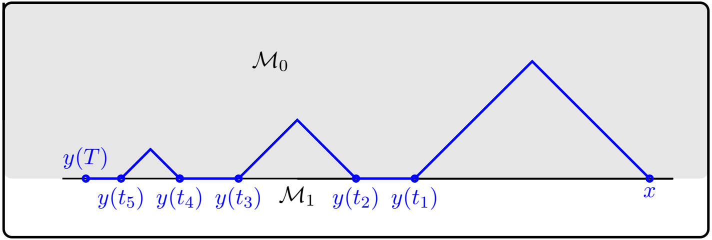
$$\varphi(y_{x,u}(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)}\varphi(y_{x,u}(t_{l+1})) + e^{\lambda t_l}\int_{t_l}^{t_{l+1}} e^{-\lambda s}\ell(y_{x,u}, u)ds.$$

Therefore $y_{x,u}$ satisfies the Strong Increasing Inequality ! !

Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there exists a partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}, \exists \mathcal{M}_i$ with $y_{x,u}(s) \in \mathcal{M}_i, \forall s \in (t_l, t_{l+1})$.



Whence, $\forall l \in \{0, \ldots, n\}$ we have

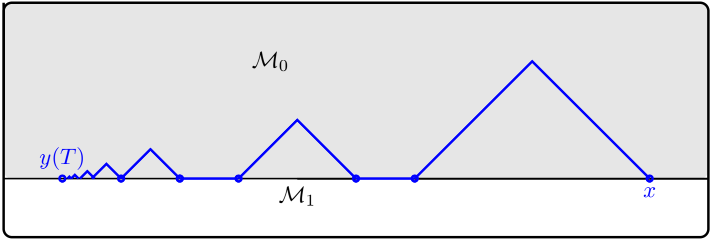
$$\varphi(y_{x,u}(t_l)) \leq e^{-\lambda(t_{l+1}-t_l)}\varphi(y_{x,u}(t_{l+1})) + e^{\lambda t_l} \int_{t_l}^{t_{l+1}} e^{-\lambda s}\ell(y_{x,u}, u)ds.$$

Therefore $y_{x,u}$ satisfies the Strong Increasing Inequality ! !

Suppose that $y_{x,u}(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there is NO partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}, \exists \mathcal{M}_i$ with $y_{x,u}(s) \in \mathcal{M}_i, \forall s \in (t_l, t_{l+1})$.

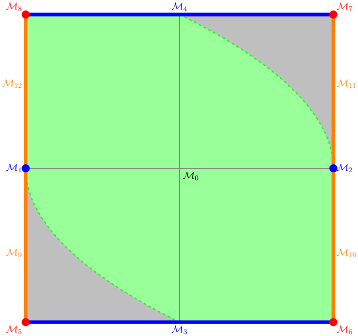


What if the trajectory "chatters" between two or more strata ? ? ?

Controllability assumption

$$(H_1) \quad \begin{cases} \forall i \in \mathcal{I} \text{ with } \operatorname{dom} \mathcal{A}_i \neq \emptyset, \ \exists \delta_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}^{(t)}(x) \cap \overline{\mathcal{M}}_i \subseteq \displaystyle\bigcup_{s \in [0, \Delta_i t]} \mathcal{R}_i^{(s)}(x), \quad \forall x \in \mathcal{M}_i, \ \forall t \in [0, \delta_i]. \end{cases}$$

$\mathcal{R}^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A})$.
$\mathcal{R}_i^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A}_i(x))$.



$$f(x_1, x_2, u) = (x_2, u)$$
$$u \in [-1, 1]$$

$$\mathcal{A}_0(x) = \mathcal{A}$$
$$\mathcal{A}_i(x) = \{0\} \text{ for } i = 1, \dots, 4$$
$$\mathcal{A}_j(x) = \emptyset \text{ for } j = 5, \dots, 12$$

Controllability assumption

$$(H_1) \quad \begin{cases} \forall i \in \mathcal{I} \text{ with } \operatorname{dom} \mathcal{A}_i \neq \emptyset, \ \exists \delta_i, \Delta_i > 0 \text{ such that} \\ \mathcal{R}^{(t)}(x) \cap \overline{\mathcal{M}}_i \subseteq \displaystyle\bigcup_{s \in [0, \Delta_i t]} \mathcal{R}_i^{(s)}(x), \quad \forall x \in \mathcal{M}_i, \ \forall t \in [0, \delta_i]. \end{cases}$$

$\mathcal{R}^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A})$.
$\mathcal{R}_i^{(t)}(\cdot)$: reachable set at time t of $x \mapsto f(x, \mathcal{A}_i(x))$.

Lemma

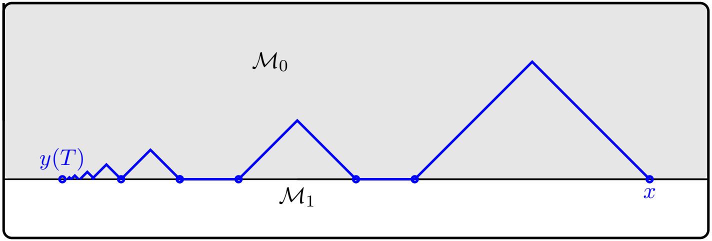
Suppose (SH) with $\mathcal{K}$ stratifiable. Assume that (H_0) and (H_1) hold.
For any $x \in \mathcal{K}$ and $T > 0$ there exists $L > 0$ such that : $\forall u \in \mathbb{A}(x), \forall \varepsilon > 0$ if $y_{x,u}(b), y_{x,u}(a) \in \mathcal{M}_i$ with $0 \leq a < b \leq T$ for some $i \in \mathcal{I}$ then,

$$\varphi(y_{x,u}(a)) \leq e^{\lambda \varepsilon} \left(e^{-\lambda(b-a)} \varphi(y_{x,u}(b)) + e^{\lambda a} \int_a^b e^{-\lambda s} \ell(y_{x,u}, u) ds \right) + L\varepsilon.$$

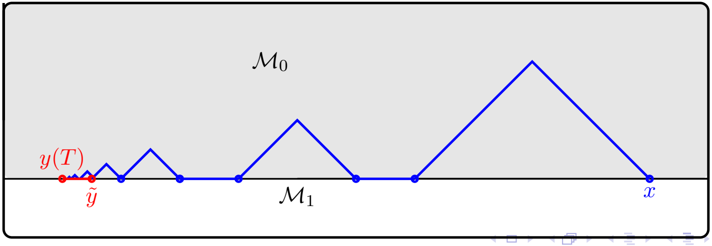
provided $\quad \operatorname{meas}(\{t \in [a, b] \mid y_{x,u}(t) \notin \mathcal{M}_i\}) < \varepsilon.$

Controllability assumption :

Chattering trajectory :



Approximated trajectory :



Characterization of the strong increasing principle

Proposition (CH - Zidani)

Suppose that (SH) with $\mathcal{K}$ stratifiable. Assume that (H_0) and (H_1) hold. Let $\varphi : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function with $\mathrm{dom}\, \mathbb{A} \subseteq \mathrm{dom}\, \varphi$ that satisfies

$$(\bigstar) \qquad \lambda\varphi(x) + H_i(x, \zeta) \leq 0 \qquad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P \varphi_i(x),\ \forall i \in \mathcal{I}.$$

Then φ is strongly increasing.

Recall

$(\bigstar)$ is equivalent to say : $\forall i \in \mathcal{I}$, $\forall x \in \mathcal{M}_i$ and $\forall g \in \mathcal{C}^1(\mathbb{R}^N)$ such that $\varphi - g$ attains a local minimum at x relative to $\mathcal{M}_i$

$$\lambda\varphi(x) + H_i(x, \nabla g(x)) \leq 0.$$

Characterization of the value function

Theorem (CH - Zidani)

Suppose (SH) with $\mathcal{K}$ stratifiable and $\lambda > \lambda_0$. Assume (H_0) and (H_1). Then the value function

$$\vartheta(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t))dt \;\middle|\; u \in \mathbb{A}(x) \right\}.$$

is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\begin{aligned} \lambda\vartheta(x) + H(x,\zeta) \geq 0 \qquad & \forall x \in \mathcal{K},\ \forall \zeta \in \partial_P \vartheta(x), \\ \lambda\vartheta(x) + H_i(x,\zeta) \leq 0 \qquad & \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P \vartheta_i(x),\ \forall i \in \mathcal{I}, \end{aligned}$$

where $\vartheta_i(x) = \begin{cases} \vartheta(x) & \text{if } x \in \overline{\mathcal{M}}_i \\ +\infty & \text{otherwise.} \end{cases}.$

Characterization of the value function : $\mathcal{M}_0 = \mathrm{int}(\mathcal{K})$.

Theorem (CH - Zidani)

Suppose (SH) with $\mathcal{K}$ stratifiable with $\mathrm{int}(\mathcal{K}) \neq \emptyset$ and $\lambda > \lambda_0$. Assume (H_0) and (H_1). Then the value function

$$v(x) := \inf \left\{ \int_0^\infty e^{-\lambda t} \ell(y_{x,u}(t), u(t)) dt \;\middle|\; u \in \mathbb{A}(x) \right\}.$$

is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

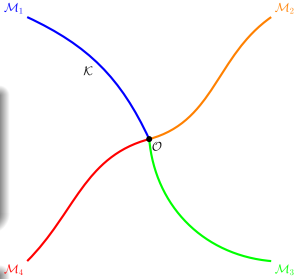
$$\begin{aligned} \lambda v(x) + H(x,\zeta) &\geq 0 && \forall x \in \mathcal{K},\ \forall \zeta \in \partial_P v(x), \\ \lambda v(x) + H(x,\zeta) &\leq 0 && \forall x \in \mathrm{int}(\mathcal{K}),\ \forall \zeta \in \partial_P v(x), \\ \lambda v(x) + H_i(x,\zeta) &\leq 0 && \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P v_i(x),\ \forall i \in \mathcal{I} \setminus \{0\}, \end{aligned}$$

where $v_i(x) = \begin{cases} v(x) & \text{if } x \in \overline{\mathcal{M}_i} \\ +\infty & \text{otherwise.} \end{cases}$

Applications to Networks

Suppose $\mathcal{K}$ is a network with one junction $\mathcal{O}$ and let $\mathcal{M}_1, \dots, \mathcal{M}_p$ be its branches.
Assume that for each $i \in \{1, \dots, p\}$, $\exists \mathcal{A}_i \subseteq \mathcal{A}$ s.t.

$$(H_3) \qquad f(x, \mathcal{A}) \cap \mathcal{T}_{\mathcal{M}_i}(x) = f(x, \mathcal{A}_i), \quad \forall x \in \mathcal{M}_i.$$



For instance, for some $v_i \in \mathbb{R}^N \setminus \{0\}$

$$\mathcal{M}_i = (0, +\infty)v_i \text{ and } f(x, u) = f_i(x, u)v_i, \quad \forall x \in \mathcal{M}_i$$

with and f_i real-valued.

Claim

The hypothesis (H_0) and (H_1) are satisfied.

Applications to Networks

Theorem (CH - Zidani)

Suppose that (SH) with $\mathcal{K}$ a network as before and $\lambda > \lambda_0$. Assume that (H_3) holds and let

$$\mathcal{A}_0 = \{u \in \mathcal{A} \mid \ f(\mathcal{O}, u) = 0\}.$$

Then the value function is the unique l.s.c. function with linear growth which is $+\infty$ on $\mathbb{R}^N \setminus \mathcal{K}$ and that satisfies :

$$\lambda v(x) + \max_{u \in \mathcal{A}_i} \{-\langle \zeta, f(x,u)\rangle - \ell(x,u)\} = 0 \quad \forall x \in \mathcal{M}_i,\ \forall \zeta \in \partial_P v(x),$$

$$\lambda v(\mathcal{O}) + H(\mathcal{O}, \zeta) \geq 0 \quad \forall \zeta \in \partial_P v(\mathcal{O}),$$

$$\lambda v(\mathcal{O}) - \min_{u \in \mathcal{A}_0} \ell(\mathcal{O}, u) \leq 0.$$

Final remarks

- The interior of $\mathcal{K}$ can always be taken as a stratum and so, the constrained Hamiton-Jacobi equation proposed by Soner is included in the set of equations proposed in the main theorem.
- The characterize of the value function neither requires its continuity nor that the state constraint set has empty interior.
- Under the continuity of the value function (on its domain) the controllability assumption can be dropped.
- The characterization for networks can be extended to a suitable notion of generalized network where the junction is replaced by a manifold.

Thanks for your attention !