A game-theoretical model of debt and bankruptcy

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Control system

Consider the control system of the borrower

$$\dot{x}(t) = I(x(t)) \cdot x(t) - u(t), \qquad x(0) = \bar{x},$$

where

- x : total debt, measured as a fraction of the yearly income of the borrower,
- $I(x)$: interest rate payed on debt at a given time,
- $u \in [0,1]$: payment rate, as a fraction of the income. This is the control variable for the borrower.

MODELS

MODEL 1: The interest rate $I(x)$ is given a priori. This yields an optimal control problem for the borrower.

MODEL 2: The interest rate $I(x)$ is determined by the CREDIT RATING of the borrower, which in turn depends on his global feedback strategy for repaying the debt. This corresponds to a differential game between the borrower (leading player) and a pool of risk-neutral lenders.

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Borrower may go bankrupt

Let $\rho : [0, M[\to [0, \infty[$ be instantaneous risk of bankruptcy which depends on the total debt such that ρ is not decreasing and

$$\rho(0) \;=\; 0, \qquad\qquad \lim_{x \to M-} \rho(x) \;=\; +\infty.$$

Here, M is a maximum size of the debt, so large that it immediately provokes bankruptcy. More precisely, if at time $\tau > 0$ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then

$$Prob.\left\{ T_b \in [\tau, \tau + \varepsilon] \,\middle|\, T_b > \tau\,,\; x(\tau) = y \right\} \;=\; \rho(y) \cdot \varepsilon + o(\varepsilon).$$

Here the random variable T_b denotes the bankruptcy time.

The larger the debt, the larger the risk of bankruptcy. Hence the interest rate charged by lenders has to increase. It is natural to assume

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Distribution function of T_b

If the size of the debt is $t \mapsto x(t)$, then the distribution function

$$\psi(t) \doteq Prob.\{T_b > t\}$$

is obtained by solving the linear Cauchy problem

$$\psi(0) = 1, \qquad \psi'(t) = -\rho(x(t))\,\psi(t).$$

Namely,

$$\psi(t) = \exp\left\{-\int_0^t \rho(x(s))\,ds\right\}.$$

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Expected cost

Assume that

- $L(u)$ is the cost to the borrower for implementing the control u such that

$$L(0) = 0, \qquad L' > 0, \quad L'' > 0, \qquad \lim_{u \to 1-} L(u) \;=\; +\infty.$$

- r is the discount rate.
- B is the bankruptcy cost to the borrower.

The total cost

$$J[u, \bar{x}] \;\doteq\; E\left[\int_0^{T_b} e^{-rt} L(u(t))\,dt + B\,e^{-rT_b}\right] \tag{1}$$

$$= \;[\text{cost of making payment}] + [\text{bankruptcy cost}]. \tag{2}$$

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Optimal control problem for the borrower

One computes

$$E\left[\int_0^{T_b} e^{-rt} L(u(t))\,dt + B\,e^{-rT_b}\right]$$
$$= \int_0^{\infty} \exp\left\{-rt - \int_0^t \rho(x(s))\,ds\right\}\left\{\rho(x(t))B + L(u(t))\right\} dt$$

(P) Optimal control problem with bankruptcy risk. *Given the initial size $\bar{x}$ of the debt, find a control $t \mapsto u(t) \in [0,1]$ which minimizes the expected cost*

$$J[\bar{x}, u] \doteq \int_0^{\infty} \exp\left\{-rt - \int_0^t \rho(x(s))\,ds\right\}\left\{\rho(x(t))B + L(u(t))\right\} dt\,, \tag{3}$$

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Hamilton-Jacobi equation

The value function

$$V(\bar{x}) \;=\; \min J[u, \bar{x}]$$

is a viscosity solution of

$$V(x) \;=\; \frac{1}{r+\rho(x)} \cdot H(x, V'(x))\,, \tag{5}$$

with

$$V(0) \;=\; 0 \quad , \quad V(M) \;=\; B\,.$$

Hamiltonian function

$$H(x,\xi) \;=\; \min_{\omega\in[0,1]} \Big\{L(\omega) - \xi\cdot\omega\Big\} + \xi\cdot I(x)x + \rho(x)B\,. \tag{6}$$

The corresponding optimal feedback control is

$$u^*(\xi) \;=\; \arg\min_{\omega\in[0,1]} \{L(\omega) - \xi\cdot\omega\}. \tag{7}$$

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A viscosity super solution

Let W(x) be the cost achieved by the strategy that keeps the debt constantly equal to x, i.e.,

$$W(x) \;=\; J[x,u] \quad \text{with} \quad u = I(x)x\,.$$

Thus,

$$W(x) \;=\; \begin{cases} \dfrac{1}{r+\rho(x)}\cdot H^{max}(x) & \text{if} \quad 0 \le y < x^*, \\ \\ +\infty & \text{if} \quad x \ge x^*\,, \end{cases} \tag{8}$$

where

$$H^{max}(x) \;\doteq\; \max_{\xi} H(x,\xi) \;=\; H(x,\xi^\sharp(x))\,.$$

and x^* is defined by $I(x^*)x^* = 1$.

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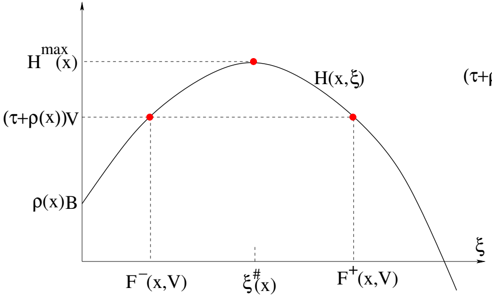
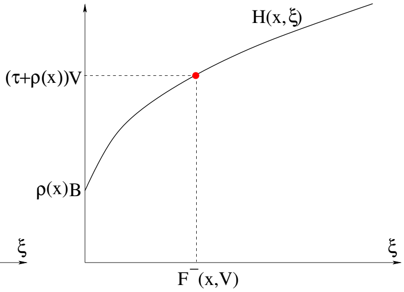
Differential inclusion

Hamilton-Jacobi equation

$$V(x) \;=\; \frac{1}{r+\rho(x)} \cdot H(x, V'(x)), \tag{9}$$





$$V' \;\in\; \left\{ F^-(x,V),\; F^+(x,V) \right\}, \tag{10}$$

where $V' = F^-$ and $V' = F^+$ are the two solutions of the equation (9).

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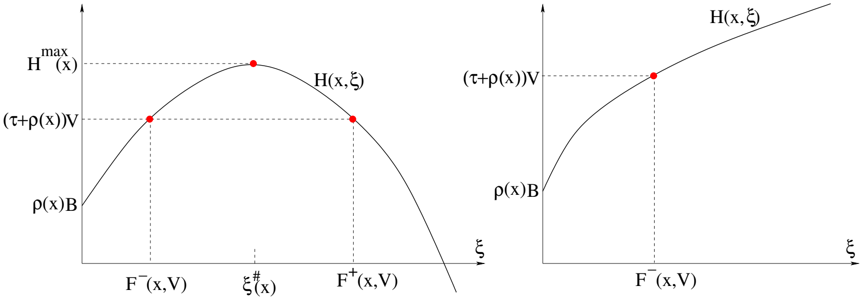

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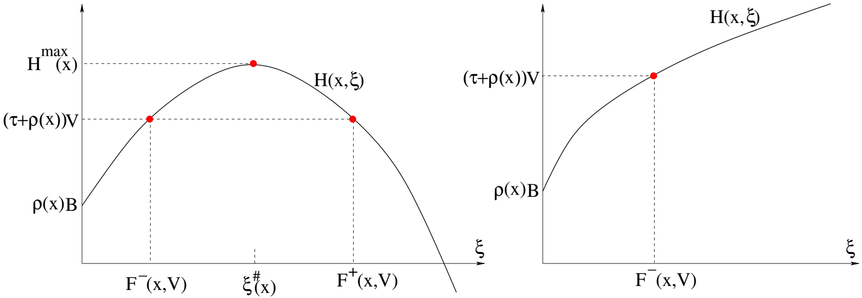

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Initial data

- The value $V' = F^{+}(x, V) \geq \xi^{\sharp}(x)$ corresponds to the choice of an optimal control such that $\dot{x} < 0$.
- The value $V' = F^{-}(x, V) \leq \xi^{\sharp}(x)$ corresponds to the choice of an optimal control such that $\dot{x} > 0$.

How to choose the appropriate initial data?

Let $t \mapsto x(t)$ be an optimal trajectory $t \mapsto x(t)$ such that

$$\lim_{t\to\infty} x(t) = x_0 .$$

Hence, as $t \to \infty$, the debt remains almost constant, close to x_0. We thus expect that $V(x(t)) \to W(x_0)$. Therefore, as initial data, we should choose

$$V(x_0) = W(x_0) \qquad \text{for some } x_0 .$$

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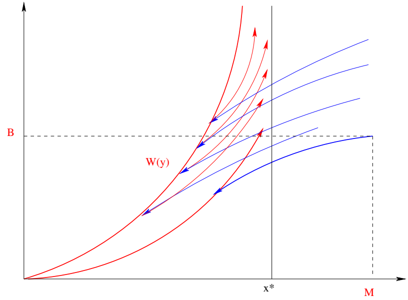
Value function

For any $x_0 \in [0, x^*[$, let $Z(\cdot, x_0)$ be the solution of

$$Z'(x) \;=\; \begin{cases} F^+(x,Z) & \text{if} \quad x_0 < x < x^*, \\ F^-(x,Z) & \text{if} \quad 0 \le x < x_0\,, \end{cases} \qquad Z(x_0) \;=\; W(x_0). \tag{11}$$

Moreover, let $Z(\cdot, M)$ be the solution of

$$Z'(x) \;=\; F^-(x,Z) \quad \text{and} \quad Z(M-) \;=\; B.$$



The value function for the optimization problem (3)–(4) is

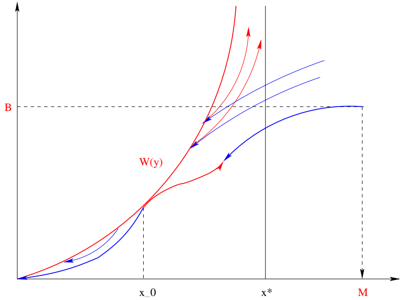
$$V(x) \;=\; \min_{x_0 \in [0,x^*[\, \cup \{M\}} Z(x; x_0). \tag{12}$$

Theorem 1. *Let the functions l, ρ, L satisfy the standard assumptions. Let W be the function in (8) and consider the set*

$$\mathcal{A} \doteq \Big\{x_0 \in [0, M]\,;\quad H^{max}(x_0) = H(x_0, W'(x_0))\Big\} \cup \{0, M\}. \tag{13}$$

Then the value function for the optimization problem (3)-(4) is given by

$$V(x) = \min_{x_0 \in \mathcal{A}} Z(x; x_0). \tag{14}$$



$$H^{max}(x_0) = H(x_0, W'(x_0))$$

$$\Updownarrow$$

$$\begin{aligned} F^+(x_0, W(x_0)) &= F^-(x_0, W(x_0)) \\ &= W'(x_0)\,. \end{aligned}$$

Proof

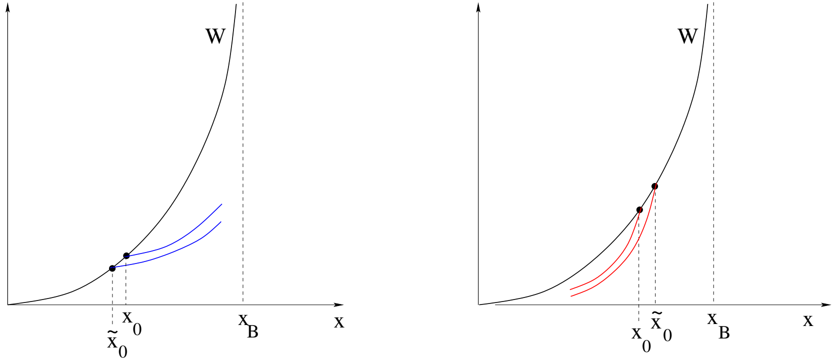


Figure : Left: if $F^{\pm}(x_0, W(x_0)) < W'(x_0)$, then we can choose a smaller initial value $\tilde{x}_0 < x_0$ and obtain a lower cost function.

Right: if $F^{\pm}(x_0, W(x_0)) > W'(x_0)$, then we can choose a bigger initial value $\tilde{x}_0 > x_0$ and obtain a lower cost function.

A game-theoretical model

Expected pay-off for a lender

A pool-risk neutral lenders charge an interest depending on the bankruptcy risk.

- σ : fraction of capital recovered by the lenders if bankruptcy occurs,

$$0 < \sigma < 1.$$

- T_e: the (random) expiration date of the loan

$$Prob.\ \{T_e > t\} = e^{-\lambda t}.$$

For a unit amount of loan, the lender will receive the payoff

$$\begin{cases} \displaystyle\int_0^{T_e} I(s)\, e^{-rs}\, ds + e^{-rT_e} & \text{if} \quad T_e < T_b, \\ \displaystyle\int_0^{T_b} I(s)\, e^{-rs}\, ds + \sigma\, e^{-rT_b} & \text{if} \quad T_e \geq T_e. \end{cases} \tag{15}$$

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An ODE for the interest rate

Risk-neutral assumption implies that

$$E\left[\int_0^{T_e\wedge T_b} I(x(s))e^{-rs}\,ds + \begin{cases} e^{-rT_e} & \text{if } T_e < T_b \\ \sigma e^{-rT_b} & \text{if } T_e \geq T_b \end{cases}\right]_{x(0)=\bar{x}} = 1. \tag{16}$$

Recalling that

$$P(t,\bar{x}) = Prob.\{T_b > t\,;\ x(0)=\bar{x}\} = \exp\left\{-\int_0^t \rho(x(s))\,ds\right\}$$

the interest rate is recovered by solving the ODEs

$$I'(\bar{x}) = \frac{\rho(\bar{x})+\lambda+r}{I(\bar{x})\bar{x}-u(\bar{x})}\cdot(I(\bar{x})-r) + \frac{\lambda(\sigma-1)\rho(\bar{x})}{I(\bar{x})\bar{x}-u(\bar{x})}\,. \tag{17}$$

System of equations

Let V be the value function of the optimal problem for the borrower and

$$H(x,\alpha,\xi) \doteq \min_{\omega\in[0,1]} \Big\{L(\omega) - \omega\,\xi\Big\} + \alpha\, x\,\xi + +\rho(x)B\,.$$

The system of equations

$$\begin{cases} V(x) = \dfrac{1}{r+\rho(x)} \cdot H(x, I(x), V'(x)), \\[2ex] I'(x) = \dfrac{\rho(x)+\lambda+r}{I(x)x - u(x)} \cdot (I(x) - r) + \dfrac{\lambda(\sigma-1)\rho(x)}{I(x)x - u(x)}, \\[2ex] u(x) = \arg\min\limits_{\omega\in[0,1]} \{L(\omega) - V'(x)\cdot\omega\}, \end{cases} \tag{18}$$

together with the boundary conditions

$$V(0) = 0\,, \qquad V(M) = B\,. \tag{19}$$

Existence of admissible solution

Definition 1. A pair of functions (V, I) defined for $x \in [0, M]$ is an *admissible solution* of (18) if the following holds

- $I : [0, M] \mapsto [0, +\infty]$ is lower semicontinuous with at most countably many points of discontinuity, and satisfies the second equation in (18) at a.e. point x where $I(x) < +\infty$.
- V is a continuous function, satisfying the boundary conditions (19). It provides a solution to the differential inclusion, with derivative V' having at most countably many points of jump. At each point y where V' is discontinuous, the left and right limits satisfy

$$V'(y-) \;\geq\; V'(y+).$$

Theorem 2. *Let the functions ρ, L satisfy the standard assumptions. Then the problem (18)-(19) has an admissible solution.*

The admissible solution constructed in Theorem 2 provides a Nash equilibrium to the debt and bankruptcy game.

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THANK YOU