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A game-theoretical model of debt and bankruptcy

Alberto Bressan and Khai T. Nguyen

Department of Mathematics, Penn State University

bressan@math.psu.edu , ktn2@psu.edu

Consider the control system of the borrower

$$\dot{x}(t) = I(x(t)) \cdot x(t) - u(t), \quad x(0) = \bar{x},$$

where

- x : total debt, measured as a fraction of the yearly income of the borrower ,
- $I(x)$: interest rate payed on debt at a given time ,
- $u \in [0, 1]$: payment rate, as a fraction of the income. This is the control variable for the borrower .

MODEL 1: The interest rate $I(x)$ is given a priori. This yields an optimal control problem for the borrower.

MODEL 2: The interest rate $I(x)$ is determined by the CREDIT RATING of the borrower, which in turn depends on his global feedback strategy for repaying the debt. This corresponds to a differential game between the borrower (leading player) and a pool of risk-neutral lenders.

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Borrower may go bankrupt

Let $\rho : [0, M[\rightarrow [0, \infty[$ be instantaneous **risk of bankruptcy** which depends on the total debt such that ρ is not decreasing and

$$\rho(0) = 0, \quad \lim_{x \rightarrow M-} \rho(x) = +\infty.$$

Here, M is a **maximum size of the debt**, so large that it immediately provokes bankruptcy. More precisely, if at time $\tau > 0$ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then

$$\text{Prob.} \left\{ T_b \in [\tau, \tau + \varepsilon] \mid T_b > \tau, x(\tau) = y \right\} = \rho(y) \cdot \varepsilon + o(\varepsilon).$$

Here the random variable T_b denotes the **bankruptcy time**.

The larger the debt, the larger the risk of bankruptcy. Hence the interest rate charged by lenders has to increase. It is natural to assume

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Distribution function of T_b

If the size of the debt is $t \mapsto x(t)$, then the distribution function

$$\psi(t) \doteq \text{Prob.}\{T_b > t\}$$

is obtained by solving the linear Cauchy problem

$$\psi(0) = 1, \quad \psi'(t) = -\rho(x(t))\psi(t).$$

Namely,

$$\psi(t) = \exp\left\{-\int_0^t \rho(x(s)) ds\right\}.$$

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Expected cost

Assume that

- $L(u)$ is the cost to the borrower for implementing the control u such that

$$L(0) = 0, \quad L' > 0, \quad L'' > 0, \quad \lim_{u \rightarrow 1^-} L(u) = +\infty.$$

- r is the discount rate.
- B is the bankruptcy cost to the borrower.

The total cost

$$J[u, \bar{x}] \doteq E \left[\int_0^{T_b} e^{-rt} L(u(t)) dt + B e^{-rT_b} \right] \quad (1)$$

$$= [\text{cost of making payment}] + [\text{bankruptcy cost}]. \quad (2)$$

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Optimal control problem for the borrower

One computes

$$\begin{aligned} E \left[\int_0^{T_b} e^{-rt} L(u(t)) dt + B e^{-rT_b} \right] \\ = \int_0^\infty \exp \left\{ -rt - \int_0^t \rho(x(s)) ds \right\} \left\{ \rho(x(t))B + L(u(t)) \right\} dt \end{aligned}$$

(P) **Optimal control problem with bankruptcy risk.** Given the initial size \bar{x} of the debt, find a control $t \mapsto u(t) \in [0, 1]$ which minimizes the expected cost

$$J[\bar{x}, u] \doteq \int_0^\infty \exp \left\{ -rt - \int_0^t \rho(x(s)) ds \right\} \left\{ \rho(x(t))B + L(u(t)) \right\} dt, \quad (3)$$

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Hamilton-Jacobi equation

The value function

$$V(\bar{x}) = \min J[u, \bar{x}]$$

is a viscosity solution of

$$V(x) = \frac{1}{r + \rho(x)} \cdot H(x, V'(x)), \quad (5)$$

with

$$V(0) = 0 \quad , \quad V(M) = B.$$

Hamiltonian function

$$H(x, \xi) = \min_{\omega \in [0,1]} \left\{ L(\omega) - \xi \cdot \omega \right\} + \xi \cdot I(x)x + \rho(x)B. \quad (6)$$

The corresponding optimal feedback control is

$$u^*(\xi) = \arg \min_{\omega \in [0,1]} \{L(\omega) - \xi \cdot \omega\}. \quad (7)$$

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A viscosity super solution

Let $W(x)$ be the cost achieved by the strategy that keeps the debt constantly equal to x , i.e.,

$$W(x) = J[x, u] \quad \text{with} \quad u = I(x)x.$$

Thus,

$$W(x) = \begin{cases} \frac{1}{r + \rho(x)} \cdot H^{\max}(x) & \text{if } 0 \leq x < x^*, \\ +\infty & \text{if } x \geq x^*, \end{cases} \quad (8)$$

where

$$H^{\max}(x) \doteq \max_{\xi} H(x, \xi) = H(x, \xi^{\#}(x)).$$

and x^* is defined by $I(x^*)x^* = 1$.

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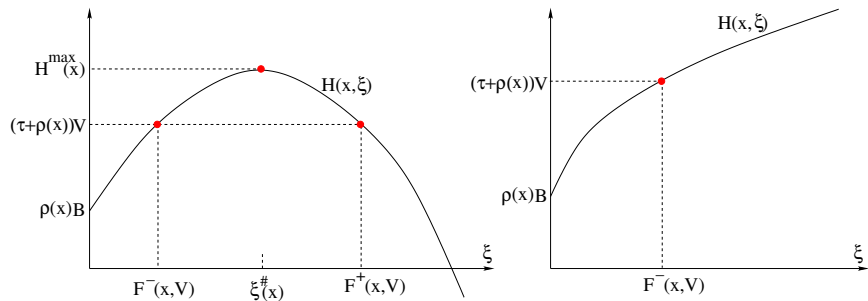
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Differential inclusion

Hamilton-Jacobi equation

$$V(x) = \frac{1}{r + \rho(x)} \cdot H(x, V'(x)), \quad (9)$$



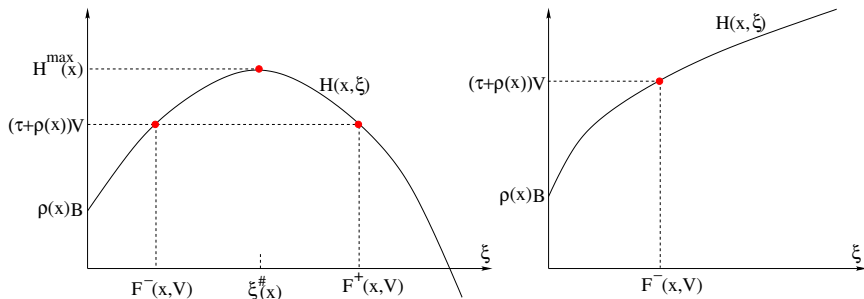
$$V' \in \left\{ F^-(x, V), F^+(x, V) \right\}, \quad (10)$$

where $V' = F^-$ and $V' = F^+$ are the two solutions of the equation (9).

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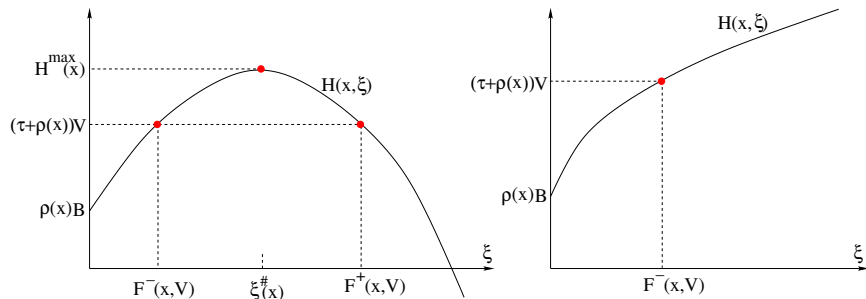
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Initial data

- The value $V' = F^+(x, V) \geq \xi^\#(x)$ corresponds to the choice of an optimal control such that $\dot{x} < 0$.
- The value $V' = F^-(x, V) \leq \xi^\#(x)$ corresponds to the choice of an optimal control such that $\dot{x} > 0$.

How to choose the appropriate initial data?

Let $t \mapsto x(t)$ be an optimal trajectory $t \mapsto x(t)$ such that

$$\lim_{t \rightarrow \infty} x(t) = x_0.$$

Hence, as $t \rightarrow \infty$, the debt remains almost constant, close to x_0 . We thus expect that $V(x(t)) \rightarrow W(x_0)$. Therefore, as initial data, we should choose

$$V(x_0) = W(x_0) \quad \text{for some } x_0.$$

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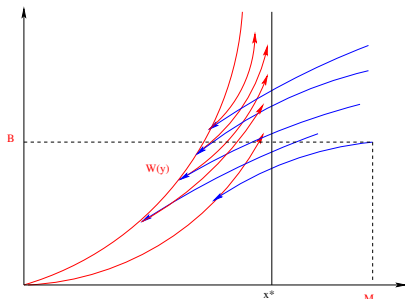
Value function

For any $x_0 \in [0, x^*[$, let $Z(\cdot, x_0)$ be the solution of

$$Z'(x) = \begin{cases} F^+(x, Z) & \text{if } x_0 < x < x^*, \\ F^-(x, Z) & \text{if } 0 \leq x < x_0, \end{cases} \quad Z(x_0) = W(x_0). \quad (11)$$

Moreover, let $Z(\cdot, M)$ be the solution of

$$Z'(x) = F^-(x, Z) \quad \text{and} \quad Z(M-) = B.$$



The value function for the optimization problem (3)–(4) is

$$V(x) = \min_{x_0 \in [0, x^*] \cup \{M\}} Z(x; x_0). \quad (12)$$

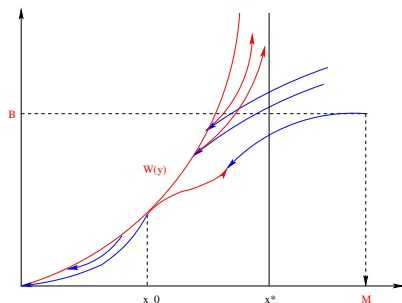
Characterization of the value function

Theorem 1. Let the functions I, ρ, L satisfy the standard assumptions. Let W be the function in (8) and consider the set

$$\mathcal{A} \doteq \left\{ x_0 \in [0, M]; H^{\max}(x_0) = H(x_0, W'(x_0)) \right\} \cup \{0, M\}. \quad (13)$$

Then the value function for the optimization problem (3)-(4) is given by

$$V(x) = \min_{x_0 \in \mathcal{A}} Z(x; x_0). \quad (14)$$



$$H^{\max}(x_0) = H(x_0, W'(x_0))$$

\Leftrightarrow

$$\begin{aligned} F^+(x_0, W(x_0)) &= F^-(x_0, W(x_0)) \\ &= W'(x_0). \end{aligned}$$

Proof

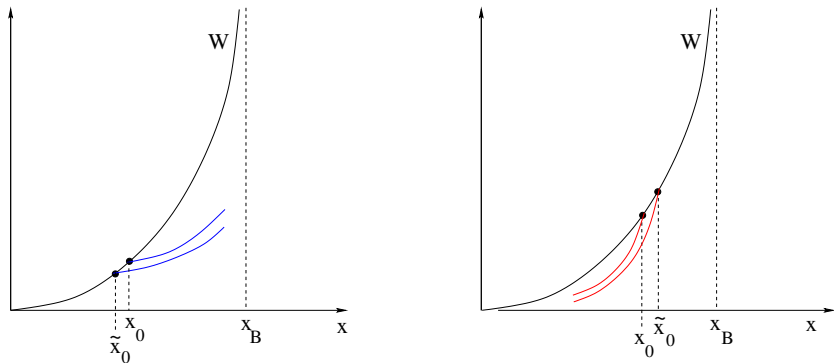


Figure : Left: if $F^\pm(x_0, W(x_0)) < W'(x_0)$, then we can choose a smaller initial value $\tilde{x}_0 < x_0$ and obtain a lower cost function.

Right: if $F^\pm(x_0, W(x_0)) > W'(x_0)$, then we can choose a bigger initial value $\tilde{x}_0 > x_0$ and obtain a lower cost function.

A game-theoretical model

Expected pay-off for a lender

A pool-risk neutral lenders charge an interest depending on the bankruptcy risk.

- σ : fraction of capital recovered by the lenders if bankruptcy occurs ,

$$0 < \sigma < 1.$$

- T_e : the (random) expiration date of the loan

$$\text{Prob. } \{T_e > t\} = e^{-\lambda t}.$$

For a unit amount of loan, the lender will receive the payoff

$$\left\{ \begin{array}{ll} \int_0^{T_e} l(s) e^{-rs} ds + e^{-rT_e} & \text{if } T_e < T_b, \\ \int_0^{T_b} l(s) e^{-rs} ds + \sigma e^{-rT_b} & \text{if } T_e \geq T_b. \end{array} \right. \quad (15)$$

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An ODE for the interest rate

Risk-neutral assumption implies that

$$E \left[\int_0^{T_e \wedge T_b} I(x(s)) e^{-rs} ds + \begin{cases} e^{-rT_e} & \text{if } T_e < T_b \\ \sigma e^{-rT_b} & \text{if } T_e \geq T_b \end{cases} \right]_{x(0)=\bar{x}} = 1. \quad (16)$$

Recalling that

$$P(t, \bar{x}) = \text{Prob.}\{T_b > t; x(0) = \bar{x}\} = \exp \left\{ - \int_0^t \rho(x(s)) ds \right\}$$

the interest rate is recovered by solving the ODEs

$$I'(\bar{x}) = \frac{\rho(\bar{x}) + \lambda + r}{I(\bar{x})\bar{x} - u(\bar{x})} \cdot (I(\bar{x}) - r) + \frac{\lambda(\sigma - 1)\rho(\bar{x})}{I(\bar{x})\bar{x} - u(\bar{x})}. \quad (17)$$

System of equations

Let V be the value function of the optimal problem for the borrower and

$$H(x, \alpha, \xi) \doteq \min_{\omega \in [0,1]} \left\{ L(\omega) - \omega \xi \right\} + \alpha x \xi + \rho(x) B.$$

The system of equations

$$\left\{ \begin{array}{l} V(x) = \frac{1}{r + \rho(x)} \cdot H(x, I(x), V'(x)), \\ I'(x) = \frac{\rho(x) + \lambda + r}{I(x)x - u(x)} \cdot (I(x) - r) + \frac{\lambda(\sigma - 1)\rho(x)}{I(x)x - u(x)}, \\ u(x) = \arg \min_{\omega \in [0,1]} \{L(\omega) - V'(x) \cdot \omega\}, \end{array} \right. \quad (18)$$

together with the boundary conditions

$$V(0) = 0, \quad V(M) = B. \quad (19)$$

Existence of admissible solution

Definition 1. A pair of functions (V, I) defined for $x \in [0, M]$ is an *admissible solution* of (18) if the following holds

- $I : [0, M] \mapsto [0, +\infty]$ is **lower semicontinuous with at most countably many points of discontinuity**, and satisfies the second equation in (18) at a.e. point x where $I(x) < +\infty$.
- V is a continuous function, satisfying the boundary conditions (19). It provides a solution to the differential inclusion, **with derivative V' having at most countably many points of jump**. At each point y where V' is discontinuous, the left and right limits satisfy

$$V'(y-) \geq V'(y+).$$

Theorem 2. *Let the functions ρ, L satisfy the standard assumptions. Then the problem (18)-(19) has an admissible solution.*

The admissible solution constructed in Theorem 2 provides a Nash equilibrium to the debt and bankruptcy game.

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THANK YOU