Bang bang trajectories with a double switching time in the minimum time problem

Laura Poggiolini, Marco Spadini, Gianna Stefani

Bang–bang trajectories with a double switching time in the minimum time problem

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The minimum time problem

$$
\begin{aligned}
& T \to \min \\
& \dot{\xi}(t) = f_0(\xi(t)) + \sum_{i=1}^{m} u_i f_i(\xi(t)), \\
& \xi(0) \in N_0, \qquad \xi(T) \in N_f, \\
& u = (u_1, \ldots, u_m) \in Q := [-1, 1]^m
\end{aligned}
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$$u = (u_1, \ldots, u_m) \in Q := [-1, 1]^m$$

Assume $(\widehat{T}, \widehat{\xi}, \widehat{u})$ is an admissible triplet satisfying PMP with an adjoint covector

$$\widehat{\lambda} \colon t \in [0, \widehat{T}] \mapsto \widehat{\lambda}(t) \in T^*M$$

The state space M is a finite dimensional manifold

Problem

Give sufficient second order conditions for $\widehat{\xi}$ to be a strong local optimal trajectory

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Definition ((time,state)-local optimality)

The trajectory $\widehat{\xi}$ is a (time,state)-local minimiser if there is a neighborhood $\widetilde{\mathcal{U}}$ of its graph such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose graph is in $\widetilde{\mathcal{U}}$.

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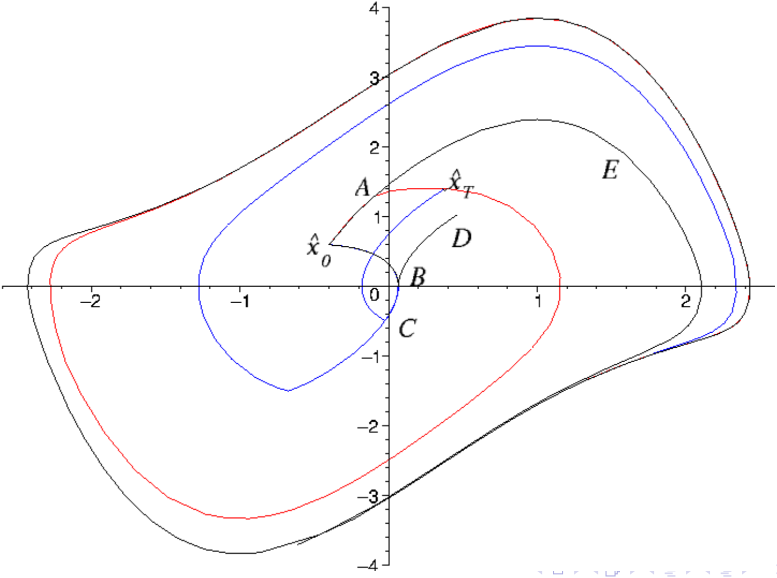
Definition (state-local optimality)

The trajectory $\widehat{\xi}$ is a state-local minimiser if there are neighborhoods $\mathcal{U}$ of its range $\widehat{\xi}([0, \widehat{T}])$, $\mathcal{U}_0$ of $\widehat{\xi}(0)$ and $\mathcal{U}_f$ of $\widehat{\xi}(\widehat{T})$ such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose range is in $\mathcal{U}$, whose initial point is in $N_0 \cap \mathcal{U}_0$ and whose final point is in $N_f \cap \mathcal{U}_f$.

Example



Hamiltonians associated to the vector fields

$$F_i : \ell = (p, q) \in T^*M \mapsto \langle p\,,\ f_i(q)\rangle\,, \quad i = 0, 1, \ldots, m$$

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Reference Hamiltonian

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Maximised Hamiltonian

$$F^{\max} : \ell \mapsto \sup_{u \in Q} \left\{ F_0(\ell) + \sum_{i=1}^{m} u_i F_i(\ell) \right\}$$

Hamiltonian vector fields

$\overrightarrow{F}^{\max}\,,\ \overrightarrow{\widehat{F}}_t\,,\ \overrightarrow{F_i}$

Pontryagin Maximum Principle (PMP)

there exist $\widehat{\lambda}\colon [0, \widehat{T}] \to T^*M, \qquad p_0 \in \{0, 1\}$

such that:

$$\|\widehat{\lambda}(0)\| + p_0 \neq 0$$

$$\pi\widehat{\lambda}(t) = \widehat{\xi}(t)$$

$$\frac{d}{dt}\widehat{\lambda}(t) = \overrightarrow{\widehat{F}}_t(\widehat{\lambda}(t))$$

$$\widehat{F}_t(\widehat{\lambda}(t)) = F^{\max}(\widehat{\lambda}(t)) \equiv p_0$$

$$\widehat{\lambda}(0)\Big|_{T_{\widehat{\xi}(0)}N_0} = 0, \qquad \widehat{\lambda}(\widehat{T})\Big|_{T_{\widehat{\xi}(\widehat{T})}N_f} = 0$$

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$$\widehat{\lambda}(0)\Big|_{T_{\widehat{\xi}(0)}N_0} = 0, \qquad \widehat{\lambda}(\widehat{T})\Big|_{T_{\widehat{\xi}(\widehat{T})}N_f} = 0$$

if $p_0 = 1$, $\widehat{\lambda}$ is called normal extremal
if $p_0 = 0$, $\widehat{\lambda}$ is called abnormal extremal

Hamiltonian approach to (time, state)–local optimality 1/2

- the flow of an over-maximised Hamiltonian

$$H \colon \ell \in T^*M \mapsto H(\ell) \in \mathbb{R}$$

$$H(\ell) \geqslant F^{\max}(\ell) \qquad H(\widehat{\lambda}(t)) = F^{\max}(\widehat{\lambda}(t))$$

$$\mathcal{H} \colon (t, \ell) \in [0, \widehat{T}] \times T^*M \mapsto \mathcal{H}_t(\ell) \in T^*M$$

$$\widehat{\lambda}(t) = \mathcal{H}_t(\widehat{\lambda}(0)), \text{ i.e. } \dot{\widehat{\lambda}}(t) = \overrightarrow{H}(\widehat{\lambda}(t))$$

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- a smooth function $\alpha\colon M \to \mathbb{R}$ such that $d\alpha(\widehat{\xi}(0)) = \widehat{\lambda}(0)$

$$\Lambda := \{(d\alpha(x), x)\}$$

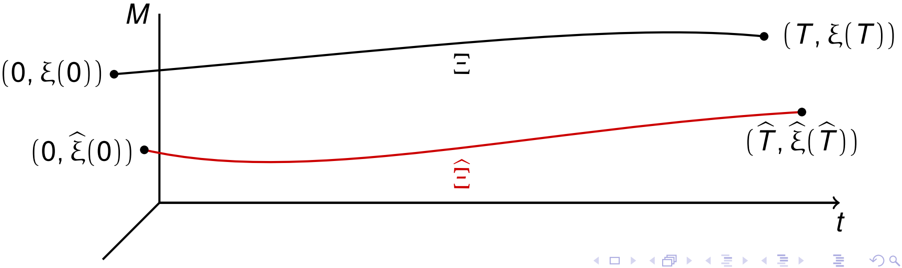
Hamiltonian approach to (time, state)–local optimality 2/2

- $$\begin{array}{ccc} [0,\widehat{T}] \times \Lambda & \xrightarrow{\text{id} \times \mathcal{H}} & [0,\widehat{T}] \times T^*M \\ & \nwarrow^{(\text{id} \times \pi\mathcal{H})^{-1}} & \downarrow{\scriptstyle \text{id} \times \pi} \\ & & [0,\widehat{T}] \times M \end{array}$$

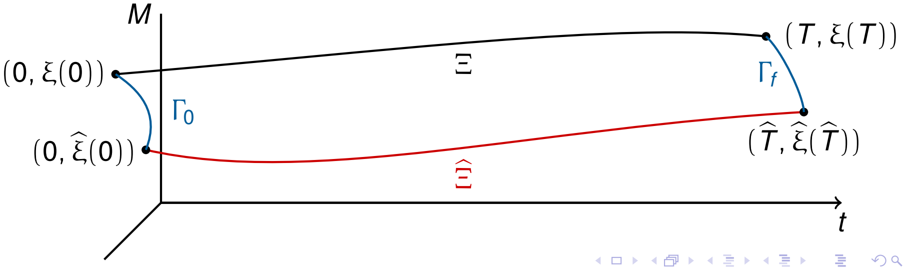
Property

$\mathcal{H}^*(p\, dq - H_t\, dt)$ is exact on $[0,\widehat{T}] \times \Lambda$

Lifting trajectories



Lifting trajectories



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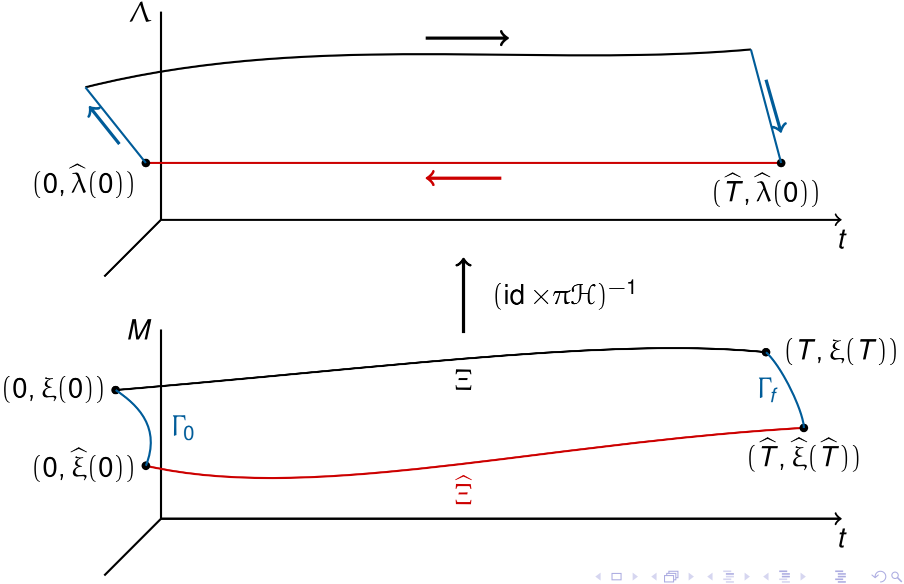

$\mathcal{H}^*(p\,\mathrm{d}q - H_t\,\mathrm{d}t)$ is exact $\implies$

$$0 = \oint \omega = \int_{(\mathrm{id} \times \pi\mathcal{H})^{-1}(\xi)} \omega + \int_{(\mathrm{id} \times \pi\mathcal{H})^{-1}(\Gamma_f)} \omega - \int_{(\mathrm{id} \times \pi\mathcal{H})^{-1}(\widehat{\xi})} \omega + \int_{(\mathrm{id} \times \pi\mathcal{H})^{-1}(\Gamma_0)} \omega$$

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$$0 \leqslant p_0(T - \widehat{T}) + O\left((T - \widehat{T})^2\right)$$

If $p_0 = 1 \implies \widehat{\xi}$ is a (time, state)-local optimal trajectory

Hamiltonian approach to state–local optimality 1/2

- the flow of an over-maximised Hamiltonian

$$H \colon \ell \in T^*M \mapsto H(\ell) \in \mathbb{R}$$

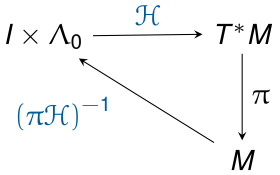
$$H(\ell) \geqslant F^{\max}(\ell) \qquad H(\widehat{\lambda}(t)) = F^{\max}(\widehat{\lambda}(t))$$

$$\mathcal{H} \colon (t, \ell) \in (-\delta, \widehat{T} + \delta) \times T^*M \mapsto \mathcal{H}_t(\ell) \in T^*M$$

$$\widehat{\lambda}(t) = \mathcal{H}_t(\widehat{\lambda}(0))$$

- a smooth function $\alpha \colon M \to \mathbb{R}$ such that
  - $d\alpha(\widehat{\xi}(0)) = \widehat{\lambda}(0)$
  - $\Lambda = \{(d\alpha(x), x)\}$ is transverse to $\{H = p_0\}$ in $\widehat{\lambda}(0)$
- $\Lambda_0 := \Lambda \cap \{H = p_0\}$ is a $(n-1)$-dim manifold of T^*M

Hamiltonian approach to state–local optimality 2/2

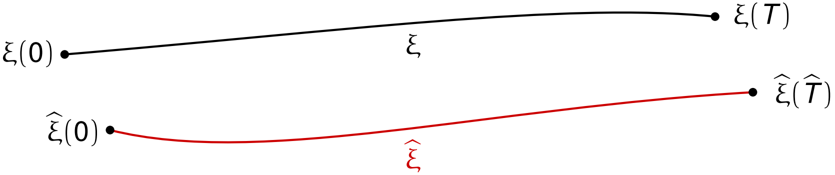
- 

$$I := (-\delta, \widehat{T} + \delta)$$

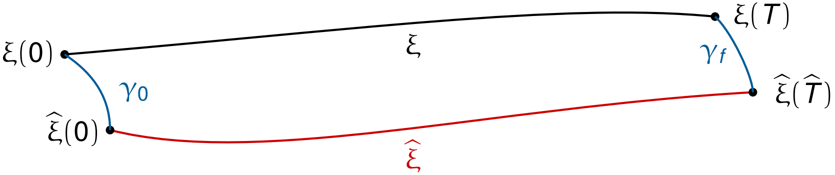
Property

$\mathcal{H}^*(p\, dq)$ is exact on $(-\delta, \widehat{T} + \delta) \times \Lambda_0$

Lifting trajectories



Lifting trajectories



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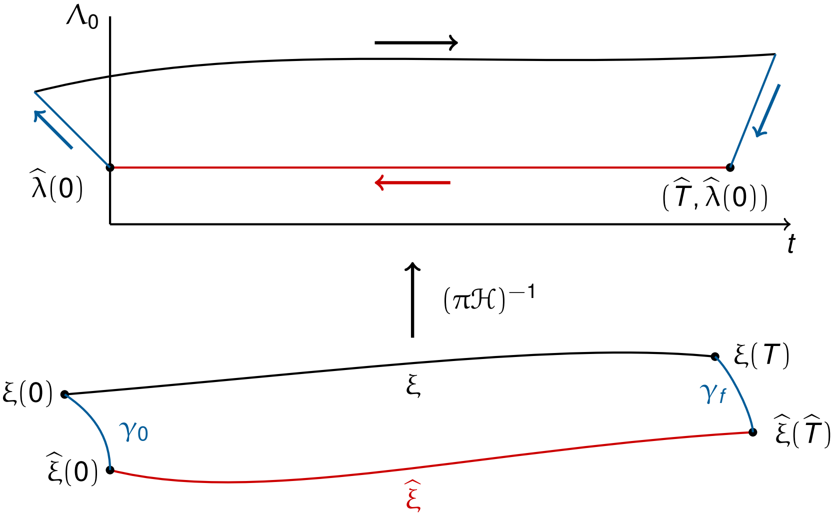

$\mathcal{H}^*(p\,dq)$ is exact $\Longrightarrow$

$$0 = \oint \omega = \int_{(\pi\mathcal{H})^{-1}(\xi)} \omega + \int_{(\pi\mathcal{H})^{-1}(\gamma_f)} \omega - \int_{(\pi\mathcal{H})^{-1}(\widehat{\xi})} \omega + \int_{(\pi\mathcal{H})^{-1}(\gamma_0)} \omega$$

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$$\begin{aligned} p_0\left(T - \widehat{T}\right) &\geqslant p_0\left(t_f - \widehat{T}\right) + \alpha(q_f) - \alpha(\widehat{\xi}(0)) + \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\overline{q}) \\ &= C(\xi(T)) - C(\widehat{\xi}(\widehat{T})) + \frac{t_0^2}{2} h_1 \cdot h_1 \cdot \alpha(\overline{q}) \end{aligned}$$

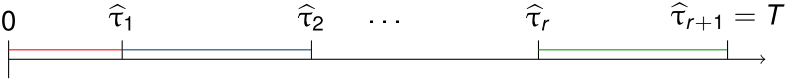
$C(x) := p_0(\psi^{\mathbb{R}}(x)) + \alpha(\pi\psi^{\Lambda_0}(x))$ $\quad\left(\psi^{\mathbb{R}}, \psi^{\Lambda_0}\right) := (\pi\mathcal{H})^{-1}$

$(t_f, q_f) := \left(\psi^{\mathbb{R}}, \pi\psi^{\Lambda_0}\right)(\xi(T))$

$(t_0, q_0) := \left(\psi^{\mathbb{R}}, \pi\psi^{\Lambda_0}\right)(\xi(0))$ $\quad \overline{q} = \exp \overline{s} h_1(q_0)$

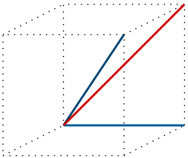
Bang-bang controls

$$\widehat{u}(t) \in \text{Vertexes}(Q) \qquad \forall t \in [0, \widehat{T}] \setminus \left\{\widehat{\tau}_j \colon j = 1, \ldots, r\right\}$$



Simple, double and multiple switches

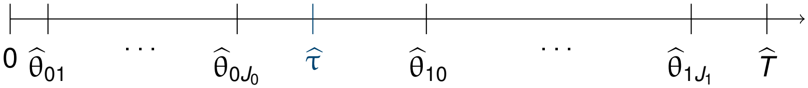
- simple switch $\widehat{\tau}$ is a simple switching time if only one control component switches at time $\widehat{\tau}$
- double switch $\widehat{\tau}$ is a double switching time if two control components switch at time $\widehat{\tau}$
- *j*-multiplicity switch $\widehat{\tau}$ is a *j* multiplicity switching time if *j* control components switch at time $\widehat{\tau}$



Bang-bang with multiple switches

Simple and double switches

- there exists only double switching time $\widehat{\tau}$;
- a certain numberof simple switching times $\widehat{\theta}_{0j}$ occurs before $\widehat{\tau}$ and a certain number of simple switching times $\widehat{\theta}_{1j}$ occur after the double switch;



Simplified case

$$
\begin{aligned}
&T \to \min, \\
&\dot{\xi}(t) = f_0(\xi(t)) + u_1(t) f_1(\xi(t)) + u_2(t) f_2(\xi(t)), \\
&\xi(0) \in N_0, \qquad \xi(T) \in N_f, \\
&|u_i(t)| \leqslant 1 \quad i = 1, 2, \text{ a.e. } t \in [0, T].
\end{aligned}
$$

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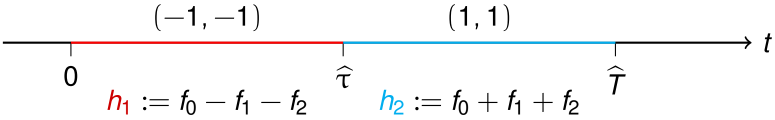

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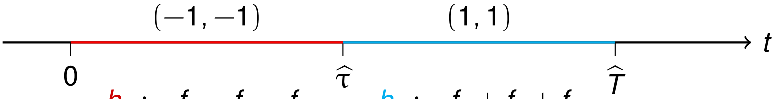

$h_1 := f_0 - f_1 - f_2$ $\qquad$ $h_2 := f_0 + f_1 + f_2$

$$H_1(\ell) := \langle \ell\,,\ h_1(\pi\ell)\rangle \qquad H_2(\ell) := \langle \ell\,,\ h_2(\pi\ell)\rangle$$

Assumptions: Regularity along the bang arcs

$$\text{PMP} \implies \begin{array}{l} \widehat{u}_1(t) F_1(\widehat{\lambda}(t)) = \widehat{u}_1(t) \langle \widehat{\lambda}(t)\,,\ f_1(\widehat{\xi}(t)) \rangle \geqslant 0 \\ \widehat{u}_2(t) F_2(\widehat{\lambda}(t)) = \widehat{u}_2(t) \langle \widehat{\lambda}(t)\,,\ f_2(\widehat{\xi}(t)) \rangle \geqslant 0. \end{array}$$

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Regularity along the bang arcs

If $t \neq \widehat{\tau}$, then $\widehat{u}$ is the only control that gives the maximised Hamiltonian:

$$\widehat{u}_1(t)F_1(\widehat{\lambda}(t)) = \widehat{u}_1(t)\langle\widehat{\lambda}(t)\,,\ f_1(\widehat{\xi}(t))\rangle > 0$$

$$\widehat{u}_2(t)F_2(\widehat{\lambda}(t)) = \widehat{u}_2(t)\langle\widehat{\lambda}(t)\,,\ f_2(\widehat{\xi}(t))\rangle > 0$$

i.e. $$\begin{aligned}&F_1(\widehat{\lambda}(t))<0 \quad F_2(\widehat{\lambda}(t))<0 \qquad \text{if } t \in [0, \widehat{\tau})\\ &F_1(\widehat{\lambda}(t))>0 \quad F_2(\widehat{\lambda}(t))>0 \qquad \text{if } t \in (\widehat{\tau}, \widehat{T}]\end{aligned}$$

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i.e. $F_1(\widehat{\lambda}(t)) < 0 \quad F_2(\widehat{\lambda}(t)) < 0 \qquad \text{if } t \in [0, \widehat{\tau})$

$F_1(\widehat{\lambda}(t)) > 0 \quad F_2(\widehat{\lambda}(t)) > 0 \qquad \text{if } t \in (\widehat{\tau}, \widehat{T}]$

$$\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}F_i(\widehat{\lambda}(t))\Big|_{t=\widehat{\tau}^-} \geqslant 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}F_i(\widehat{\lambda}(t))\Big|_{t=\widehat{\tau}^+} \geqslant 0$$

Assumptions: Regularity at the switching time 1/2

Regularity at the switching times

$$\frac{\mathrm{d}}{\mathrm{d}t} F_i(\widehat{\lambda}(t))\Big|_{t=\widehat{\tau}^-} > 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t} F_i(\widehat{\lambda}(t))\Big|_{t=\widehat{\tau}^+} > 0 \qquad i = 1, 2.$$

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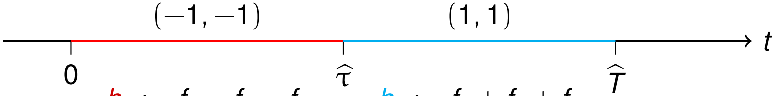

$$h_1 := f_0 - f_1 - f_2 \qquad h_2 := f_0 + f_1 + f_2$$

$$k_1 := f_0 + f_1 - f_2 = h_1 + 2f_1 = h_2 - 2f_2$$

$$k_2 := f_0 - f_1 + f_2 = h_1 + 2f_2 = h_2 - 2f_1$$

$$K_1(\ell) = \langle \ell\,,\ k_1(\pi\ell)\rangle \qquad K_2(\ell) = \langle \ell\,,\ k_2(\pi\ell)\rangle$$

Assumptions: Regularity at the switching time 2/2

$$\langle \widehat{\lambda}(\widehat{\tau}),\ [h_1, k_1](\widehat{\xi}(\widehat{\tau}))\rangle = \sigma\left(\overrightarrow{H_1}, \overrightarrow{K_1}\right)(\widehat{\lambda}(\widehat{\tau})) > 0$$

$$\langle \widehat{\lambda}(\widehat{\tau}),\ [h_1, k_2](\widehat{\xi}(\widehat{\tau}))\rangle = \sigma\left(\overrightarrow{H_1}, \overrightarrow{K_2}\right)(\widehat{\lambda}(\widehat{\tau})) > 0$$

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$$\langle \widehat{\lambda}(\widehat{\tau}),\ [k_2, h_2](\widehat{\xi}(\widehat{\tau}))\rangle = \sigma\left(\overrightarrow{K_2}, \overrightarrow{H_2}\right)(\widehat{\lambda}(\widehat{\tau})) > 0$$

The maximised flow

- for $\nu = 1, 2$ let $\tau_\nu(\ell)$ be the unique solution to

$$2F_\nu \circ \exp \tau_\nu(\ell)\overrightarrow{H_1}(\ell) = (K_\nu - H_1) \circ \exp \tau_\nu(\ell)\overrightarrow{H_1}(\ell) = 0$$

in a neighborhood of $(\widehat{\tau}, \widehat{\lambda}(0))$;

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- choose
$$\theta_1(\ell) := \min\{\tau_1(\ell), \tau_2(\ell)\};$$

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- choose

$$\theta_1(\ell) := \min\{\tau_1(\ell), \tau_2(\ell)\};$$

- for $\nu = 1, 2$, let $\tau^2_\nu(\ell)$ be the unique solution to

$$2F_{3-\nu} \circ \exp\left(\tau^2_\nu(\ell) - \tau_\nu(\ell)\right)\overrightarrow{K_\nu} \circ \exp \tau_\nu(\ell)\overrightarrow{H_1}(\ell)$$
$$= (H_2 - K_\nu) \circ \exp\left(\tau^2_\nu(\ell) - \tau_\nu(\ell)\right)\overrightarrow{K_\nu} \circ \exp \tau_\nu(\ell)\overrightarrow{H_1}(\ell) = 0$$

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The maximised flow

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  in a neighborhood of $(\widehat{\tau}, \widehat{\lambda}(0))$;
- choose

$$\theta_2(\ell) = \begin{cases} \tau^2_1(\ell) & \text{if } \tau_1(\ell) \leqslant \tau_2(\ell), \\ \tau^2_2(\ell) & \text{if } \tau_2(\ell) < \tau_1(\ell). \end{cases}$$

- $S_0 := \{(t,\ell)\colon t \in [-\delta, \theta_1(\ell)]\}$

$$\mathcal{F}_t^{\max}(\ell) = \exp t\overrightarrow{H_1}(\ell);$$

- $S_1 := \{(t,\ell)\colon \theta_1(\ell) = \tau_1(\ell),\ t \in (\theta_1(\ell), \theta_2(\ell)]\}$

$$\mathcal{F}_t^{\max}(\ell) = \exp(t - \theta_1(\ell))\overrightarrow{K_1} \circ \exp \theta_1(\ell)\overrightarrow{H_1}(\ell);$$

- $S_1^2 := \{(t,\ell)\colon \theta_1(\ell) = \tau_1(\ell),\ t \in (\theta_2(\ell), \widehat{T} + \delta]\}$

$$\mathcal{F}_t^{\max}(\ell) = \exp(t - \theta_2(\ell))\overrightarrow{H_2}(\ell) \circ \exp(\theta_2(\ell) - \theta_1(\ell))\overrightarrow{K_1} \circ \exp \theta_1(\ell)\overrightarrow{H_1}(\ell);$$

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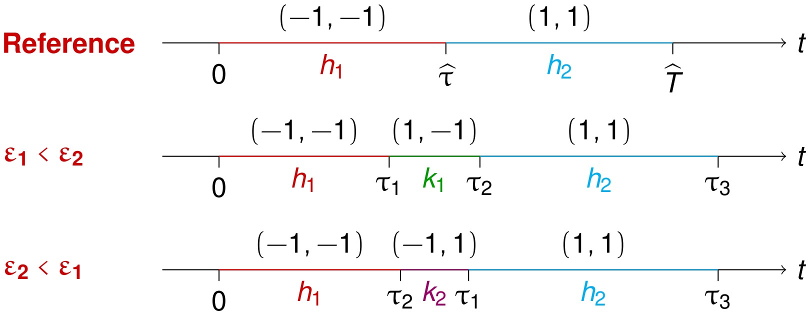
Finite-dimensional subproblem

Allow for different perturbed switching times of the control components

$$\tau_1 := \widehat{\tau} + \varepsilon_1 \qquad \tau_2 := \widehat{\tau} + \varepsilon_2$$

and for variations of the final time

$$\tau_3 := \widehat{T} + \varepsilon_3.$$



if $\varepsilon_1 < \varepsilon_2$

$$T \to \min$$

$$\dot{\xi}(t) = \begin{cases} h_1(\xi(t)) & t < \tau_1 \\ k_1(\xi(t)) & \tau_1 < t < \tau_2 \\ h_2(\xi(t)) & \tau_2 < t < \tau_3 \end{cases}$$

$$\xi(0) \in N_0, \qquad \xi(T) \in N_f,$$

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$$\xi(0) \in N_0, \qquad \xi(T) \in N_f,$$

if $\varepsilon_2 < \varepsilon_1$

$$T \to \min$$

$$\dot{\xi}(t) = \begin{cases} h_1(\xi(t)) & t < \tau_2 \\ k_2(\xi(t)) & \tau_2 < t < \tau_1 \\ h_2(\xi(t)) & \tau_1 < t < \tau_3 \end{cases}$$

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Assumption: coercivity of the second order approximations

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Each second order approximation J''_ν $\nu = 1, 2$ is coercive on the half-space of the linearised constraints of the finite-dimensional subproblem.

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$$J''_\nu[\delta x, a_1, b, a_2] = 2\, \delta x \cdot (a_1 g_1 + b\, j_\nu + a_2 g_2) \cdot \widehat{\beta}(\widehat{\xi}(0)) + \\ + (a_1 g_1 + b\, j_\nu + a_2 g_2)^2 \cdot \widehat{\beta}(\widehat{\xi}(0)) \\ + a_1 b\, [g_1, j_\nu] \cdot \widehat{\beta}(\widehat{\xi}(0)) + a_1 a_2\, [g_1, g_2] \cdot \widehat{\beta}(\widehat{\xi}(0)) + b\, a_2\, [j_\nu, g_2] \cdot \widehat{\beta}(\widehat{x}_0),$$

$$\widehat{N}_f := \widehat{S}_{\widehat{T}}^{-1}(N_f) \qquad g_\nu := \widehat{S}_{\widehat{\tau}*}^{-1} h_\nu \circ \widehat{S}_{\widehat{\tau}} \qquad j_\nu := \widehat{S}_{\widehat{\tau}*}^{-1} k_\nu \circ \widehat{S}_{\widehat{\tau}}$$

$$\widehat{\beta} : M \to \mathbb{R} : \ d\widehat{\beta}(\widehat{\xi}(0)) = -\widehat{\lambda}(0)$$

The coercivity of J''_ν allows one to prove that there exists a function α such that

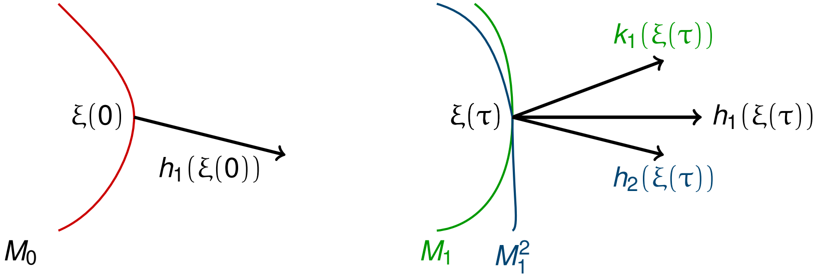
- $\Lambda_0 := \{ d\alpha(x) \colon x \in M\} \bigcap \{H_1 = p_0\}$ and $M_0 := \pi\Lambda_0 = \{x \in M \colon h_1 \cdot \alpha(x) = p_0\}$ are $(n-1)$-dimensional manifolds
- For $\nu = 1, 2$ define

$$M_\nu := \left\{ \pi \mathcal{F}^{\max}_{\tau_\nu(\ell)}(\ell) \colon \ell \in \Lambda_0, \quad \tau_\nu(\ell) = \theta_1(\ell) \right\},$$
$$M^2_\nu := \left\{ \pi \mathcal{F}^{\max}_{\tau^2_\nu(\ell)}(\ell) \colon \ell \in \Lambda_0, \quad \tau_\nu(\ell) = \theta_1(\ell) \right\}.$$

Then these four half-manifolds separate the images of the five smooth pieces of the projected maximised Hamiltonian flow.

Picture for $\nu = 1$

$$M_1 := \left\{ \pi \mathcal{F}^{\max}_{\tau_1(\ell)}(\ell) \colon \ell \in \Lambda_0 \right\}, \qquad M_1^2 := \left\{ \pi \mathcal{F}^{\max}_{\tau_1^2(\ell)}(\ell) \colon \ell \in \Lambda_0 \right\}.$$



Invertibility of the projected maximised flow 1: The degenerate case $d\tau_1|_{T_{\widehat{\lambda}(0)}\Lambda_0} \equiv d\tau_2|_{T_{\widehat{\lambda}(0)}\Lambda_0}$

- **Clarke's inverse function theorem**
  if the convex hull of the Jacobians of each of the five smooth pieces at $(\widehat{\tau}, \widehat{\lambda}(0))$ is made of invertible matrices only
  $\Longrightarrow$ $\pi\mathcal{F}^{\max} \colon (-\delta, \widehat{T} + \delta) \times \Lambda_0 \to M$ is locally one-to-one from a neighborhhod of $(\widehat{\tau}, \widehat{\lambda}(0))$ onto a neighborhood of $\widehat{\xi}(\widehat{\tau})$

Invertibility of the projected maximised flow
2: The nondegenerate case $d\tau_1|_{T_{\widehat{\lambda}(0)}\Lambda_0} \neq d\tau_2|_{T_{\widehat{\lambda}(0)}\Lambda_0}$

- Consider the tangent cones to the five sectors where $\mathcal{F}^{\max}$ has a different smooth expression: the interior of each cone is non-empty

Invertibility of the projected maximised flow 2: The nondegenerate case $d\tau_1|_{T_{\widehat{\lambda}(0)}\Lambda_0} \neq d\tau_2|_{T_{\widehat{\lambda}(0)}\Lambda_0}$

- Consider the tangent cones to the five sectors where $\mathcal{F}^{\max}$ has a different smooth expression: the interior of each cone is non-empty
- Show that the linearisation of each of the five smooth pieces of $\pi\mathcal{F}^{\max}$ is a proper linear function with positive determinant and degree = 1: for each couple of neighbouring tangent cones take $(\delta t_1, \delta\ell_1)$ and $(\delta t_2, \delta\ell_2)$ in each of the two cones and show that their images under the linerisation of the projected maximised flow do not overlap

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The Hamiltonian methods can be applied:

- if $p_0 = 1$ then $\widehat{\xi}$ is a state-locally optimal trajectory;
- if $p_0 = 0$ then $\widehat{\xi}$ is isolated among admissible trajectories.

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$$+ a_1 b\, [g_1, j_\nu] \cdot \widehat{\beta}(\widehat{\xi}(0)) + a_1 a_2\, [g_1, g_2] \cdot \widehat{\beta}(\widehat{\xi}(0)) + b\, a_2\, [j_\nu, g_2] \cdot \widehat{\beta}(\widehat{x}_0),$$

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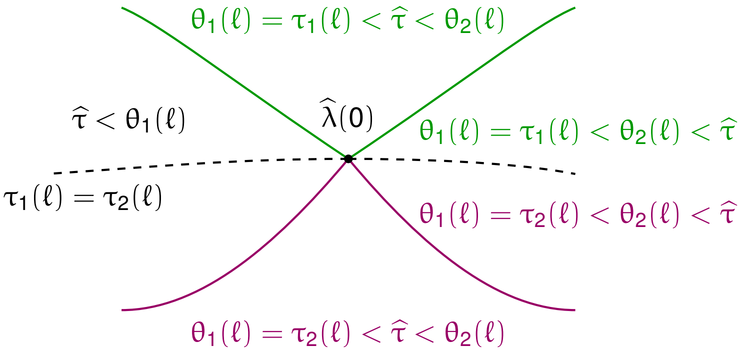
$$\mathrm{id} \times \pi\mathcal{F}^{\max} \colon [0, \widehat{T}] \times \{\mathrm{d}\alpha(x) \colon x \in M\} \to [0, \widehat{T}] \times M$$

is locally one-to-one a each $(t, \widehat{\lambda}(0))$, $t \in [0, \widehat{T}]$, i.e.

$$\pi\mathcal{F}_t^{\max} \colon \{\mathrm{d}\alpha(x) \colon x \in M\} \to M$$

is locally one-to-one at $\widehat{\lambda}(0)$, for each $t \in [0, \widehat{T}]$.

Freeze at $t = \widehat{\tau}$



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