Traffic on networks: modeling and analysis

Régis Monneau

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Régis Monneau

(joint work with C. Imbert)

Paris-Est University

Tours; June 27, 2014

Great thanks

for the whole organizing committee ! !

A mathematical point of view

$$u_t + H(x, Du, D^2u) = 0$$

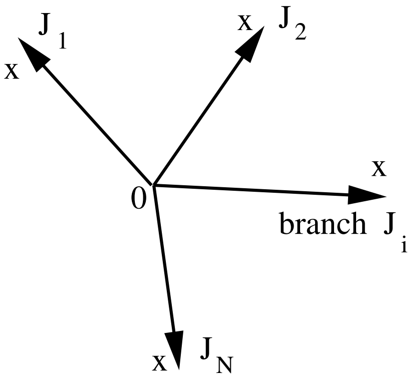
H discontinuous in x

A mathematical point of view

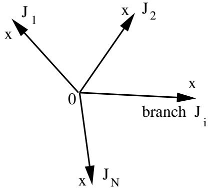
$$u_t + H(x, Du) = 0$$

H discontinuous in x

A junction



Hamilton-Jacobi equations

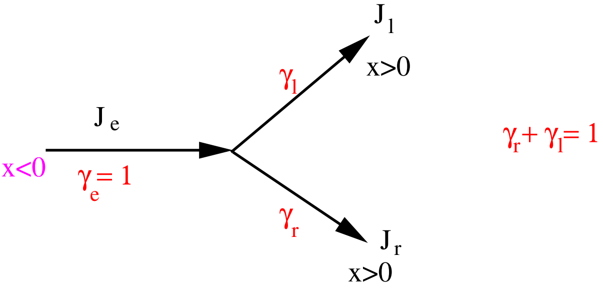


$$\begin{cases} u^i_t + H_i(u^i_x) = 0, & x > 0, \quad i = 1, \ldots, N \\ u^i = u^j =: u, & x = 0, \\ u_t + F(u^1_x, \ldots, u^N_x) = 0, & x = 0 \end{cases}$$

Plan

1. Motivation, Hamilton-Jacobi equations and optimal control
2. Doubling of variables revisited
3. General junction conditions
4. Homogenization of networks

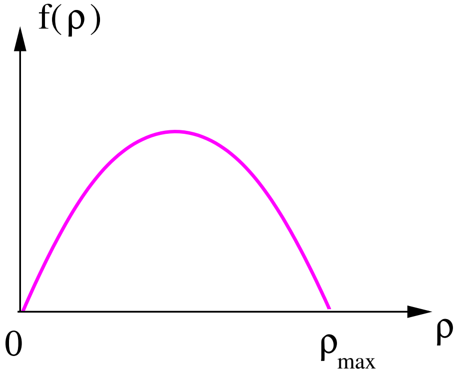
Motivation



LWR model (Lighthill, Whitham 1955 ; Richards 1956) :

$$\rho_t + (f(\rho))_x = 0$$

The fundamental diagram



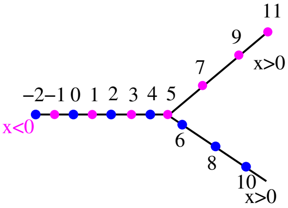
Some references

Existence for $N \geq 1$
[Garavello, Piccoli (2006)]
Uniqueness result for $N = 2$ branches
[Garavello, Natalini, Piccoli, Terracina (2007)]
[Andreianov, Karlsen, Risebro (2011)]

Uniqueness for $N \geq 1$ for Hamilton-Jacobi equations
[Achdou, Camilli, Cutri, Tchou ; (2013)] : particular Hamiltonians.
[Camilli, Schieborn ; (2013)] : eikonal equations.
[Imbert, M., Zidani (2013)] : particular junction condition
[Imbert, M. preprint (2013-2014)] : general junction conditions
Generalizations
[Barles, Briani, Chasseigne ; (2013), (2014)] : hyperplane (higher dimensions).
[Rao, Zidani ; (2013)], [Rao, Siconolfi, Zidani ; (2013)] : multi-domains.

How to get Hamilton-Jacobi equations ?

$-u^i(x,t) =$ continuous index of the car at (x,t) on the branch i



Getting HJ equations

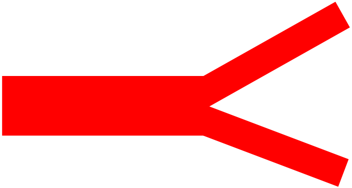
- Take some (space) primitive of the density ρ^i with prefactor $1/\gamma_i$
- change some signs and the orientation of the incoming roads

$$u_t^i + H_i(u_x^i) = 0, \quad x > 0, \quad i = e, l, r$$

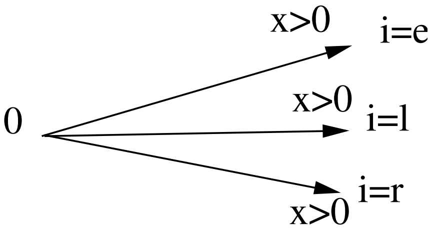
with

$$\begin{cases} \text{convex} \quad H_i \quad \text{if } f \text{ concave,} \\ \\ H_i(p) = -\dfrac{1}{\gamma_i} f(\gamma_i p), \quad i = e \\ \\ H_i(p) = -\dfrac{1}{\gamma_i} f(-\gamma_i p), \quad i = l, r \end{cases}$$

$$f(\rho^i(0^+,t)) = \gamma_i \; f(\rho^e(0^-,t)) \quad \text{for} \quad i=l,r \quad \text{and} \quad \gamma_e = 1$$



What is the condition at $x = 0$?



$$u_t + F(u_x^e, u_x^l, u_x^r) = 0$$

Optimal control

If

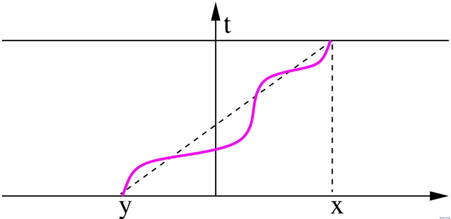
$$\begin{cases} u_t + H(u_x) = 0 & \text{for} \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for} \quad x \in \mathbb{R} \end{cases}$$

with

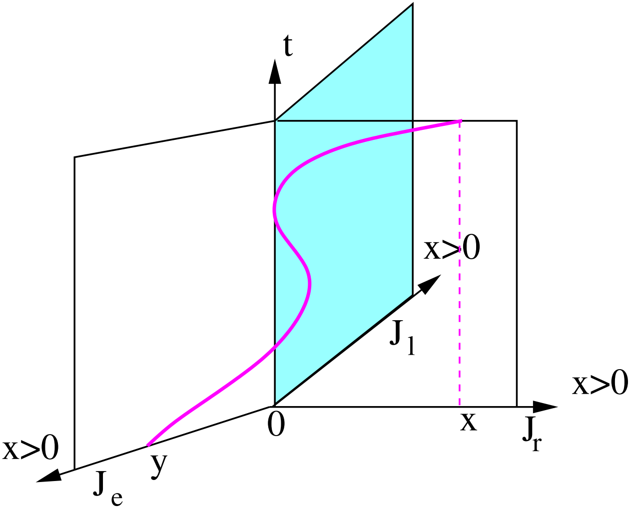
$$H(p) = L^*(p) := \sup_{q \in \mathbb{R}} \{pq - L(q)\}$$

then

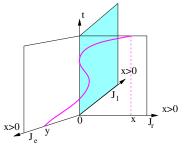
$$u(x,t) = \inf_{\left\{ \begin{array}{l} y \in \mathbb{R}, \\ X(0) = y, \\ X(t) = x \end{array} \right\}} \left\{ u_0(y) + \int_0^t ds \ L(\dot{X}(s)) \right\}$$



Key idea



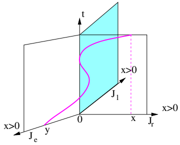
Representation formula



$$J = \bigcup_{i=e,l,r} J_i$$

$$u(x,t) = u_{oc}(x,t) := \inf_{\left\{\begin{array}{l} y \in J, \\ X(0) = y, \\ X(t) = x \end{array}\right\}} \left\{ u_0(y) + \int_0^t ds\; L(X(s), \dot{X}(s)) \right\}$$

Representation formula

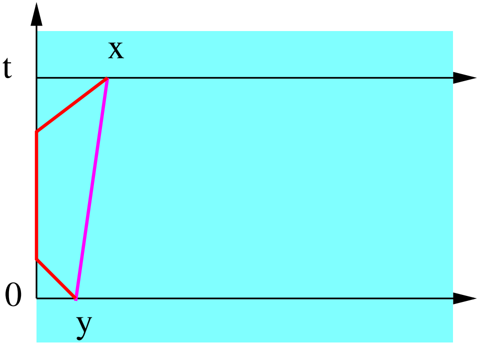


$$J = \bigcup_{i=e,l,r} J_i$$

$$u(x,t) = u_{oc}(x,t) := \inf_{\left\{\begin{array}{l} y \in J, \\ X(0) = y, \\ X(t) = x \end{array}\right\}} \left\{ u_0(y) + \int_0^t ds\ L(X(s), \dot{X}(s)) \right\}$$

with a Lagrangian

$$L(x,p) = \begin{cases} L_i(p) & \text{if} \quad x \in J_i \setminus \{0\}, \quad i = e,l,r \\ L_0 & \text{if} \quad x = 0 \end{cases}$$

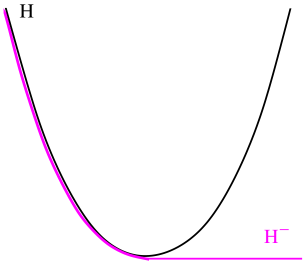

x
t
0
y

Junction condition with flux limiter A

$$u_t(0,t) + F_A = 0$$

with

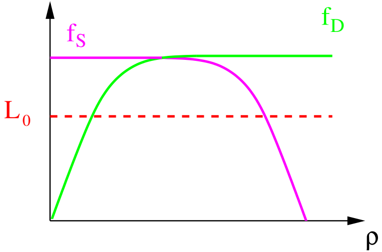
$$\begin{cases} F_A := \max\left\{A, \quad \max_{i=e,l,r} H_i^-(u_x^i(0^+,t))\right\} \quad \text{with} \quad A = -L_0 \\ \\ H_i^-(p) = \sup_{q\le 0}\{pq - L_i(q)\} \end{cases}$$



Traffic interpretation of the junction condition

$$u_t(0,t) = \text{passing flux} = \min\left(L_0, \quad \frac{1}{\gamma_e} f_D(\rho^e), \quad \min_{i=l,r} \frac{1}{\gamma_i} f_S(\rho^i)\right)$$

Lebacque condition :
minimum of Demand (f_D) and Supply (f_S)



Riemann Solver of Garavello, Piccoli,
Boundary condition of [Bardos, LeRoux, Nedelec (1979)]

Doubling of variables revisited

The standard doubling of variables method does not work !

$$\sup \left\{ u(x,t) - v(y,t) - \frac{(x-y)^2}{\varepsilon} - \dots \right\} \text{ with } x \text{ and } y \text{ on different branches !}$$

The standard doubling of variables method does not work !

$$\sup \left\{ u(x,t) - v(y,t) - \frac{(x-y)^2}{\varepsilon} - \dots \right\}$$ with x and y on different branches !

Adaptation with the vertex test function G

$$\sup \left\{ u(x,t) - v(y,t) - \varepsilon \ G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \dots \right\}$$

Doubling of variables

$$0 < \eta \leq H(y, -G_y) - H(x, G_x)$$

Contradiction if

$$H(y, -G_y) \leq H(x, G_x)$$

Construction of the test function G

Set of particular solutions for $p = (p^1, \dots, p^N)$

$$u_{p,\lambda}(x,t) = -\lambda t + p^i x \quad \text{on} \quad J_i$$

Germ

$$\mathcal{G}_A = \{(p, \lambda) \qquad \text{with} \quad F_A(p) = H_i(p^i) = \lambda \geq A\}$$

Construction of the test function G

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Germ

$$\mathcal{G}_A = \{(p,\lambda) \qquad \text{with} \quad F_A(p) = H_i(p^i) = \lambda \geq A\}$$

Let

$$w = w_{p,\lambda}(x,t;y,s) = u_{p,\lambda}(x,t) - u_{p,\lambda}(y,s)$$

which solves

$$H(y, -w_y) = \lambda = H(x, w_x)$$

Construction of the test function G

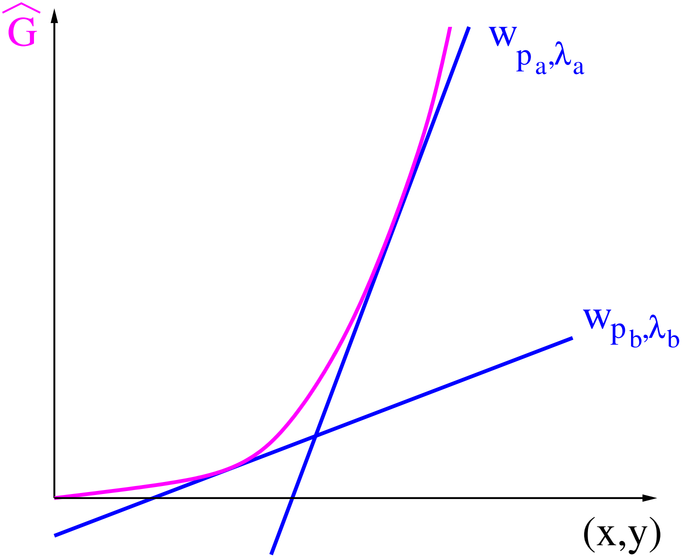
The function

$$\widehat{G}(x,t;y,s) = \sup_{(p,\lambda)\in\mathcal{G}_A} \{w_{p,\lambda}(x,t;y,s)\}$$

satisfies

$$H(y,-\widehat{G}_y) = H(x,\widehat{G}_x) \qquad \text{where } \widehat{G} \text{ is smooth}$$

Properties of $\widehat{G}$



Construction of the test function G

Let

$$G^0(x,y) = A + \widehat{G}(x,1;y,0)$$

s.t.

$$G^0(x,y) \geq 0 = G^0(x,x)$$

Construction of the test function G

Let

$$G^0(x,y) = A + \widehat{G}(x,1;y,0)$$

s.t.

$$G^0(x,y) \geq 0 = G^0(x,x)$$

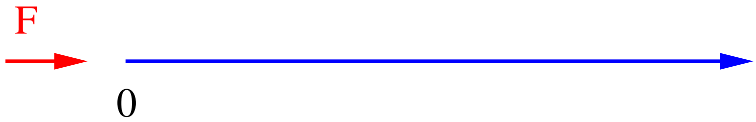
But G^0 is smooth except on the diagonal $\{x = y\}$.

Define $G = G^\delta$ as a smoothing of G^0 on the diagonal such that

$$\begin{cases} G(x,y) \geq 0 \quad \text{and} \quad 0 \leq G(x,x) \leq \delta, \\ H(y,-G_y) \leq H(x,G_x) + \delta \end{cases}$$

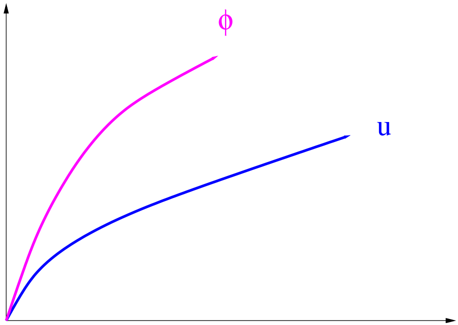
General junction conditions for $N = 1$

Monotonicity of F



$$\begin{cases} u_t + H(u_x) = 0, & x > 0 \\ u_t + F(u_x) = 0, & x = 0 \end{cases}$$

Monotonicity of F

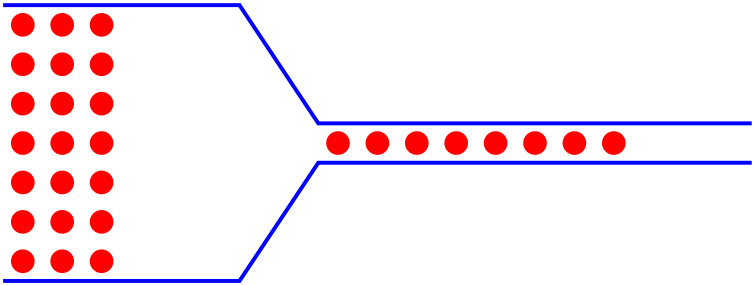


$$\begin{cases} u_t + F(u_x) = 0, & x = 0 \\ \phi_t + F(\phi_x) \le 0, & x = 0 \end{cases}$$

$F(\rho)$ non increasing in ρ

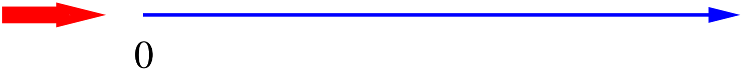
Meaning of the junction condition

Junction condition not satisfied if the imposed flux is too large



Relaxation

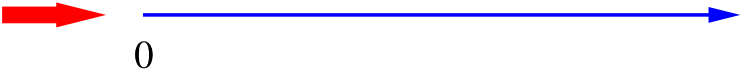
Relaxed junction condition



Relaxed junction condition for subsolutions

$$\left\{\begin{array}{llll} u_t + F(u_x) & \leq 0, & x = 0 & \text{(Boundary condition)} \\ & & & \\ \text{OR} & & & \\ & & & \\ u_t + H(u_x) & \leq 0, & x = 0 & \text{(PDE)} \end{array}\right.$$

Relaxed junction condition



Relaxed junction condition for supersolutions

$$\begin{cases} u_t + F(u_x) \quad \geq 0, \quad x = 0 & \text{(Boundary condition)} \\ \text{OR} & \\ u_t + H(u_x) \quad \geq 0, \quad x = 0 & \text{(PDE)} \end{cases}$$

Theorem (Existence, uniqueness, reduction to F_A-solutions)

If

$$\begin{cases} H \text{ convex and coercive} \quad (i.e. \quad H(p) \to +\infty \text{ as } |p| \to +\infty) \\ F \text{ continuous and non increasing} \end{cases}$$

Then there exists a unique (viscosity) relaxed solution u of

$$\begin{cases} u_t + H(u_x) = 0, & x > 0, \\ u_t + F(u_x) = 0, & x = 0 \end{cases}$$

with Lipschitz initial data

$$u_{|t=0} = u_0$$

Moreover there exists a real A such that u is also solution with F replaced by

$$F_A(u_x) = \max(\ A,\ \ H^-(u_x)),$$

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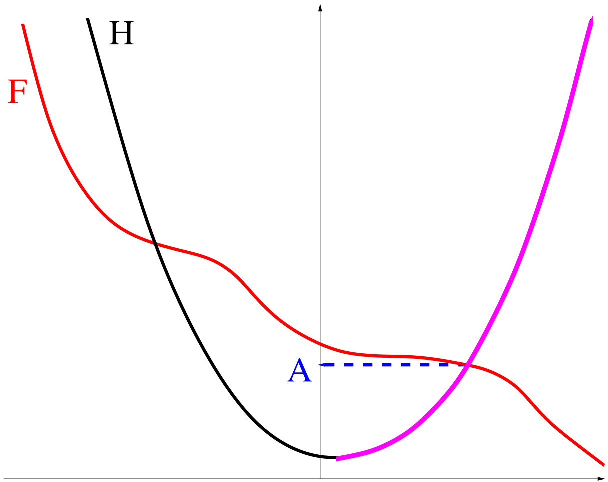
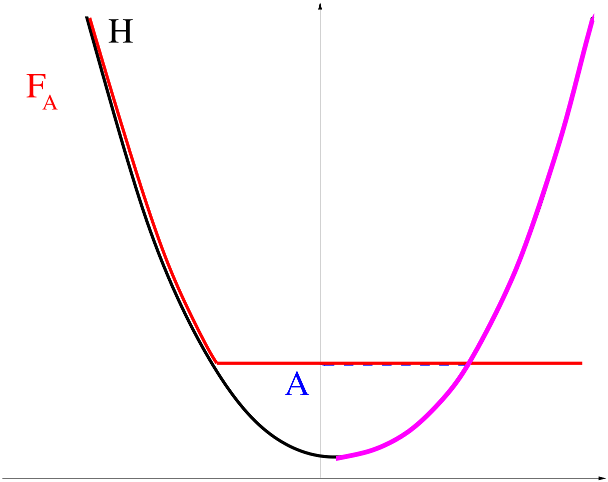
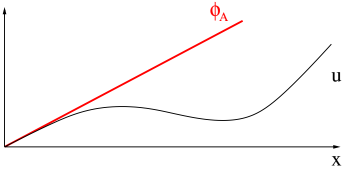

H
F
A


H
F_A
A

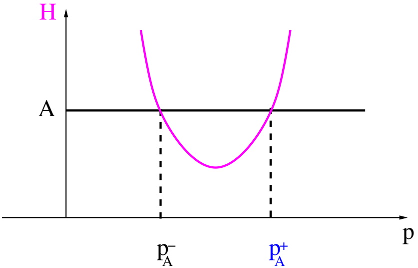
Ramp metering on the A19 in Middlesbrough, UK.

Reduction to a single test function



Check with test functions

$$\varphi(x,t) = \psi(t) + \phi_A(x) \qquad \text{with} \qquad \phi_A(x) = p_A^+ x$$



For H convex (or quasi-convex) and coercive,

$$\begin{cases} u_t + H(u_x) \geq 0, & x \geq 0, \\ u_t + H(u_x) \leq 0, & x > 0, \end{cases} \qquad \text{(state constraint)}$$

if and only if

$$\begin{cases} u_t + H(u_x) = 0, & x > 0, \\ u_t + H^-(u_x) = 0, & x = 0, \end{cases} \qquad (F_A \text{ condition with } A = -\infty)$$

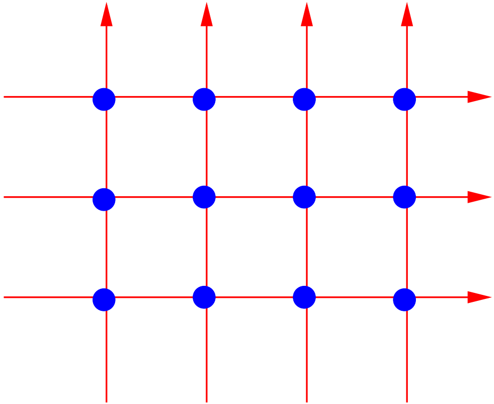
Applications for $N \geq 1$ branches

- relation with Ishii solutions
- characterization of subsolutions on the junction point only
- restriction of solutions on a subdomain
- homogenization
- ...

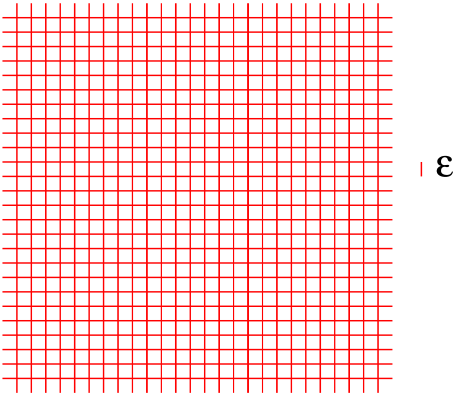
Homogenization of networks

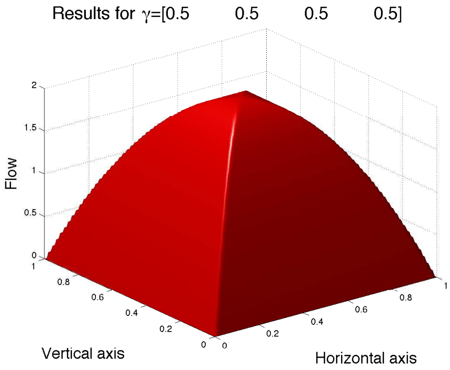
(Numerical simulations done by G. Costeseque)

A periodic network



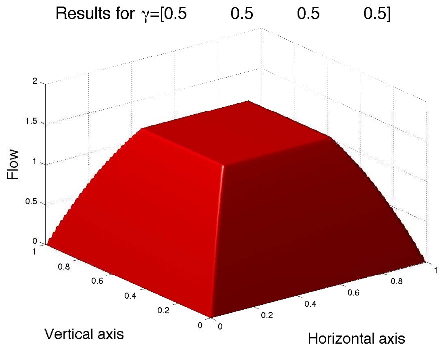
Rescaling of the network





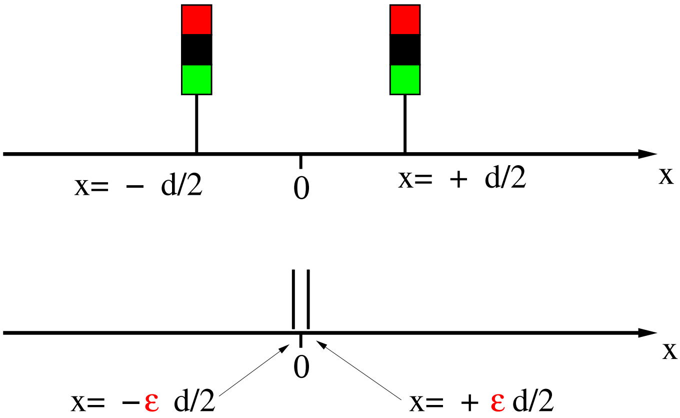
$$\gamma_h = \gamma_v = 1/2, \quad \text{no flux limiter}$$

Graph of $-\bar{H}$

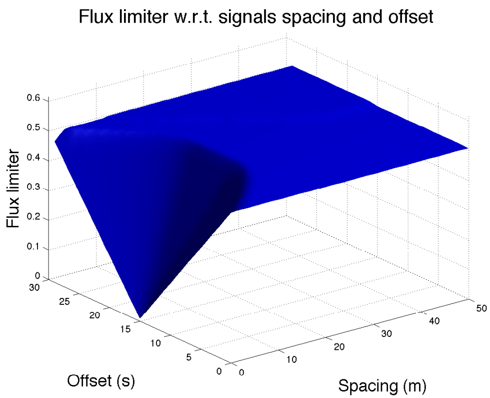


$$\gamma_h = \gamma_v = 1/2, \quad \text{with flux limiter} \quad A$$

Homogenization of traffic lights on a single road



$-\bar{A}(d,\theta)$



Flux limiter w.r.t. signals spacing and offset
Flux limiter
0.6
0.5
0.4
0.3
0.2
0.1
0
Offset (s)
30
25
20
15
10
5
0
Spacing (m)
0
10
20
30
40
50

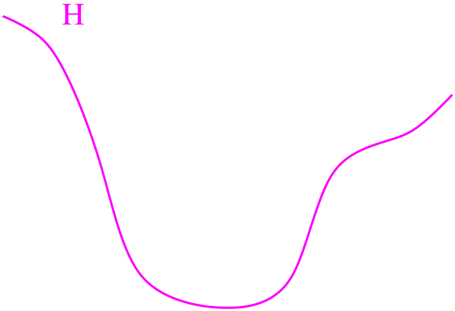
Conclusion :

what we know

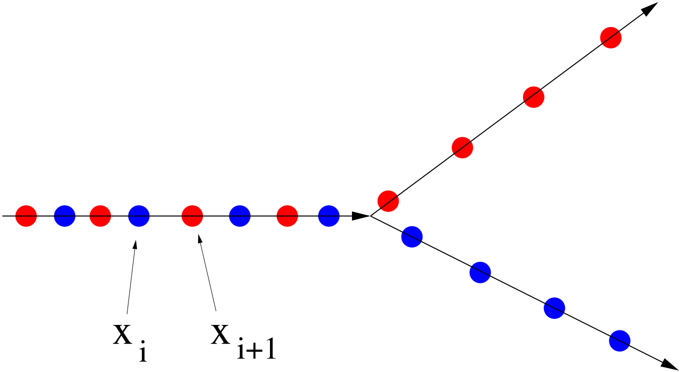
What we can do in general

Infinite networks with Hamiltonians $H(x,t,u_x)$ and $F(t,\{u_x^i\}_i)$

Hamiltonians $H(x,t,u_x)$ are convex in u_x, or quasi-convex



Open problems



$$\frac{dx_i}{dt} = V_{s(i)}(\{x_j\}_{j \geq i}) \quad \text{with } s(i) = \text{blue or red}$$

weakly related to :

Achdou, Tchou, (preprint : 2014),

Galise, Imbert, M. (preprint 2014)

Open problems

- traffic : time evolution/ optimization of the proportions γ_i at each node

Open problems

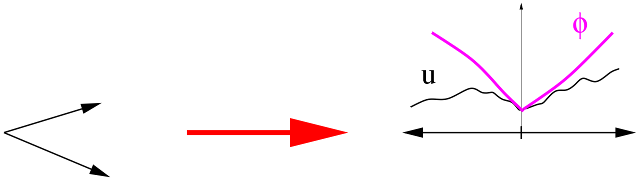
- traffic : time evolution/ optimization of the proportions γ_i at each node

- more general non quasi-convex Hamiltonians in 1D
- general junction conditions in higher dimensions : submanifolds, multi-domains.
- mixed first order/second order PDEs

Thank you

The case with $N \geq 1$ branches

Viscosity solution at the junction point

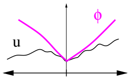


Junction

$$J = \bigcup_{i=1,\ldots,N} J_i$$

test function

$$\phi \in C^1_* \iff \begin{cases} \phi \in C(J \times [0,T)), \\ \phi \in C^1(J_i \times (0,T)), \quad i = 1, \ldots, N \end{cases}$$



Definition (Relaxed subsolution at the junction point)

The USC function u is a relaxed subsolution at the junction point, if

$$u \leq \phi \quad \textit{with equality at } P_0 = (x_0, t_0) \textit{ with } x_0 = 0$$

then at P_0 :

$$\begin{cases} \phi_t + F(\phi_x^1, ..., \phi_x^N) \leq 0, \\ \textit{OR} \\ \phi_t + H_i(\phi_x^i) \leq 0 \quad \textit{for some index } i \end{cases}$$

with

$$\phi^i = \phi_{|J_i \times (0,T)}$$

We rewrite

$$\begin{cases} u_t^i + H_i(u_x^i) = 0, & \text{for} \quad (x,t) \in (0,+\infty) \times (0,T), \quad i = 1,...,N \\ u_t + F(u_x^1,...,u_x^N) = 0, & \text{for} \quad (x,t) \in \{0\} \times (0,T) \end{cases}$$

as

$$u_t + H(x,u_x) = 0 \quad \text{on} \quad J \times (0,T) \quad \text{with} \quad J = \bigcup_{i=1,...,N} J_i \tag{1}$$

with

$$H(x,p) = \begin{cases} H_i(p) & \text{convex coercive} \quad \text{if} \quad x \in J_i \setminus \{0\} \\ F(p^1,...,p^N) & \text{non increasing} \quad \text{if} \quad x = 0 \end{cases}$$

The PDE

$$u_t + H(x, u_x) = 0 \quad \text{on} \quad J \times (0,T) \tag{1}$$

Initial condition

$$u(x,0) = u_0(x) \quad \text{for all} \quad x \in J, \quad \text{with} \quad u_0 \in \mathrm{Lip}(J) \tag{2}$$

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Theorem (Existence, uniqueness, reduction to F_A-solutions)

There exists a unique (viscosity) relaxed solution u of (1)-(2).
Moreover there exists a real A such that u is also solution with F replaced by

$$F_A(u_x^1, \ldots, u_x^N) = \max\Big(A, \max_{i=1,\ldots,N} H_i^-(u_x^i)\Big)$$

where

$$A = F((H_1^+)^{-1}(A), \ldots, (H_N^+)^{-1}(A)) \qquad \text{or} \qquad A = A_0 := \max_{i=1,\ldots,N} \left(\min_{\mathbb{R}} H_i \right)$$