# Propagation of singularities for semiconcave solutions of Hamilton-Jacobi equations

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# Propagation of Singularities for Semiconcave Solutions of Hamilton-Jacobi Equations

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# Outline of the talk

Local propagation of singularities

Minimizing generalized characteristics

Global propagation of singularities

## Semiconcave functions

Let $Q \subset \boldsymbol{R}^N$ be open and $u : Q \to \boldsymbol{R}$ be a semiconcave function with semiconcavity constant $C$, i.e.

$$\lambda u(X) + (1-\lambda)u(Y) - u(\lambda X + (1-\lambda)Y) \leq C\lambda(1-\lambda)\frac{|X-Y|^2}{2}$$

for every $\lambda \in [0,1]$ and $X, Y \in Q$ such that $[X, Y] \subset Q$.

Denote by $\Sigma(u)$ the set of points where $u$ is not differentiable.

We are interested in the structure of the set $\Sigma(u)$.

# Generalized gradients

The set of limiting gradients and the superdifferential of $u$ at $X \in Q$ are defined respectively by

$$D^*u(X) = \left\{ P \in \boldsymbol{R}^N \ : \ Q \setminus \Sigma(u) \ni X_i \to X, \ Du(X_i) \to P \right\}$$

and

$$D^+u(X) = \left\{ P \in \boldsymbol{R}^N \ : \ \limsup_{Y \to X} \frac{u(Y) - u(X) - \langle P, Y - X \rangle}{|Y - X|} \leq 0 \right\}.$$

If $u$ is semiconcave with constant $C$, then

- $$D^+u(X) = \mathrm{co}\ D^*u(X)\,;$$
- $$\langle P - Q, X - Y \rangle \leq C|X - Y|^2$$
  for every $P \in D^+u(X)$, $Q \in D^+u(Y)$ and $X, Y \in Q$ such that $[X, Y] \subset Q$.

# Local propagation for semiconcave functions

**Theorem (Albano - Cannarsa (1999))**

*Let $u$ be semiconcave and $X_0 \in \Sigma(u)$. If*

$$\partial D^+ u(X_0) \setminus D^* u(X_0) \neq \emptyset \,,$$

*then there exist $T > 0$ and a nonconstant Lipschitz continuous arc $\xi : [0, T] \to Q$ such that $\xi(0) = X_0$ and $\xi(t) \in \Sigma(u)$ for all $t \in [0, T]$.*

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When $u$ is the solution of a Hamilton-Jacobi equation, the condition above turns out to be necessary and sufficient for the propagation of the singularity at $X_0$.

# Local propagation for solutions of HJ equations

$$F(Du(X)) = 0 \qquad \text{a.e. } X \in Q$$

Let $F \in C^1(\mathbb{R}^N)$ satisfy

- $F$ is convex;
- the sublevel sets of $u$ are strictly convex.

**Theorem (Albano - Cannarsa (2002), Cannarsa - Yu (2009))**

*Let $u$ be a semiconcave solution of*

$$F(Du(X)) = 0 \qquad \textit{a.e. } X \in Q$$

*and $X_0 \in \Sigma(u)$. If*

$$0 \notin DF(D^+u(X_0))\,,$$

*then there exist $T > 0$ and a nonconstant Lipschitz continuous arc $\xi : [0, T] \to Q$ such that $\xi(0) = X_0$ and $\xi(t) \in \Sigma(u)$ for all $t \in [0, T]$.*

# Evolutionary HJ equation

Let $N = 1 + n$.
Consider

$$F(P) = \tau + H(p)$$

with $P = (\tau, p) \in \boldsymbol{R} \times \boldsymbol{R}^n$ and $Q = (0, +\infty) \times \Omega$, $\Omega \subset \boldsymbol{R}^n$.
This case correspond to the HJ equation

$$\begin{cases} u_t(t,x) + H(\nabla u(t,x)) = 0 & \text{a.e. } (t,x) \in Q \\ u(t,x) = \varphi(t,x) & \text{for } (t,x) \in \partial Q\,. \end{cases}$$

Suppose that $\varphi : \bar{Q} \to \boldsymbol{R}$ is continuous and its restriction to $\{0\} \times \Omega$ is Lipschitz continuous.

## Hopf representation formula

$$u(t,x) = \min_{\substack{(s,y) \in \partial Q \\ s < t}} \left[ (t-s)\, H^* \left( \frac{x-y}{t-s} \right) + \varphi(s,y) \right].$$

# Evolutionary HJ equation

Question: Is it possible to propagate singularities globally in time?
i.e., given $X_0 = (t_0, x_0) \in \Sigma(u)$, does a Lipschitz continuous arc $\gamma : [0, +\infty) \to \boldsymbol{R}^n$ exist such that $\gamma(0) = x_0$ and

$$(t_0 + s, \gamma(s)) \in \Sigma(u) \qquad \forall\, s \in [0, +\infty)\,?$$

# Evolutionary HJ equation

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$$(t_0 + s, \gamma(s)) \in \Sigma(u) \qquad \forall\, s \in [0, +\infty)\,?$$

Yes, in the case $n = 1$ (Dafermos, 1977).

# Outline of the talk

Local propagation of singularities

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Global propagation of singularities

# Characteristics

Consider the Hamilton-Jacobi equation

$$(HJ) \qquad F(Du(X)) = 0 \qquad X \in Q\,,$$

with $F$ of class $C^1$ and convex.

If $u \in C^1(Q)$ is a classical solution of (HJ), a curve $\xi : [0, T] \to Q$ is a characteristic if it satisfies

$$\xi'(t) = DF(Du(\xi(t))) \qquad \text{a.e. } \; t \in [0, T]\,.$$

# Generalized characteristics

Let $u$ be a semiconcave solution of (HJ).

A Lipschitz continuous curve $\xi : [0, T] \to Q$ is a generalized characteristic if it satisfies

$$\xi'(t) \in \text{co } DF(D^+u(\xi(t))) \qquad \text{a.e. } t \in [0, T] \,.$$

The proof of (Dafermos, 1977) crucially depends on the dimension 1.

# Minimizing generalized characteristics

Notice that $F(P) = 0$ for every limiting gradient $P$.
Consequently, $X \in \Sigma(u)$ if and only if

$$\min_{P \in D^+u(X)} F(P) < 0\,.$$

Sufficient conditions are provided in the literature for the existence of generalized characteristics that are "energy minimizing" (Cannarsa - Yu, 2009), (Stromberg, 2013).

# Minimizing generalized characteristics

**Theorem (Cannarsa - Yu (2009))**

*If for any $X_0 \in Q$ there exists a unique generalized characteristic starting from $X_0$, then any generalized characteristic $\xi : [0, T_0) \to Q$ admits right derivative $\dot{\xi}^+(s)$ for all $s \in [0, T_0)$, this is right-continuous and is given by*

$$\dot{\xi}^+(s) = DF(P(s)),$$

*where $P(s) \in D^+u(\xi(s))$ is such that*

$$F(P(s)) \leq F(P) \qquad \forall\, P \in D^+u(\xi(s)).$$

# Evolutionary HJ equation

In the case of the evolutionary HJ equation

$$F(P) = \tau + \frac{1}{2}Ap \cdot p$$

with $P = (\tau, p) \in \boldsymbol{R} \times \boldsymbol{R}^n$, $Q = (0, +\infty) \times \Omega$, $\Omega \subset \boldsymbol{R}^n$ and $A$ positive definite, the uniqueness of generalized characteristics, given the initial data, is a consequence of Gronwall's Lemma.

Following (Cannarsa - Yu, 2009), it is possible to propagate a singularity $X_0$ locally along the minimizing generalized characterstic

$$\dot{\xi}^+(s) = DF(\tau(s), p(s)) = \begin{pmatrix} 1 \\ Ap(s) \end{pmatrix}, \qquad \xi(0) = X_0\,.$$

# Outline of the talk

# Global propagation of singularities

Question: Is it possible to propagate singularities globally in time along generalized characteristics?

So far, besides (Dafermos, 1977), an affirmative answer has been given only in a few particular cases.

# The eikonal equation

$$\begin{cases} |Du(X)|^2 = 1 & \text{a.e. } X \in Q \\ u(X) = 0 & \text{on } \partial Q \end{cases}$$

The unique nonnegative viscosity solution is $u = d_{\mathbf{R}^N \setminus Q}$.

**Theorem (Albano - Cannarsa - Khai Nguyen - Sinestrari (2013))**

*Let $X_0 \in Q$. There exists a unique Lipschitz continuous solution of*

$$\xi'(t) \in D^+u(\xi(t)) \qquad a.e.\ t \in [0,+\infty), \qquad \xi(0) = X_0\,.$$

*Moreover, if $X_0 \in \Sigma(u)$ then $\xi(t) \in \Sigma(u)$ for all $t \in [0,+\infty)$.*

# The eikonal equation

- Application: $Q$ is homotopically equivalent to $\Sigma(u)$.
- The result holds true on riemannian manifolds.

# The eikonal equation

- Application: $Q$ is homotopically equivalent to $\Sigma(u)$.
- The result holds true on riemannian manifolds.

The proof strongly relies on the semiconcavity of $u^2$ with constant $2$. This yields

$$\langle u(X)P - u(Y)Q, X - Y\rangle \leq |X - Y|^2$$

for every $P \in D^+u(X)$, $Q \in D^+u(Y)$ and $X, Y \in \boldsymbol{R}^n$.

$$u_t(t,x) + \frac{|\nabla u(t,x)|^2}{2} = 0$$

Fixed $t > 0$, the function $u(t,\cdot)$ is semiconcave with constant $\frac{1}{t}$. This implies

$$\langle p - q, x - y \rangle \leq \frac{|x-y|^2}{t}$$

for every $p \in \nabla^+ u(t,x)$, $q \in \nabla^+ u(t,y)$ and $x, y \in \boldsymbol{R}^n$.
We need an estimate on the monotonicity of $D^+ u$ jointly in time and space.

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Using the Hopf formula, we find

$$\left\langle \begin{pmatrix} \tau - \sigma \\ p - q \end{pmatrix}, \begin{pmatrix} t - s \\ x - y \end{pmatrix} \right\rangle \leq \frac{|x-y|^2}{t}$$

$$-\frac{(t-s)^2}{t}\sigma - \frac{t-s}{t}[\langle p, x-y\rangle + u(t,x) - u(s,y)]$$

$\forall\, (\tau, p) \in D^+u(t,x)$, $(\sigma, q) \in D^+u(s,y)$ and $t, s > 0$, $x, y \in \boldsymbol{R}^n$.

$$u_t + H(\nabla u) = 0 \ \text{ a.e. } (t,x) \in (0,+\infty)\times\Omega =: Q, \qquad H(p) = \frac{1}{2}Ap\cdot p$$

For any $(t_0, x_0) \in Q$ there exist $T_0 > 0$ and a Lipschitz continuous arc $\xi : [0, T_0) \to Q$ such that $\xi(0) = (t_0, x_0)$, the right derivative $\dot{\xi}^+(s)$ does exist for all $s \in [0, T_0)$, it is right-continuous and satisfies

$$\dot{\xi}^+(s) = DF(\tau(s), p(s)) = \begin{pmatrix} 1 \\ \nabla H(p(s)) \end{pmatrix},$$

where $(\tau(s), p(s)) \in D^+u(\xi(s))$ is such that

$$F(\tau(s), p(s)) \le F(\tau, p) \qquad \forall\, (\tau, p) \in D^+u(\xi(s)).$$

Using the monotonicty above, it is possible to obtain (formally)

$$\frac{d}{ds}F(\tau(s), p(s)) \le -\frac{2}{s}F(\tau(s), p(s))\,.$$

# Local propagation of singularities

$$u_t + H(\nabla u) = 0 \text{ a.e. } (t,x) \in (0,+\infty)\times\Omega =: Q, \quad H(p) = \frac{1}{2}Ap\cdot p$$

## Theorem

*Let* $(t_0, x_0) \in Q$ *and* $\bar{t} < t_0$ *be such that*

$$u(t,x) = \min_{y\in\Omega}\left[(t-\bar{t})H^*\left(\frac{x-y}{t-\bar{t}}\right) + u(\bar{t},y)\right]$$

*There exist* $T_1 > 0$ *and a Lipschitz continuous arc* $\gamma : [0, T_1) \to \Omega$ *starting from* $x_0$ *and such that*

$$\min_{(\tau,p)\in D^+u(t_0+s,\gamma(s))} F(\tau,p) \le \left(\frac{t_0-\bar{t}}{t_0+s-\bar{t}}\right)^2 \min_{(\tau_0,p_0)\in D^+u(t_0,x_0)} F(\tau_0,p_0)$$

*for every* $s \in [0, T_1)$.

# Example

The estimate above is somehow mild:

Let $n = 1$, $A = Id$, $\Omega = \mathbb{R}^N$ and $\varphi(0,x) = \frac{(|x|-1)^2}{2\varepsilon}$.

The Hopf formula yields

$$u(t,x) = \frac{1}{2}\frac{(|x|-1)^2}{t+\varepsilon}\,.$$

We obtain

$$\arg\min_{(\tau,p)\in D^+u(t_0+s,\gamma(s))} F(\tau,p) = \left\{\left(-\frac{1}{2(t_0+s+\varepsilon)^2}, 0\right)\right\}$$

$$\frac{\min_{(\tau,p)\in D^+u(t_0+s,\gamma(s))} F(\tau,p)}{\min_{(\tau_0,p_0)\in D^+u(t_0,0)} F(\tau_0,p_0)} = \left(\frac{t_0+\varepsilon}{t_0+s+\varepsilon}\right)^2$$

# Global propagation of singularities

A maximality argument yields the global propagation:

**Theorem**

*Let $(t_0, x_0)$ be a singular point of $u$. Then there exist $T \in (0, +\infty]$ and a Lipschitz continuous arc $\gamma : [0, T) \to \mathbf{R}^n$ starting from $x_0$, satisfying*

$$(t_0 + s, \gamma(s)) \in \Sigma(u) \qquad \forall\, s \in [0, T)$$

*and such that $\lim_{s \to T} \gamma(s) \in \partial\Omega$ whenever $T < +\infty$.*

# Future directions

- Generalize to semiconcave functions on riemannian manifolds: up to now, we can prove a global propagation on manifolds with nonnegative curvature;
- Generalize to the case of more complex structures of the function $H$.

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- Generalize to the case of more complex structures of the function $H$.

*...the end.*

*Thank you!*