The Minimum Principle for Delayed Optimal Control Problems with State Constraints

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Helmut Maurer[1], Laurenz Göllmann[2], Richard Vinter [3]

[1]Institute of Computational and Applied Mathematics,
University of Münster, Germany

[2]Department of Mechanical Engineering,
Münster University of Applied Sciences, Steinfurt, Germany

[3]Imperial College London, Dept. of Electrical and Electronic Engineering

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Challenges for Optimal Control Problems with Delays

Theory and Numerics for non-delayed optimal control problems with control and state constraints are in a mature state-of-art:

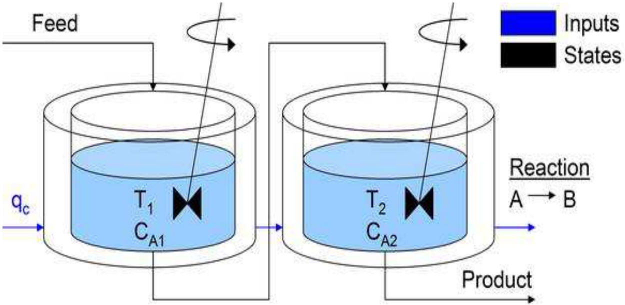
1. Necessary and sufficient conditions,
2. Stability and sensitivity analysis,
3. Numerical methods: Boundary value methods, Discretization and NLP, Semismooth Newton methods,
4. Real-time control techniques for perturbed extremals.

CHALLENGE: Establish similar theoretical and numerical methods for delayed (retarded) optimal control problems.

Overview

1. Example: Two-stage Continuous Stirred Tank Reactor (CSTR)
2. Formulation of Optimal Control Problems with Delays in State and Control Variables
3. Minimum Principle
4. Numerical Treatment: Discretize and Optimize
5. Example: Optimal Control of the Innate Immune Response

Two-Stage Continuous Stirred Tank Reactor (CSTR)



Time delays are caused by transport between the two tanks.

Delayed Optimal Control Problem with State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$, Delays $d_x,\ d_u \geq 0$.

Dynamics and Boundary Conditions

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-d_x), u(t), u(t-d_u)), \ \text{ a.e. } t \in [0, t_f]\,, \\ x(t) &= x_0(t), \quad t \in [-d_x, 0]\,, \\ u(t) &= u_0(t), \quad t \in [-d_u, 0), \\ \psi(x(t_f)) &= 0 \in \mathbb{R}^q, \quad (0 \leq q \leq n). \end{aligned}$$

Control and State Constraints

$$u(t) \in U \subset \mathbb{R}^m, \ \ S(x(t)) \leq 0, \quad t \in [0, t_f] \quad (S : \mathbb{R}^n \to \mathbb{R}^k)\,.$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), x(t-d_x), u(t), u(t-d_u))\, dt$$

Literature on optimal control with time-delays

Time delays in state variables and pure control constraints:

Kharatishvili (1961), Oguztöreli (1966), Banks (1968), Halanay (1968), Soliman, Ray (1970, chemical engineering), Warga (1972, abstract theory, optimization in Banach spaces), Guinn (1976, transform delayed problems to standard problems), Colonius, Hinrichsen (1978), Clarke, Wolenski (1991), Dadebo, Luus (1992), Mordukhovich, Wang (2003–).

Time delays in state variables and pure state constraints:

Angell, Kirsch (1990).

State and control delays and mixed control–state constraints:

single delay	:	Göllmann, Kern, Maurer (OCAM 2009)
multiple delays	:	Göllmann, Maurer (JIMO 2014)

Optimal control problems with state constraints

Use the transformation method of Guinn (1976) and transform an optimal control problem with delays and state constraints to a standard non–delayed optimal control problem with state constraints. Then apply the necessary conditions for non-delayed problems:

- Jacobson, Lele, Speyer (1975): KKT conditions in Banach spaces.
- Maurer (1979) : Regularity of multipliers for state constraints.
- Hartl, Sethi, Thomsen (SIAM Review 1995): Survey on Maximum Principles.
- Vinter (2000): (Nonsmooth) Optimal Control

Use methodology for mixed control-state constraints in

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The smooth case: all functions are diffeentiable

Hamiltonian (Pontryagin) Function

$$H(x, y, \lambda, u, v) := \lambda_0 f_0(t, x, y, u, v) + \lambda f(t, x, y, u, v)$$

- y variable with $y(t) = x(t - d_x)$
- v variable with $v(t) = u(t - d_u)$
- $\lambda \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{R}$ adjoint (costate) variable

Let $(u, x) \in \mathcal{L}^\infty([0, t_f], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ be a locally optimal pair of functions. Then there exist

- an adjoint function $\lambda \in \mathcal{BV}([0, t_f], \mathbb{R}^n)$ and $\lambda_0 \geq 0$,
- a multiplier $\rho \in \mathbb{R}^q$ (associated with terminal conditions),
- a multiplier function (measure) $\mu \in \mathcal{BV}([0, t_f], \mathbb{R}^k)$,

such that the following conditions are satisfied for a.e. $t \in [0, t_f]$:

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such that the following conditions are satisfied for a.e. $t \in [0, t_f]$:

Minimum Principle

(i) Advanced adjoint equation and transversality condition:

$$\lambda(t) = \int_t^{t_f} \left(H_x(s) + \chi_{[0,t_f-d_x]}(s)\, H_y(s+d_x) \right) \mathrm{d}s + \int_t^{t_f} S_x(x(s))\, \mathrm{d}\mu(s)$$
$$+ (\lambda_0 g + \rho\psi)_x(x(t_f)), \qquad (\text{ if } \; S(x(t_f)) < 0\,)$$

where $H_x(t)$ and $H_y(t+d_x)$ denote evaluations along the optimal trajectory and $\chi_{[0,t_f-d_x]}$ is the characteristic function.

(ii) Minimum Condition:

$$H(t) + \chi_{[0,t_f-d_u]}(t)\, H(t+d_u)$$
$$= min_{\,w\in U} \left[\, H(x(t), y(t), \lambda(t), w, v(t)) \right.$$
$$\left. + \chi_{[0,t_f-d_u]}(t)\, H(t+d_u) H(x(t+d_u), y(t), \lambda(t+d_u), u(t+d_u), w) \,\right]$$

(iii) Multiplier condition and complementarity condition:

$$\mathrm{d}\mu(t) \geq 0, \qquad \int_0^{t_f} S(x(t))\, \mathrm{d}\mu(t) = 0$$

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(iii) Multiplier condition and complementarity condition:

$$d\mu(t) \geq 0, \qquad \int_0^{t_f} S(x(t))\, d\mu(t) = 0$$

Regularity conditions for $d\mu(t) = \eta(t)dt$ if $d_u = 0$

Boundary arc	:	$S(x(t)) = 0$	for	$t_1 \le t \le t_2$.
Assumption	:	$u(t) \in int(U)$	for	$t_1 < t < t_2$.

Under certain regularity conditions we have $d\mu(t) = \eta(t)\,dt$ with a multiplier $\eta(t)$ for all $t_1 < t < t_2$.

Adjoint equation and jump conditions

$$\dot{\lambda}(t) = -H_x(t) - \chi_{[0,t_f - d_x]}(t)\,H_y(t + d_x) - \eta(t) S_x(x(t))$$

$$\lambda(t_k+) = \lambda(t_k-) - \nu_k S_x(x(t_k)), \quad \nu_k \ge 0$$

at each contact or junction time t_k , $\nu_k = \mu(t_k+) - \mu(t_k-)$

Minimum condition on the boundary

$$H_u(t) = 0\,.$$

This condition allows to compute the multiplier $\eta = \eta(x, \lambda)$.

Numerical Treatment: Discretize and Optimize

1. Choose a suitable stepsize $h = t_f/N$, $N \in \boldsymbol{N}$, adapted to the delays d_x, d_y (commensurability). Use the implicit or explicit Euler integration scheme for the dynamic equations and define the associated NLP having

$$u(t_i) \approx u_i \in \mathbb{R}^m, \quad x(t_i) \approx x_i \in \mathbb{R}^n, \quad i = 0, \ldots, N$$

as decision variables

2. Apply NLP-Solvers (eg. IPOPT, LOQO embedded in AMPL source code) to solve the NLP

3. The associated Lagrange multipliers $(\hat{\lambda}_i)$ for the discretized dynamic equations and state constraint $(\hat{\mu}_i)$ give a consistent approximation of the costate and the multiplier function

$$\hat{\lambda}(t_i) \approx \hat{\lambda}_i, \quad \hat{\eta}(t_i) \approx \frac{1}{h}\hat{\eta}_i \quad (i = 0, .., N), \quad \hat{\nu} \approx \hat{\nu}_N \,.$$

Göllmann, Kern, Maurer (2009) for two single state/control delays, Göllmann, Maurer (2014) for multiple delays.

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Model of the immune response

Dynamic model of the immune response:

Asachenko A, Marchuk G, Mohler R, Zuev S,
Disease Dynamics, Birkhäuser, Boston, 1994.

Optimal control:

Stengel RF, Ghigliazza R, Kulkarni N, Laplace O,
Optimal control of innate immune response,
Optimal Control Applications and Methods **23**, 91–104 (2002),

Simultaneous optimization of all control variables and delays:

L. Göllmann, H. Maurer: Optimal control problems with multiple time-delays, JIMO 2014.

Here we consider a more realistic modified optimal control problem.

Innate Immune Response: state and control variables

State variables:

$x_1(t)$: concentration of **pathogen**
(=concentration of associated **antigen**)

$x_2(t)$: concentration of **plasma cells**,
which are carriers and producers of antibodies

$x_3(t)$: concentration of **antibodies**, which kill the pathogen
(=concentration of **immunoglobulins**)

$x_4(t)$: relative characteristic of a **damaged organ**
(**0 = healthy, 1 = dead**)

Control variables:

$u_1(t)$: pathogen killer

$u_2(t)$: plasma cell enhancer

$u_3(t)$: antibody enhancer

$u_4(t)$: organ healing factor

Innate Immune Response: state and control variables

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Generic dynamical model of the immune response

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\begin{aligned}
\dot{x}_1(t) &= (1 - x_3(t))x_1(t) - u_1(t - d_u)x_1(t), \\
\dot{x}_2(t) &= 3A(x_4(t))x_1(t - d_x)x_3(t - d_x) - (x_2(t) - 2) + u_2(t)x_2(t)\,, \\
\dot{x}_3(t) &= x_2(t) - (1.5 + 0.5x_1(t))x_3(t) + u_3(t)x_3(t)\,, \\
\dot{x}_4(t) &= x_1(t) - x_4(t) - u_4(t)x_4(t)\,.
\end{aligned}
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Immune deficiency function triggered by target organ damage

$$
A(x_4) = \left\{ \begin{array}{ll} \cos(\pi x_4)\,, & 0 \le x_4 \le 0.5 \\ 0 & 0.5 \le x_4 \end{array} \right\}.
$$

For $0.5 \le x_4(t)$ the production of plasma cells stops.

State delay $d_x \ge 0$ in variables x_1 and x_3.

Control delay $d_u \ge 0$ in variable u_1.

Initial conditions $(d = 0)$: $x_2(0) = 2,\ x_3(0) = 4/3,\ x_4(0) = 0$

Case 1 :	$x_1(0) = 1.5,$	decay, requires no therapy (control)
Case 2 :	$x_1(0) = 2.0,$	slower decay, requires no therapy
Case 3 :	$x_1(0) = 3.0,$	diverges without control (lethal case)

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Optimal control model: cost functional

L^2-functional quadratic in control: Stengel et al.

$$\text{Minimize} \quad J_2(x,u) = x_1(t_f)^2 + x_4(t_f)^2 + \int_0^{t_f} (\, x_1^2 + x_4^2 + u_1^2 + u_2^2 + u_3^2 + u_4^2 \,)\, dt$$

Control constraints: $0 \le u_i(t) \le u_{\max} = 1, \; i = 1, .., 4$

L^1-functional linear in control

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State Constraint

As result of organ damage triggered Immune deficiency function

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with ceasing plasma cells production for $0.5 \leq x_4(t)$, we impose

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$x_4(t) \leq \alpha$ with threshold $\alpha < 0.5$ i.e.

$$S(x_4(t)) := x_4(t) - \alpha \leq 0$$

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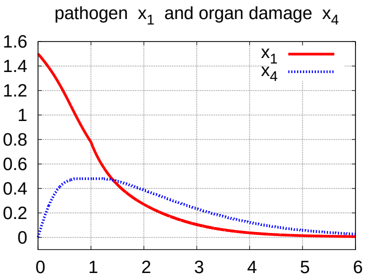
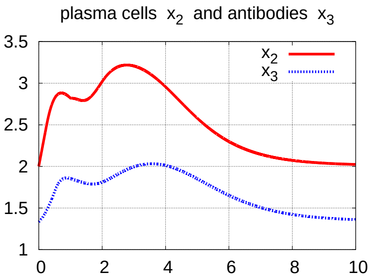


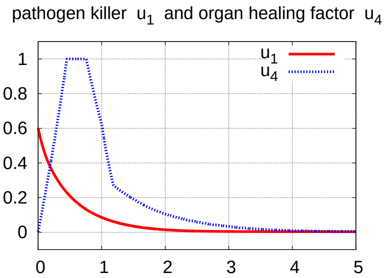


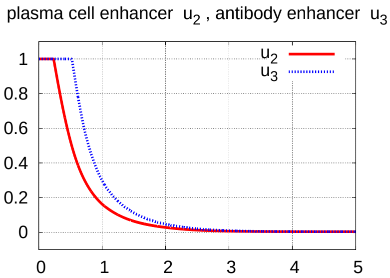
pathogen concentration
organ damage

plasma cell concentration
immunoglobulins

Results for L^2–functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Optimal control variables:



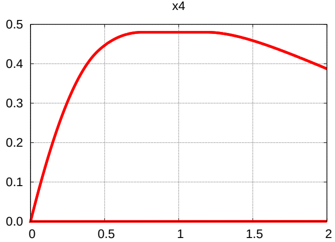


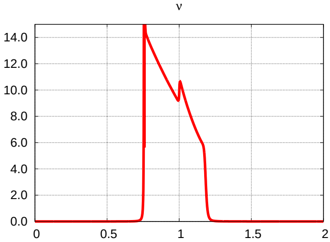
pathogen killer
organ healing factor

plasma cell enhancer
antibody enhancer

Results for L^2–functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Zoom: Optimal state x_4: Boundary arc observed





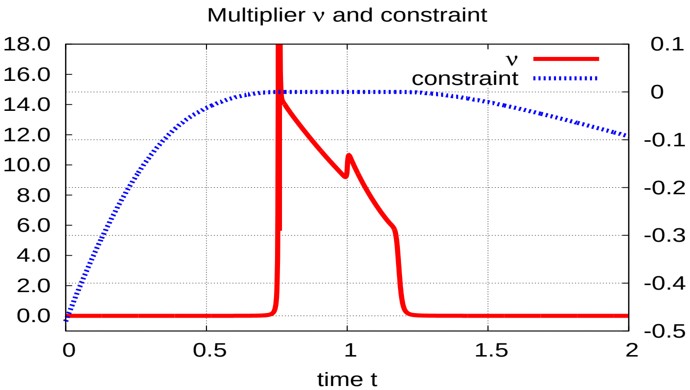
organ healing factor multiplier

First order necessary conditions, in particular $H_{u_4} = 0$, yield the multiplier on boundary arc $x_4(t) = \alpha$:

$$\eta(x, y, v_1, \lambda) = -2\alpha + 3\pi\lambda_2 \sin(\pi\alpha) y_1 y_2 + \lambda_4 \frac{x_1}{\alpha}$$
$$- \frac{2}{\alpha^2}((1 - x_3)x_1 - v_1 x_1$$

Results for L^2–functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Zoom into boundary arc $x_4(t) = 4.8$:

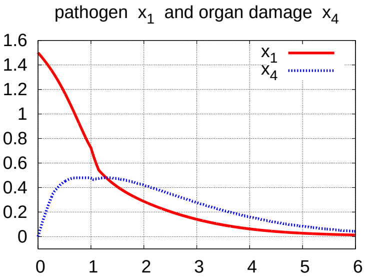


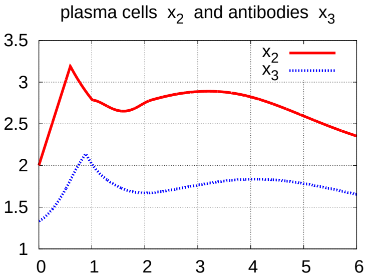
right axis: constraint $S(x_4) - \alpha \leq 0$.

left axis: corresponding multiplier $\eta(t)$.

Results for L^1–functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Optimal state variables:



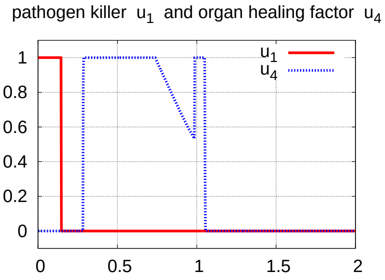


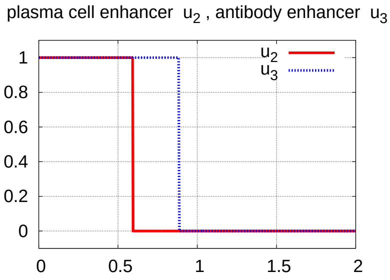
pathogen concentration
organ damage

plasma cell concentration
immunoglobulins

Results for L^1–functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Optimal control variables:



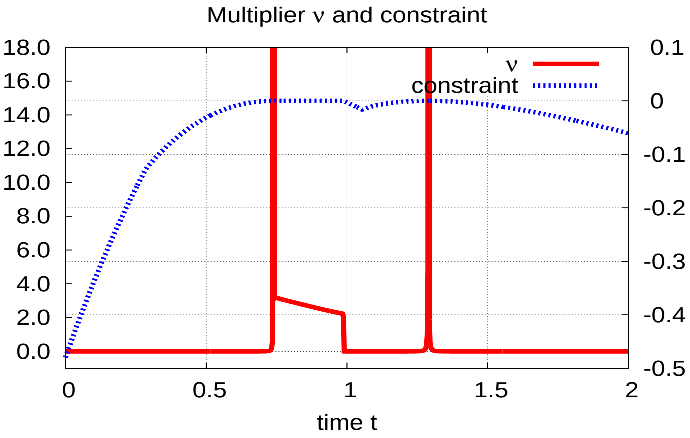


pathogen killer
organ healing factor

plasma cell enhancer
antibody enhancer

Results for L^1–functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

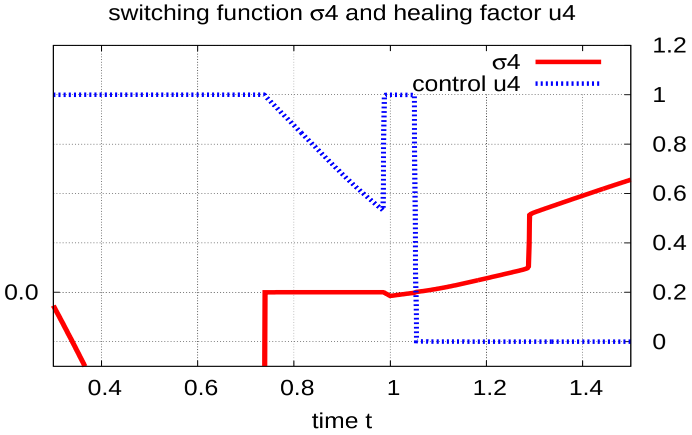
Zoom into boundary arc $x_4(t) = 4.8$:



overlay: constraint $S(x_4) - \alpha \leq 0$ (right axis)
and corresponding multiplier $\eta(t)$ (left axis)

Results for L^1–functional, $d_x = d_u = 1.0$, $\alpha = 4.8$

Zoom into boundary arc $x_4(t) = 4.8$:



right axis: control u_4 .

left axis: corresponding switching function $\sigma(t) = \frac{\partial H}{\partial u_4}$.

Further applications and future work

1. Optimal investment and dividend decisions of a firm
2. Expediting the transition from non-renewable to renewable energy (Maurer, Semmler)
3. Biomedical applications: optimal protocols in cancer treatment and immunology, optimal tapering of steroids
4. Verifiable sufficient conditions
5. Time and state dependent delays

Conclusion

Thank you for your attention !