Discrete mean field games: the short-stage limit

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Outline

Mean field games

Mean field games models aim to understand the behavior of a large number of identical players, where each tries to optimize its position in space and time, but whose preferences are determined by the choices of the other players.

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- Introduced by Huang, Caines and Malhamé (2003, 2006) and by Lasry and Lions (2006,2007).
- Two important features: dynamics and anticipation (backward-forward structure.)
- Most of the studied models are in continuous time: the "backward" part corresponding to a Hamilton-Jacobi PDE and the "forward" part corresponding to a Fokker-Planck PDE.

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- Our model builds on results from Adlakha, Johari and Weintraub (2012).

Notation

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- We reserve capital letters for random variables and lower case letters for their realizations.

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- Each player i receives the payoff $\ell(x^i_{t,N}, a^i_{t,N}, m_{t,N})$.
- The new state $X^i_{t+1,N}$ is chosen randomly using the transition function $Q(x^i_{t,N}, a^i_{t,N}, m_{t,N})$.

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- At the beginning of stage $t+1$, the realization of $X^i_{t+1,N}$ and $M_{t+1,N}$, denoted $x^i_{t+1,N}$ and $m_{t+1,N}$ respectively, are observed, and the situation is repeated.

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- At stage $t = T$ a final payoff $g(x^i_{T,N}, m_{T,N})$ is allocated.

Strategies

- A **behavioral strategy** for player i is a vector $\pi^i = (\pi^i_t)_{t=0}^{T-1}$ where $\pi^i_t : \mathcal{H}_t \to \mathcal{P}(A)$ and $\mathcal{H}_t = (\mathcal{X} \times A \times \mathcal{P}(\mathcal{X}))^t$ is the set of all possible histories up to date t. Denote by $\mathcal{S}$ the set of behavioral strategies.

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- A **Markovian strategy** for player i is a vector $\sigma^i = (\sigma^i_t)_{t=0}^{T-1}$ such that $\sigma^i_t : \mathcal{X} \to \mathcal{P}(A)$.
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- A **stationary strategy** is a function $\sigma : \mathcal{X} \to \mathcal{P}(A)$.
- A **strategy profile** is a vector $\pi = (\pi^i)_{i \in I}$, where π^i is a behavioral strategy of player i.

Payoff

The payoff of player i, when using the strategy π^i and when his adversaries use the strategy profile $\pi^{-i} \in \mathcal{S}^{N-1}$ is

$$J_N^i(x, m_0, \pi^i, \pi^{-i}) := \mathbb{E}_\pi^Q \left[\sum_{t \in \mathcal{T}} \ell(x_{t,N}^i, a_{t,N}^i, m_{t,N}) + g(x_{T,N}^i, m_{T,N}) \right].$$

Nash equilibrium

Definition

An $\epsilon-$**Nash equilibrium** *where* $\epsilon > 0$, *is a strategy profile* $(\pi^i)_{i\in I}$ *such that, for all player i and all behavioral strategy* τ,

$$J_N^i(x, m_0, \tau, \pi^{-i}) \leq J_N^i(x, m_0, \pi^i, \pi^{-i}) + \epsilon.$$

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- Compute an optimal strategy for a single player with $\mathbf{m}$ fixed.
- Everyone's strategy creates a vector of state distributions $\mathbf{m}' = (m_0, m_1', \ldots m_T')$
- If $\mathbf{m} = \mathbf{m}'$, we are happy.

Dynamic programming

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$$V(s,x,\mathbf{m}) \quad = \quad \max_{a\in A}\left\{\ell(x,a,m_s)+\mathbb{E}^Q V(s+1,x_{s+1},\mathbf{m})\right\}$$

with terminal condition $V(T,x,\mathbf{m})=g(x,m_T)$.

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Now consider a Markovian strategy $\sigma \in \Sigma$ and m_0 fixed and let $m_0^\sigma := m_0$. We define, for $t \geq 0$:

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$$m_{t+1}^\sigma(x) := \sum_{y \in \mathcal{X}} Q(y, \sigma_t(y), m_t^\sigma)(x) \cdot m_t^\sigma(y).$$

Definition: MFE

Definition

Let m_0 given. A **mean field equilibrium** *is a pair* $(\sigma, \mathbf{m}) = \left((\sigma_t)_{t=0}^{T-1}, (m_t)_{t=1}^{T}\right)$ *such that:*

1. σ *is the optimal strategy in the one player game* $\Gamma_{\mathbf{m}}$ *, computed using the dynamic programming equation.*
2. $\mathbf{m}$ *is the trajectory followed by* m_0 *according to the mass equation for the strategy* σ*.*

Existence

Proposition

(M.,2013) There exists a mean field equilibrium for the finite horizon game in the following cases:

- *If there exists a unique maximizer for the right hand side of dynamic programming equation for* $\mathbf{m}$ *and for each* (s, x).

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- *If there exists a unique maximizer for the right hand side of dynamic programming equation for* $\mathbf{m}$ *and for each* (s, x)*.*
- *The transitions are independent of the state distribution, i.e.*

$$Q(x, a, m)(y) =: Q(x, a)(y), \ \forall (x, y, a, m).$$

Approximation

Proposition

(M., 2013) Let x be a fixed initial state, $(\sigma, \mathbf{m})$ a mean field equilibrium and $(a^i_{t,N})_{t\in\mathcal{T}}$ an arbitrary sequence of actions of player i. Consider the following two trajectories:

1. *The trajectory of player i defined by $X^i_{t+1,N} \sim Q(x^i_{t,N}, a^i_{t,N}, m_{t,\sigma,N})$.*
2. *The trajectory defined by $X^i_{t+1} \sim Q(x^i_t, a^i_{t,N}, m_t)$.*

The following estimate holds:

$$\max_{i=1,\ldots,N} \mathbb{E}\left(\max_{s\leq T} \|X^i_s - X^i_{s,N}\|_\infty\right) \leq \frac{L_Q T|\mathcal{X}| \exp(T(\|Q\|_\infty + L_Q + 1))}{\sqrt{N}}$$

Some remarks

- In continuous time mean field games, the complementary approach of studying the limit behaviour of equilibria of N player games as $N \to +\infty$ has been developed by Bardi (2012) for the linear-quadratic case and by Lasry and Lions (2007) and Feleqi (2013) for games with several populations of players and ergodic payoffs.

Some remarks(cont.)

- This construction is "robust" with respect to the number of players: players can "play well" even if they do not know the exact number of players.

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- In general, the set of Nash equilibria of N player might contain equilibria that depend on N as in the driving game.

Driving game

- N players have to choose whether to drive on the left or on the right in a two-way road.

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- N players have to choose whether to drive on the left or on the right in a two-way road.
- Some equilibria: everyone on the left, everyone on the right, if N is even, everyone on the left; N odd, everyone on the right.
- The N player game has $2^{\aleph_0}$ equilibria, while the game with infinitely many players has only two.

The discounted N-player game

The $\lambda-$ discounted N player game is the game with payoff:

$$J_N^{\lambda,i}(x, m_0, \pi^i, \pi^{-i}) := \mathbb{E}_\pi^Q \left[\sum_{t=1}^{\infty} (1-\lambda)^{t-1} \ell(x_{t,N}^i, a_{t,N}^i, m_{t,N}) \mid x_0^i = x \right].$$

for $\lambda \in (0, 1]$.

Discounted payoff: One player

For the discounted case, one can define a mean field equilibrium as follows: The value of the one-player game Γ_m^λ, ($m \in \mathcal{P}(\mathcal{X})$ is fixed) with payoff:

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satisfies:

$$V_\lambda(x,m) = \max_{a \in A} \left\{ \ell(x,a,m) + (1-\lambda) \sum_{y \in \mathcal{X}} V_\lambda(y,m) Q(x,a,m)(y) \right\}.$$

Stationary mean field equilibrium

The mean field equilibrium in this case is a fixed point of the maps $\Psi_\lambda : \mathcal{P}(\mathcal{X}) \mapsto \Sigma_s$ and $\Phi_\lambda : \Sigma_s \rightsquigarrow \mathcal{P}(\mathcal{X})$:

$$\begin{array}{rcl} m & \rightsquigarrow & \text{Optimal stationary strategies in } \Gamma_m^\lambda, \\ \sigma \in \Sigma_s & \rightsquigarrow & \text{Inv. dist. of the MC with transition} Q(\cdot, \sigma(\cdot), m) \end{array}$$

Existence

Proposition

$\Phi_\lambda \circ \Psi_\lambda$ has a fixed point, i.e. there exists a stationary mean field equilibrium in the following cases:

- *If for every stationary strategy σ and all $m \in \mathcal{P}(\mathcal{X})$, the Markov chain with transition law $Q(\cdot, \cdot, \sigma(x), m)$ has a unique stationary distribution.*

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The short-stage equilibrium

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We adapt an idea recently introduced by Neyman(2013) and Cardaliaguet et al.(2013) to our model. The aim (informally) is to construct an approximation by Friedman/Fleming discretization of a game in continuous time.
Consider a function $\mu : \mathcal{X} \times \mathcal{X} \times A \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ bounded and such that, for all (x, a, m):

$$\mu(x, y, a, m) \geq 0,\ x \neq y, \quad \mu(x, x, a, m) = -\sum_{y \neq x} \mu(x, y, a, m).$$

Short-stage, one-player game

Let $\rho > 0$. The payoff in continuous time we want to approximate is:

$$\int_0^\infty e^{-\rho t}\ell(x_t, \sigma(x_t), m)dt$$

where $(x_t)_{t\geq 0}$ is a continuous time Markov chain whose transition semigroup has generator $\mu(\cdot, \sigma(\cdot), m)$.

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$$\rho V_{\rho,\delta}(x,m) = \max_{a \in A} \left\{ \ell(x,a,m) + e^{-\rho\delta} \sum_{y \in \mathcal{X}} \mu(x,y,a,m) V_{\rho,\delta}(y,m) \right\}$$

The short-stage limit

Proposition

The equation

$$\rho f(x,m) = \max_{a \in A} \left\{ \ell(x,a,m) + \sum_{y \in \mathcal{X}} \mu(x,y,a,m) f(y,m) \right\}$$

has a unique fixed point, denoted V_ρ. Moreover, $V^\delta_{\rho\delta} \to V_\rho$ uniformly as $\delta \to 0$.

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has a unique fixed point, denoted V_ρ. *Moreover,* $V^{\delta}_{\rho\delta} \to V_\rho$ *uniformly as* $\delta \to 0$.

Proof.

Use the stationary strategy σ^ρ given by this equation in the game with stage δ. □

The limit mass equation

Now let σ_δ be a fixed stationary strategy for the game with stage δ and let $m' \in \mathcal{P}(\mathcal{X})$. Let $L[\sigma_\delta, m'] \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ be defined by

$$L[\sigma_\delta, m']_{x,y} = \mu(x, y, \sigma_\delta(x), m').$$

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For σ, m' given, the associated invariant distribution m must solve

$$\delta L[\sigma, m'] \cdot m = 0 \iff L[\sigma, m'] \cdot m = 0$$

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Combining the two limit equations we obtain the following:

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Definition

A **limit stationary mean field equilibirum** *is a pair* $(\sigma, m) \in \Sigma_s \times \mathcal{P}(\mathcal{X})$ *such that* σ *is the stationary strategy derived from*

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and m *solves:*

$$L[\sigma^\rho, m] \cdot m = 0.$$

Approximation

Theorem

(M.2013). For every $\epsilon > 0$ there exists $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that, for all $\delta < \delta_0$ and $N > N_0$, the strategy provided by the limit stationary mean field equilibrium is a 2ϵ-Nash equilibrium of the discounted mean field game with discount factor $\lambda = 1 - e^{-\rho\delta}$.

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- Finally, take N_0 as in the bound derived for the error in the game with K_0 stages.

$\square$

Example: Online hotel booking

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v) *Frequent interaction is desirable in this example!*

Some remarks

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- Mean field games provide an extremely simple strategy that does not need to keep track of the other players.
- However, the mean field equilibrium might not be unique! Unless the players agree somehow on which equilibrium to play, it is hard to predict anything.

A possible way out (Repeated driving game)

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Consider the repeated version of the driving game with N players with the following adaptation mechanism:

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ii) *On the second stage, observing the realizations of the first stage, each player looks at everyone's choice (and recalls its own) and imitates the choice of the majority.*
iii) *Thus, from stage three, the players follow a mean field equilibrium.*

Possible extensions

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- Validation of this model in applications: Several economic applications in the paper of Adlakha, Johari, Weintraub(2012), applications for dynamic auctions by Iyer, Johari, Sundararajan (2011).

Thank you!