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# Stability of solutions to Hamilton-Jacobi equations on closed domains arising in optimal control

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# Hamilton-Jacobi equation

Consider a Hamiltonian  $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  that is convex in the last variable and the Hamilton-Jacobi equation

$$-v_t + H(t, x, -v_x) = 0, \quad v(T, \cdot) = \varphi(\cdot). \quad (1)$$

Let  $H^*(t, x, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be the Fenchel conjugate of  $H(t, x, \cdot)$  and consider the Calculus of Variations problem

$$v(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^T H^*(t, x(t), \dot{x}(t)) dt : \right.$$

$$\left. x \in W^{1,1}([t_0, T], x(t_0) = x_0) \right\}.$$

# Faithful Representation of Hamiltonian

It is well known that under some assumptions on  $H$ ,  $v$  is the unique viscosity solution of (1). However,  $H^*$  may have infinite values.

**Question:** can we associate to  $H$  mappings  $f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$  and  $\ell : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$  inheriting Lipschitz type regularity properties of  $H$  and such that

$$H(t, x, p) = \max_{u \in U} (\langle p, f(t, x, u) \rangle - \ell(t, x, u)), \quad (2)$$

where  $U$  is a compact subset of a finite dimensional space.

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where  $U$  is a compact subset of a finite dimensional space.

**The answer is indeed positive.**

That is  $H$  is equal to the Hamiltonian of a Bolza optimal control problem.

## Value function of state constrained Bolza problem

Let  $K$  be a closed nonempty subset of  $\mathbb{R}^n$ .

$$V(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^T \ell(t, x(t), u(t)) dt \mid (x, u) \in S_{[t_0, T]}(x_0) \right\}$$

Here,  $S_{[t_0, T]}(x_0)$  denotes the set of all trajectory-control pairs of the control system under state constraint

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)), & u(s) \in U \text{ a.e. in } [t_0, T] \\ x(t_0) = x_0 \\ x(s) \in K, & \forall s \in [t_0, T] \end{cases}$$

Under appropriate assumptions,  $V$  is the unique solution to the Hamilton-Jacobi equation on the set  $[0, T] \times K$ .

# Assumptions

**(H1)**  $H(t, x, \cdot)$  is convex for any  $(t, x) \in [0, T] \times \mathbb{R}^n$ .

**(H2)** For any  $R > 0$  there exists an integrable function  $c_R \in L^1([0, T])$  such that for all  $x, y \in B_R$  and  $p \in \mathbb{R}^n$

$$|H(t, x, p) - H(t, y, p)| \leq c_R(t)(1 + |p|)|x - y|.$$

**(H3)** There exists an integrable function  $c \in L^1([0, T])$  such that

$$|H(t, x, p) - H(t, x, q)| \leq c(t)(1 + |x|)|p - q|$$

for all  $(t, x) \in [0, T] \times \mathbb{R}^n$  and  $p, q \in \mathbb{R}^n$ .

**(H4)**  $H^*(t, x, \cdot)$  is bounded on its domain for all  $(t, x) \in [0, T] \times \mathbb{R}^n$ .

**(H5)** For every  $R > 0$  there exists  $M_R > 0$  such that for all  $(t, x) \in [0, T] \times B_R$  and  $v \in \text{dom}(H^*(t, x, \cdot))$  we have

$$H^*(t, x, v) = \max_{p \in B(0, M_R)} (\langle v, p \rangle - H(t, x, p)).$$

**(H6)** For every  $R > 0$  there exists an absolutely continuous  $a_R \in W^{1,1}(0, T)$  such that for all  $x \in B_R$ ,  $p \in \mathbb{R}^n$  and  $t, s \in [0, T]$

$$|H(t, x, p) - H(s, x, p)| \leq (1 + |p|)|a_R(t) - a_R(s)|.$$



# Regularity of dynamics and the cost function

## Theorem

If (H1)-(H6) hold true, then  $\exists f : [0, T] \times \mathbb{R}^n \times B \rightarrow \mathbb{R}^n$ , such that for  $\ell : [0, T] \times \mathbb{R}^n \times B \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$\ell(t, x, u) = H^*(t, x, f(t, x, u))$  we have

**(A1)**  $H(t, x, p) = \max_{u \in B} (\langle p, f(t, x, u) \rangle - \ell(t, x, u))$ ,  $\forall (t, x, p)$ .

**(A2)** For any  $R > 0$  and for all  $t \in [0, T]$ ,  $x, y \in B_R$ ,  $u, v \in B$

$$\begin{cases} |f(t, x, u) - f(t, y, u)| \leq 10nc_R(t)|x - y| \\ |f(t, x, u) - f(t, x, v)| \leq 5n(1 + R)c(t)|u - v|. \end{cases}$$

**(A3)**  $|f(t, x, u)| \leq c(t)(1 + |x|)$  for all  $(t, x, u) \in [0, T] \times \mathbb{R}^n \times B$ .

## Regularity of dynamics and the cost function

### Theorem

**(A4)**  $\ell$  takes finite values and for any  $R > 0$ ,  $t \in [0, T]$ ,  $x, y \in B_R$ ,  $u, v \in B$

$$\begin{cases} |\ell(t, x, u) - \ell(t, y, u)| \leq (1 + 11nM_R)c_R(t)|x - y|, \\ |\ell(t, x, u) - \ell(t, x, v)| \leq 5nM_R(1 + R)c(t)|u - v|. \end{cases}$$

and

$$|\ell(t, x, u) - \ell(s, x, u)| \leq (1 + 11nM_R)|a_R(t) - a_R(s)|.$$

We associate to  $f, \ell, \varphi$  the Bolza optimal control problem.

## Stability of representations

If assumptions (H1)-(H6) hold true for continuous  $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i \geq 1$ , then there exist  $f_i : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ ,  $\ell_i : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$  satisfying (A1)-(A4), which are standard in control theory.

### Theorem

*If  $H_i$  converge uniformly on compacts to some  $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and satisfy (H1) – (H6) with the same  $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$ , then the representations  $f_i, \ell_i$  can be chosen to converge to some  $f, \ell$  and satisfying (A1)-(A4) with the same  $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$ .*

## State constraints

We assume that the closed sets  $K$  and  $K_i$  are defined by the multiple inequality constraints

$$K \doteq \bigcap_{j=1}^m \{x : g^j(x) \leq 0\}$$

$$K_i \doteq \bigcap_{j=1}^m \{x : g_i^j(x) \leq 0\},$$

where  $g_i^j : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g^j : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $j = 1, \dots, m$ ,  $i = 1, 2, \dots$  are continuously differentiable functions satisfying

### (G1)

i) For any  $R > 0$  there exists  $A_R > 0$  such that  $|\nabla g_i^j(x)| \leq A_R$ , for any  $x \in RB$ ,  $\nabla g_i^j$  is  $A_R$ -Lipschitz on  $RB$ .

ii)  $\nabla g_i^j \rightarrow \nabla g^j$  uniformly on compacts and  $g_i^j(0) \rightarrow g^j(0)$ , when  $i \rightarrow \infty$ , for any  $j = 1, \dots, m$ .

## Inward pointing condition

For any  $x \in \mathbb{R}^n$  denote by  $I(x)$  the set of active indices at  $x$  for  $g(\cdot) = (g^1(\cdot), \dots, g^m(\cdot))$ , i.e.

$$I(x) = \{j : g^j(x) = 0\}.$$

**(IPC)** For any  $R > 0$  there exists  $\rho_R > 0$  such that for every  $x \in K \cap RB$  with  $I(x) \neq \emptyset$  and every  $s \in [0, T]$

$$\inf_{v \in \text{dom}(H^*(t, x, \cdot))} \max_{j \in I(x)} \langle \nabla g^j(x), v \rangle \leq -\rho_R.$$

# Viscosity solutions of Hamilton-Jacobi equation

## Definition

A continuous function  $V : [0, T] \times K \rightarrow \mathbb{R}$  is called a viscosity solution of Hamilton-Jacobi equation on the closed set  $[0, T] \times K$  if  $V(T, \cdot) = \varphi(\cdot)$  and

i) for all  $(t, x) \in [0, T] \times \text{Int}K$  and all  $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0$$

ii) for all  $(t, x) \in [0, T] \times \text{Int}K$  and all  $(p_t, p_x) \in \partial_+ V(t, x)$

$$-p_t + H(t, x, -p_x) \leq 0,$$

where  $\partial_{\pm} V(t, x)$  are the Frechet super/subdifferential of  $V$  at  $(t, x)$ .

# Bilateral solution of Hamilton-Jacobi equation

## Definition

$V : [0, T] \times K \rightarrow \mathbb{R}$  is called a bilateral solution of Hamilton-Jacobi equation on the closed set  $[0, T] \times K$  if  $V(T, \cdot) = \varphi(\cdot)$  and

i) for all  $(t, x) \in [0, T] \times \text{Int}K$  and all  $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) = 0$$

ii) for all  $(t, x) \in [0, T] \times \partial K$  and all  $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0.$$

# Uniqueness of solutions of Hamilton-Jacobi equation

## Theorem

*If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique bilateral solution of the Hamilton-Jacobi equation on  $[0, T] \times K$  in the class of continuous functions.*

## Theorem

*If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique viscosity solution of the Hamilton-Jacobi equation on  $[0, T] \times K$  in the class of continuous functions.*



# Stability of Solutions to Hamilton-Jacobi equation

## Theorem

Suppose that  $H_i$  satisfy (H1)-(H6) with the same  $a_R(\cdot)$ ,  $c_R(\cdot)$ ,  $c(\cdot)$ ,  $M_R$  and converge uniformly on compacts to a Hamiltonian  $H$ . Assume that (IPC) holds true. If  $\varphi_i$  converge to  $\varphi$  uniformly on compacts, then the unique solutions  $V_i$  to Hamilton-Jacobi equation with  $H_i, \varphi_i, K_i$  converge uniformly on **compacts contained in the interior of  $K$**  to the unique solution of Hamilton-Jacobi equation with  $H, \varphi, K$ .

## Stability of Value functions: Corollary

### Corollary

Suppose that  $H_i$  satisfy (H1)-(H6) with the same  $a_R(\cdot)$ ,  $c_R(\cdot)$ ,  $c(\cdot)$ ,  $M_R$  and converge uniformly on compacts to a Hamiltonian  $H$ . Assume that (IPC) holds true. Then

$$\text{Lim}_{i \rightarrow \infty} \text{graph} V_i = \text{graph} V$$

and

$$\text{Lim}_{i \rightarrow \infty} \text{epi} V_i = \text{epi} V,$$

where the limit is taken in the Kuratowski sense.

Thank you for your attention.