Reconstruction of Independent sub-domains in a Hamilton-Jacobi Equation and its Use for Parallel Computation

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ITN SADCO
Initial Training Network
Sensitivity Analysis for Deterministic Controller Design

Introduction

Patchy Decomposition

- Cacace, Cristiani, Falcone and Picarelli, *A patchy dynamic programming scheme for a class of Hamilton-Jacobi-Bellman equation*, SIAM J. Sc. Comp. (2012)

- *Reconstruction of some “Sub-Domains of Invariance” through the resolution of the problem on a course grid passing by the synthesis of the controls*
- **Goal:** solve the problem in parallel on a fine grid
- Good point: Some cases of interest where this idea works well
- **Open questions:**
  - Convergence, error introduced,
  - Extension of this idea to a wider class of problems

Patchy Decomposition: example

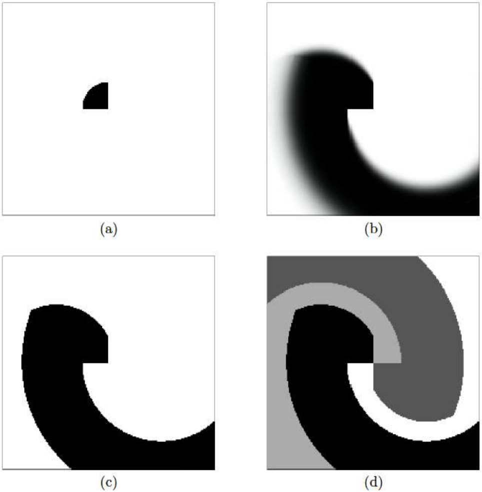


Figure: Some steps of the *Patchy Algorithm* (thanks to the authors)

Decomposition for Differential Games

(Deduced from F. Vinter, preprint 2014)

Let us consider, for an H not necessarily convex

$$\begin{cases} \lambda v(x) + H(x, Dv(x)) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \Gamma \end{cases}$$

Considered a **decomposition of the boundary** $\Gamma := \bigcup_{i \in \mathbb{I}} \Gamma_i$, with $\mathcal{I} := \{1, ...m\} \subset \mathbb{N}$, we call $v_i : \bar{\Omega} \to \mathbb{R}$ a Lipschitz continuous viscosity solution of the problem

$$\begin{cases} \lambda v_i(x) + H\left(x, Dv_i(x)\right) = 0 & x \in \Omega \\ v_i(x) = g_i(x) & x \in \Gamma \end{cases}$$

where the functions $g_i : \Gamma \to \mathbb{R}$ is a regular function such that

$$\begin{aligned} g_i(x) = g(x), &\ \text{if } x \in \Gamma_i, \\ g_i(x) > g(x), &\ \text{otherwise.} \end{aligned}$$

Define

$$\begin{aligned} I(x) &:= \{i \in \{1,\dots,m\} | v_i(x) = \min_j v_j(x)\}, \\ \Sigma &:= \{x \in \mathbb{R}^N | card(I(x)) > 1\}. \end{aligned}$$

Theorem

Assume the following condition satisfied: for arbitrary $x \in \Sigma$, any convex combination $\{\alpha_i \,|\, i \in I(x)\}$ and any collection of vectors $\{p_i \in \partial^L v_i(x) \,|\, i \in I(x)\}$ we have

$$\lambda \bar{v} + H\left(x, \sum_i \alpha_i p_i\right) \leq 0\,. \qquad \text{(E)}$$

Then, for all $x \in \mathbb{R}^N \backslash \mathcal{T}$,

$$v(x) \;=\; \bar{v}(x) \;:=\; \min_i \{v_1(x), \dots, v_m(x)\}.$$

Example (I):

The equation considered is, with $\Omega := (-1,1) \times (-1,1)$,

$$\begin{cases} \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1 & x \in \Omega \\ v(x) = 0 & x_1 \in \partial\Omega; \end{cases}$$

(evidently the solution is the **distance function** from the boundary), we associate, considered to the easiest division $\partial\Omega = \Gamma := \cup_i \Gamma_i = \cup[(\pm 1, \pm 1), (\pm 1, \pm 1)]$, to the $\Gamma_1 := [(-1,-1), (-1,1)]$ the function $g_1 : \partial\Omega \to \mathbb{R}$ defined as

$$\begin{cases} g_1(x) := 0 & x \in \Gamma_1 \\ g_1(x) := \gamma(1 + x_2) & x \in \Gamma \setminus \Gamma_1; \end{cases}$$

with $\gamma > 0$.

Example (II):

The unique viscosity solution of such a problem is

$$v_1(x) = (1+\gamma) - \max(|x_1 - \gamma|, |x_2|)$$

and finally, the original value function $v(x) = 1 - \max(|x_1|, |x_2|)$ is recovered as $v(x) = \min_{i=1,\ldots,4} v_i(x)$.

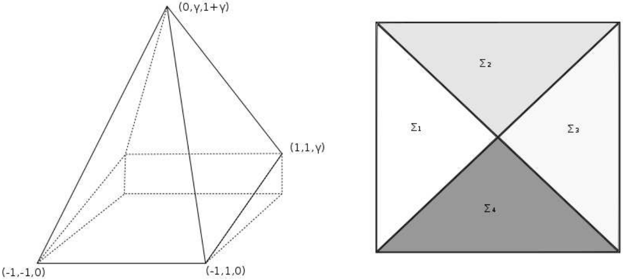


Figure: the auxiliary solution v_1 and the “regions of coincidence”.

Independent subdomains reconstruction

Differential Games Problem

Let the dynamics be given by

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)), & a.e. \\ y(0) = x, \end{cases}$$

$x \in \Omega \subseteq \mathbb{R}^n$ open, $a, b \in \mathcal{A}, \mathcal{B} = \{\mathbb{R}^+ \to A,\ measureable\}$, A, B compact sets. A solution is a trajectory $y_x(t, a(t), b(t))$.

The goal is to find the *sup – inf* **optimum** over $\mathcal{A}$, $\mathcal{B}$ of

$$J_x(a, b) := \int_0^{\tau_x(a,b)} l(y_x(s, a(s), b(s)), a(s), b(s))e^{-\lambda s} ds$$
$$+ e^{-\lambda \tau_x(a,b)} g(y_x(\tau_x(a, b))), \quad \lambda \geq 0,$$

where $\tau_x(a, b) := \min\{t \in [0, +\infty) \mid y_x(t, a(t), b(t)) \notin \Omega\}$.

the value function of this problem is

$$v(x) := \sup_{\phi \in \Phi} \inf_{a \in \mathcal{A}} J_x(a, \phi(a)),$$

$$\Phi := \{\phi : \mathcal{A} \to \mathcal{B} : t > 0, a(s) = \tilde{a}(s) \text{ for all } s \leq t \\ \text{implies } \phi[a](s) = \phi[\tilde{a}](s) \text{ for all } s \leq t\}.$$

we will assume the *Isaacs' conditions* verified.

Theorem

The value function of the problem is a viscosity solution of the HJ equation associated with

$$H(x, p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x, a, b) \cdot p - l(x, a, b)\}.$$

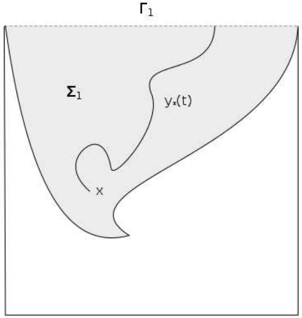
Independent Sub-Domains

Definition

A closed subset $\Sigma \subseteq \overline{\Omega}$ is an independent sub-domain of the problem (11) if, given a point $x \in \Sigma$ and an **optimal control** $(\overline{a}(t), \bar{\phi}(\overline{a}(t))$

(i.e. $J_x(\overline{a}, \bar{\phi}(\bar{a})) \leq J_x(a, \bar{\phi}(a))$ for every choice of $a \in \mathcal{A}$, and $J_x(\bar{a}, \bar{\phi}(\bar{a})) \geq J_x(\bar{a}, \phi(\bar{a}))$ for every choice of $\phi \in \Phi$),

the trajectory $y_x(\bar{a}(t), \bar{\phi}(\bar{a}(t))) \in \Sigma$ for $t \in [0, \tau_x(\bar{a}, \bar{\phi}(\bar{a}))]$.



Independent Domains Decomposition

Proposition

Given a collection of $n-1$ dimensional subsets $\{\Gamma_i\}_{i=\mathcal{I}}$ such that $\Gamma = \cup_{i=1}^{m}\Gamma_i$, the sets defined as

$$\Sigma_i := \left\{x \in \overline{\Omega} | v_i(x) = v(x)\right\}, \quad i = 1, ..., m$$

where v_i, v are defined accordingly to Theorem (1), are independent sub-domains of the original problem.

Proof.

By contradiction using the DPP. $\square$

Numeric framework, SemiLagrangian scheme

Consider a **structured grid of simplices** S_j, such that $\overline{\Omega} \in \cup_j S_j$, denoting x_m, $m = 1, ..., N$, the nodes of the triangulation,

$$\Delta x := \max_j diam(S_j)$$

G is *internal nodes,* ∂G *boundary points,* Ψ *ghost nodes*.

Mapping the values at the nodes onto $V = (V_1, ..., V_N)$, it is possible to obtain the following scheme in **fixed point form**

$$V = T(V),$$

$$[T(V)]_i = \begin{cases} \max_{b \in B} \min_{a \in A} \left\{ \frac{1}{1+\lambda h} \mathbb{I}[V](x_i - hf(x_i, a, b)) - hl(x_i, a, b) \right\} & x_i \in G, \\ g(x_i) & x_i \in \partial G, \\ +\infty & x_i \in \Psi. \end{cases}$$

An estimate needed

Theorem

Let v and V be the solutions of the continuous and the discrete problem. Under some regularity hypotheses

$$||v - V||_\infty \leq C(\Delta x)^q,$$

where C is a positive constant independent from Δx, $q \in \mathbb{R}^+$ depending on the regularity of the problem.

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Examples in the SL case: differential games with a Lipschitz solution

$$\|v - V\|_\infty \leq Ch^{\frac{1}{2}} \left(1 + \left(\frac{\Delta x}{h}\right)^2\right),$$

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Examples in the SL case: differential games with a Lipschitz solution

$$\|v - V\|_\infty \leq Ch^{\frac{1}{2}}\left(1 + \left(\frac{\Delta x}{h}\right)^2\right),$$

Optimal Control problem with $\lambda > 0$:

$$\|v - V\|_\infty \leq 2(M_v + M_{v_h})h^{\frac{1}{2}} + \left(\frac{L_l}{\lambda(\lambda - L_f)}\frac{\Delta x}{h}\right).$$

A necessary condition

Proposition

Assumed $x \in \Omega$ such that $v_i(x) = v(x)$ for a certain $i \in \mathbb{I}$, an $\epsilon \in [0, \Delta x)$ and a direction $d \in B(0,1)$, such that, for a x_j node of the grid G, $x_j = x + \epsilon d$ and $v(x_j) < v_i(x_j)$; then

$$|V_i(x_j) - V(x_j)| \leq 2(C(\Delta x)^q + M\Delta x)$$

C as in the previous statement and $M := \max\{L_{v_i}, i \in \mathcal{I}\}$ where L_{v_i} is the Lipschitz constant of the function v_i.

Independent Subdomains Reconstruction

INDEPENDENT SUB-D. RECONSTRUCTION ALGORITHM (IRA).

- Given a collection of indices vectors such that $union(x_i, i = 1, ..., m) = y$

1) (Resolution of auxiliary problems)
   for $i = 1...m$ solve iteratively the problem
   $V_i = T_i(V_i)$ with $\partial G := \{x_j | j \in y_i\}$
   end

2) **(Reconstruction of the value function)**
   obtain V as $V = \min_{i=1...m}\{V_i\}$

3) (Reconstruction of the Sub-Domains)
   for i=1...m
   initialize $w_i = \emptyset$
   for j=1...N
   if $\| V_i(j) - V(j) \| \leq \varepsilon(\Delta x)$
   then add j to vector w_i
   end
   the index set of the $i-$subset is w_i
   end

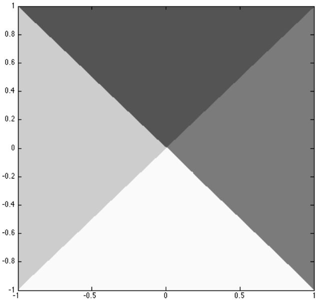
Some Observations

(IRA) builds an approximation of the independent sub-domains.
Moreover we have:

- the approximation **exceeds the desired** set, i.e. $\Sigma_i \subseteq \overline{\Sigma}_i$.
- The same relation holds for the **projection of the approximation** on a finer grid, i.e. for $\Delta x_1 \geq \Delta x_2$,

$$Proj_{\Delta x_2} \overline{\Sigma}_i^{\Delta x_1} \supseteq \Sigma_i$$

Example of Reconstruction (I)



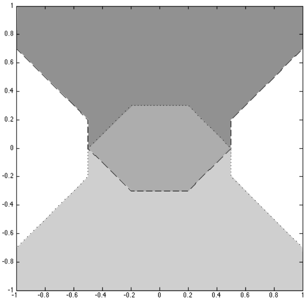


Figure: Distance function: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of 15^2 points.

Example of Reconstruction (II)

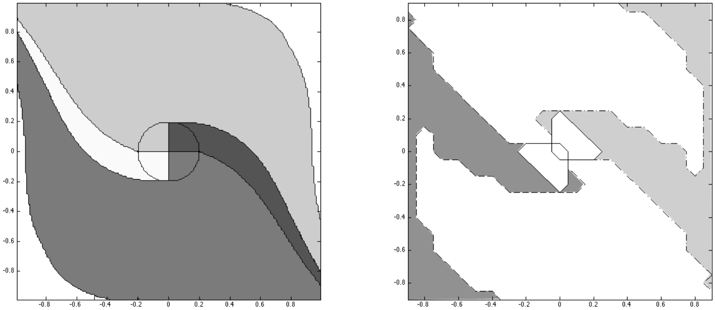


Figure: Van Der Pol: Exact decomposition and two (of the four) approximated independent subsets with a course grind of 15^2 points.

Use for Parallel Computing

Consider:

- a coarse grid K of discretization step Δx_k
- a **fine grid** G of step Δx_G

on the domain Ω.
The two triangulations are chosen such that $\Delta x_G \ll \Delta x_K$.

Independent Sub-Domain Parallelization

INDEPENDENT-SETS ALGORITHM (ISA).

- Given a collection of indices vectors such that $union(z_i, i = 1, ..., M) = y$

1) **(Reconstruction of the approximated independent sub-domains)**
   Using IRA get a collection w_i, $i = 1, ..., M$ of indexes sets.

2) (Projection on the fine grid)
   Extend indexes sets w_i on the fine grid G_j getting W_i

3) **(Resolution on the fine grid)**
   for $i = 1, ..., M$ solve iteratively the problem
   $$V_i = T(V_i)$$
   end

4) (Assembly of the final Solution)
   for $j = 1, ..., N_2$
   $$\overline{V}(x_j) = min\{V_i \text{ s.t. } j \in w_i\}$$
   end

Convergence Result:

Proposition

Called $\overline{V}$ the ***exact discrete solution*** *of the (ISA) algorithm (i.e. all $V_i = T_i(V_i)$ are verified exactly) and v the exact solution of the* ***original continuous problem****, then, for a $C > 0$, independent from Δx, the following estimate holds*

$$\|\overline{V} - v\|_\infty \leq C(\Delta x_G)^q.$$

Convergence Result:

Proposition

*Called $\overline{V}$ the **exact discrete solution** of the (ISA) algorithm (i.e. all $V_i = T_i(V_i)$ are verified exactly) and v the exact solution of the **original continuous problem**, then, for a $C > 0$, independent from Δx, the following estimate holds*

$$\|\overline{V} - v\|_\infty \leq C(\Delta x_G)^q.$$

Proof.

We know that the inclusion property holds. Then for $x_j \in G$, it is assured the existence of $i \in \{1, ..., M\}$ s.t. $v(x_j) = v_i(x_j)$, and $|\overline{V}(x_j) - V_i(x_j)| = 0$. Then

$$|\overline{V}(x_j) - v(x_j)| \leq |\overline{V}(x_j) - V_i(x_j)| + |V_i(x_j) - v(x_j)| \leq \|V_i - v_i\|_\infty \leq C(\Delta x_G)^q$$

for the arbitrariness of the choice of x_j we have the thesis. $\square$

A convergence result for a domain partition:

Corollary

Let us consider a partition of the grid K in subsets B_i $i = 1, ..., M$ such that for every $i = 1, ..., M$

- $Z_i \subseteq B_i$
- $\forall z_j \in B_i$, $|V_i(j) - V(j)| \leq 2(C(\Delta x_K)^q + M\Delta x_K)$
- *each B_i is connected (for each $z_j \in B_i$, there exists a direction d s.t. for $r \in [0, \Delta x_k]$, $z_j + dr = z_{\bar{j}} \in B_i$ with $j \neq \bar{j}$)*

then the following estimate for $\hat{V} := \min\{V_i(j) | x_j \in B_i\}$, where the V_i is calculated on the projection on G of the set B_i:

$$\|\hat{V} - v\|_\infty \leq C\Delta x_G + 2(C(\Delta x_K)^q + M\Delta x_K)$$

Tests: Distance function

Eikonal equation on the set $\Omega := (-1,1)^2$ with the boundary value fixed to zero on $\Gamma := \partial\Omega$.
$\lambda = 1$ (non linear monotone scaling of the solution);
the correct viscosity solution is

$$v(x) + \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1, \qquad v(x) = 1 - \frac{\min\{e^{|x_1|}, e^{|x_2|}\}}{e}.$$

N. of variables	Δx	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert / \lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert / \lvert\Omega\rvert$
5^2	0.4	$1 \cdot 10^{-3}$	50%	
7^2	0.28	$2 \cdot 10^{-3}$	43%	
10^2	0.2	$4 \cdot 10^{-3}$	38%	
15^2	0.133	$2 \cdot 10^{-2}$	35%	25%
20^2	0.1	$5 \cdot 10^{-2}$	33%	
40^2	0.05	3	29%	
50^2	0.04	11	28.3%	

Tests: Distance function

Table: Time necessary for the resolution on the original domain and the reduced one with various discretization steps (15^2 (IRA))

N. of variables	Δx	Time on Ω	Time on $\max_i \lvert\bar{\Sigma}_i\rvert$
25^2	0.08	0.13	0.035
50^2	0.04	7.02	0.68
75^2	0.026	57.5	4.8
100^2	0.02	$1.5 \cdot 10^3$	16.6
200^2	0.01	–	$3 \cdot 10^3$

Table: Approximation error Error in norm $\Delta\infty$ (and Δ_1)

	50^2	100^2	200^2
original	$1.2 \cdot 10^{-2}(1.1 \cdot 10^{-2})$	$6.5 \cdot 10^{-3}(3.6 \cdot 10^{-3})$	$2.5 \cdot 10^{-3}(1.6 \cdot 10^{-3})$
2-subs.	$1.2 \cdot 10^{-2}(7.2 \cdot 10^{-3})$	$6.5 \cdot 10^{-3}(3.7 \cdot 10^{-3})$	$2.5 \cdot 10^{-3}(1.4 \cdot 10^{-3})$
4-subs.	$9 \cdot 10^{-3}(7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3}(3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3}(1.3 \cdot 10^{-3})$
8-subs.	$9 \cdot 10^{-3}(7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3}(3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3}(1.3 \cdot 10^{-3})$

Tests: Van Der Pool Oscillator

We consider $\Gamma := \partial B(0,\rho)$ (in this case $\rho = 0.2$) and $\Omega := (-1,1)^2 \setminus \bar{B}(0,\rho)$. The nonlinear system will be:

$$f(x,a) = \begin{pmatrix} x_2 \\ (1-x_1^2)x_2 - x_1 + a \end{pmatrix}.$$

The others parameters of the system are:

$$A = [-1,1], \quad \lambda = 1, \quad l(x,y,a) = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad g(x) \equiv 0.$$

Table: Van Der Pol: Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	Δx	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	$1.4 \cdot 10^{-3}$	62%	
10^2	0.2	0.011	55%	
20^2	0.1	0.103	47%	42.2%
30^2	0.06	1.47	45%	
40^2	0.05	5.6	44.6%	
50^2	0.04	16.3	44.1%	

Tests: Van Der Pool Oscillator

Table: Van Der Pol: Approximation error Error in norm $\Delta\infty$ (and Δ_1)

	50^2	100^2	200^2
original	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
2-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
4-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
8-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$

Tests: A Pursuit-Evasion game

We will consider a Pursuit evasion game, where two agents have the opposite goal to reduce/postpone the time of capture.

The dynamics considered are the following:

$$f(x,a,b) := \begin{pmatrix} f_1(x)(a_1 - b_1) \\ f_2(x)(a_2 - b_2) \end{pmatrix}$$

where the functions f_1, f_2 are $f_1(x) := x_2 + 1$ and $f_2(x) := 1$.
The running cost $l(x,a,b) := x_1 + 0.1$

Controls are $A = B(0,1)$ for the pursuer and $B = B(0,1/2)$ for the evader.
Capture happens trajectory is driven to touch $B(0,0.2)$, then $\Gamma := \partial B(0,0.2)$

Tests: A Pursuit-Evasion game

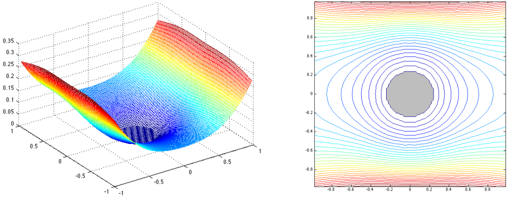


Figure: Approximated value function

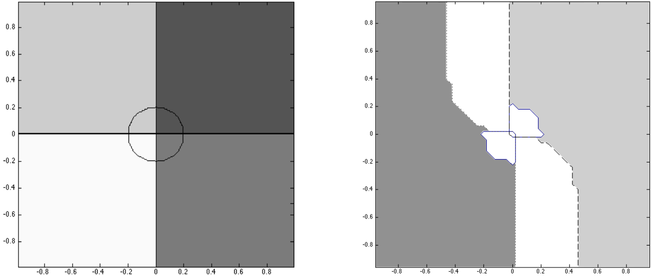


Figure: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of 40^2 points.

Tests: A Pursuit-Evasion game

Table: Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	Δx	Time elapsed	$\max_i \lvert\overline{\Sigma}_i\rvert/\lvert\Omega\rvert$	$\max_i \lvert\Sigma_i\rvert/\lvert\Omega\rvert$
5^2	0.4	10^{-3}	60%	
10^2	0.2	0.008	46%	
30^2	0.06	1.38	38%	25%
50^2	0.04	15.9	36.1%	

The HJ equation associated verifies the decomposability condition (E); because considering the norm (it is possible because $|f_i(x)| > 0$ for $i = 1, 2$)

$$\|p\|_* := \max_{a \in B(0,1)} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} a^T \cdot p$$

the Hamiltonian associated is equivalent to

$$H(x,p) := \|p\|_* - \frac{\|p\|_*}{2} - (x_1^2 + 0.1)$$

evidently convex everywhere with respect to p.

Independent Domain Parallelization: Summary

- The gain is variable, but can be considerable high (no communication between the sub-domains)
- There is a direct dependence of the technique on the **initial decomposition** of the boundary
- Convergence for $\Delta_G \to 0^+$ is guaranteed and the error in the approximation is bounded by an estimate
- It is proved also **convergence for a partition decomposition** (as the *Patchy Dec.*) but it is guaranteed a convergence only for $\Delta_K \to 0^+$, and relative estimation of the error.

Conclusions and Perspectives

- We have shown a constructive manner to obtain a decomposition of the domain of a HJ eq. verifying (E), in independent subdomains, subsets which have the properties of being computed, independently from each others.
- A critical point is **the estimation of the function $\varepsilon(\cdot)$**.
- The critical occurrence about the balance of the dimension of the subsets can be solved with a recursive refinement of the division of Γ.
- A more serious limit appears in **presence of flat regions**: in that case it is, for the moment, impossible to obtain a satisfactory reduction of the dimension of the decomposed domains without solving the problem on a sufficiently fine grid.

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Thank you.