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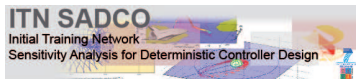
Reconstruction of Independent sub-domains in a Hamilton-Jacobi Equation and its Use for Parallel Computation

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NetCo 2014, Tours, France.



Introduction

Patchy Decomposition

- Cacace, Cristiani, Falcone and Picarelli, *A patchy dynamic programming scheme for a class of Hamilton-Jacobi-Bellman equation*, SIAM J. Sc. Comp. (2012)

- ▶ *Reconstruction of some “Sub-Domains of Invariance” through the resolution of the problem on a coarse grid passing by the synthesis of the controls*
- ▶ **Goal:** solve the problem in parallel on a fine grid
- ▶ **Good point:** Some cases of interest where this idea works well
- ▶ **Open questions:**
 - ▶ Convergence, error introduced,
 - ▶ Extension of this idea to a wider class of problems

Patchy Decomposition: example

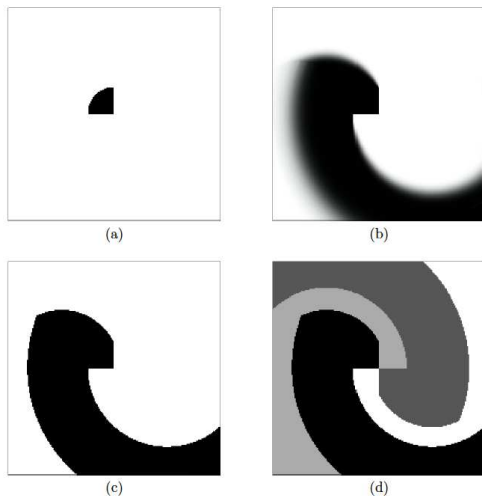


Figure: Some steps of the *Patchy Algorithm* (thanks to the authors)

Decomposition for Differential Games

(Deduced from F. Vinter, preprint 2014)

Let us consider, for an H not necessarily convex

$$\begin{cases} \lambda v(x) + H(x, Dv(x)) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \Gamma \end{cases}$$

Considered a **decomposition of the boundary** $\Gamma := \bigcup_{i \in \mathcal{I}} \Gamma_i$, with $\mathcal{I} := \{1, \dots, m\} \subset \mathbb{N}$, we call $v_i : \bar{\Omega} \rightarrow \mathbb{R}$ a **Lipschitz continuous viscosity solution** of the problem

$$\begin{cases} \lambda v_i(x) + H(x, Dv_i(x)) = 0 & x \in \Omega \\ v_i(x) = g_i(x) & x \in \Gamma \end{cases}$$

where the functions $g_i : \Gamma \rightarrow \mathbb{R}$ is a regular function such that

$$\begin{aligned} g_i(x) &= g(x), \text{ if } x \in \Gamma_i, \\ g_i(x) &> g(x), \text{ otherwise.} \end{aligned}$$

Define

$$I(x) := \{i \in \{1, \dots, m\} \mid v_i(x) = \min_j v_j(x)\},$$
$$\Sigma := \{x \in \mathbb{R}^N \mid \text{card}(I(x)) > 1\}.$$

Theorem

Assume the following condition satisfied: for arbitrary $x \in \Sigma$, **any convex combination** $\{\alpha_i \mid i \in I(x)\}$ and any collection of vectors $\{p_i \in \partial^L v_i(x) \mid i \in I(x)\}$ we have

$$\lambda \bar{v} + H \left(x, \sum_i \alpha_i p_i \right) \leq 0. \quad (\text{E})$$

Then, for all $x \in \mathbb{R}^N \setminus \mathcal{T}$,

$$v(x) = \bar{v}(x) := \min_i \{v_1(x), \dots, v_m(x)\}.$$

Example (I):

The equation considered is, with $\Omega := (-1, 1) \times (-1, 1)$,

$$\begin{cases} \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1 & x \in \Omega \\ v(x) = 0 & x_1 \in \partial\Omega; \end{cases}$$

(evidently the solution is the **distance function** from the boundary), we associate, considered to the easiest division $\partial\Omega = \Gamma := \cup_j \Gamma_j = \cup [(\pm 1, \pm 1), (\pm 1, \pm 1)]$, to the $\Gamma_1 := [(-1, -1), (-1, 1)]$ the function $g_1 : \partial\Omega \rightarrow \mathbb{R}$ defined as

$$\begin{cases} g_1(x) := 0 & x \in \Gamma_1 \\ g_1(x) := \gamma(1 + x_2) & x \in \Gamma \setminus \Gamma_1; \end{cases}$$

with $\gamma > 0$.

Example (II):

The **unique viscosity solution** of such a problem is

$$v_1(x) = (1 + \gamma) - \max(|x_1 - \gamma|, |x_2|)$$

and finally, the original value function $v(x) = 1 - \max(|x_1|, |x_2|)$ is recovered as $v(x) = \min_{i=1, \dots, 4} v_i(x)$.

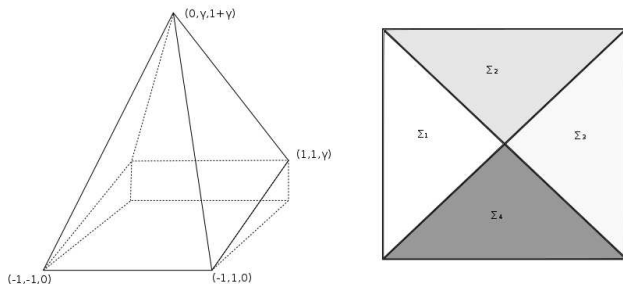


Figure: the auxiliary solution v_1 and the “regions of coincidence”.

Independent subdomains reconstruction

Differential Games Problem

Let the dynamics be given by

$$\begin{cases} \dot{y}(t) = f(y(t), a(t), b(t)), & a.e. \\ y(0) = x, \end{cases}$$

$x \in \Omega \subseteq \mathbb{R}^n$ open, $a, b \in \mathcal{A}, \mathcal{B} = \{\mathbb{R}^+ \rightarrow A, \text{ measurable}\}$, A, B compact sets. A solution is a **trajectory** $y_x(t, a(t), b(t))$.

The goal is to find the **sup – inf optimum** over \mathcal{A}, \mathcal{B} of

$$J_x(a, b) := \int_0^{\tau_x(a, b)} l(y_x(s, a(s), b(s)), a(s), b(s)) e^{-\lambda s} ds \\ + e^{-\lambda \tau_x(a, b)} g(y_x(\tau_x(a, b))), \quad \lambda \geq 0,$$

where $\tau_x(a, b) := \min \{t \in [0, +\infty) \mid y_x(t, a(t), b(t)) \notin \Omega\}$.

the value function of this problem is

$$v(x) := \sup_{\phi \in \Phi} \inf_{a \in \mathcal{A}} J_x(a, \phi(a)),$$

$\Phi := \{\phi : \mathcal{A} \rightarrow \mathcal{B} : t > 0, a(s) = \tilde{a}(s) \text{ for all } s \leq t$
implies $\phi[a](s) = \phi[\tilde{a}](s) \text{ for all } s \leq t\}$.

we will assume the *Isaacs' conditions* verified.

Theorem

The value function of the problem is a **viscosity solution** of the HJ equation associated with

$$H(x, p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} \{-f(x, a, b) \cdot p - l(x, a, b)\}.$$

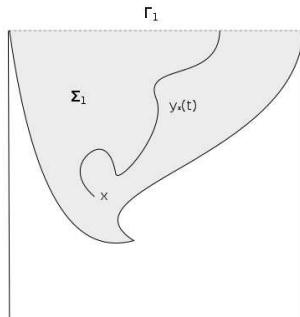
Independent Sub-Domains

Definition

A closed subset $\Sigma \subseteq \bar{\Omega}$ is an **independent sub-domain** of the problem (11) if, given a point $x \in \Sigma$ and an **optimal control** $(\bar{a}(t), \bar{\phi}(\bar{a}(t)))$

(i.e. $J_x(\bar{a}, \bar{\phi}(\bar{a})) \leq J_x(a, \bar{\phi}(a))$ for every choice of $a \in \mathcal{A}$, and $J_x(\bar{a}, \bar{\phi}(\bar{a})) \geq J_x(\bar{a}, \phi(\bar{a}))$ for every choice of $\phi \in \Phi$,

the trajectory $y_x(\bar{a}(t), \bar{\phi}(\bar{a}(t))) \in \Sigma$ for $t \in [0, \tau_x(\bar{a}, \bar{\phi}(\bar{a}))]$).



Independent Domains Decomposition

Proposition

Given a collection of $n - 1$ dimensional subsets $\{\Gamma_i\}_{i \in \mathcal{I}}$ such that $\Gamma = \cup_{i=1}^m \Gamma_i$, the sets defined as

$$\Sigma_i := \{x \in \bar{\Omega} \mid v_i(x) = v(x)\}, \quad i = 1, \dots, m$$

where v_i, v are defined accordingly to Theorem (1), are *independent sub-domains* of the original problem.

Proof.

By contradiction using the DPP. □

Numeric framework, SemiLagrangian scheme

Consider a **structured grid of simplices** S_j , such that $\bar{\Omega} \in \cup_j S_j$, denoting x_m , $m = 1, \dots, N$, the nodes of the triangulation,

$$\Delta x := \max_j \text{diam}(S_j)$$

G is *internal nodes*, ∂G *boundary points*, Ψ *ghost nodes*.

Mapping the values at the nodes onto $V = (V_1, \dots, V_N)$, it is possible to obtain the following scheme in **fixed point form**

$$V = T(V),$$

$$[T(V)]_i = \begin{cases} \max_{b \in B} \min_{a \in A} \left\{ \frac{1}{1+\lambda h} \mathbb{I}[V](x_i - hf(x_i, a, b)) - hl(x_i, a, b) \right\} & x_i \in G, \\ g(x_i) & x_i \in \partial G, \\ +\infty & x_i \in \Psi. \end{cases}$$

An estimate needed

Theorem

Let v and V be the solutions of the continuous and the discrete problem. Under some regularity hypotheses

$$\|v - V\|_{\infty} \leq C(\Delta x)^q,$$

where C is a positive constant independent from Δx , $q \in \mathbb{R}^+$ depending on the regularity of the problem.

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$$\|v - V\|_{\infty} \leq C(\Delta x)^q,$$

where C is a positive constant independent from Δx , $q \in \mathbb{R}^+$ depending on the regularity of the problem.

Examples in the SL case: differential games with a Lipschitz solution

$$\|v - V\|_{\infty} \leq Ch^{\frac{1}{2}} \left(1 + \left(\frac{\Delta x}{h} \right)^2 \right),$$

An estimate needed

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Examples in the SL case: differential games with a Lipschitz solution

$$\|v - V\|_{\infty} \leq Ch^{\frac{1}{2}} \left(1 + \left(\frac{\Delta x}{h} \right)^2 \right),$$

Optimal Control problem with $\lambda > 0$:

$$\|v - V\|_{\infty} \leq 2(M_v + M_{v_h})h^{\frac{1}{2}} + \left(\frac{L_l}{\lambda(\lambda - L_f)} \frac{\Delta x}{h} \right).$$

A necessary condition

Proposition

Assumed $x \in \Omega$ such that $v_i(x) = v(x)$ for a certain $i \in \mathbb{I}$, an $\epsilon \in [0, \Delta x)$ and a direction $d \in B(0, 1)$, such that, for a x_j node of the grid G , $x_j = x + \epsilon d$ and $v(x_j) < v_i(x_j)$; then

$$|V_i(x_j) - V(x_j)| \leq 2(C(\Delta x)^q + M\Delta x)$$

C as in the previous statement and $M := \max\{L_{v_i}, i \in \mathbb{I}\}$ where L_{v_i} is the Lipschitz constant of the function v_i .

Independent Subdomains Reconstruction

INDEPENDENT SUB-D. RECONSTRUCTION ALGORITHM (IRA).

- Given a collection of indices vectors such that $\text{union}(x_i, i = 1, \dots, m) = y$
 - 1) (Resolution of auxiliary problems)
for $i = 1 \dots m$ solve iteratively the problem
 $V_i = T_i(V_i)$ with $\partial G := \{x_j | j \in y_i\}$
end
 - 2) (Reconstruction of the value function)
obtain V as $V = \min_{i=1 \dots m} \{V_i\}$
 - 3) (Reconstruction of the Sub-Domains)
for $i=1 \dots m$
 initialize $w_i = \emptyset$
 for $j=1 \dots N$
 if $\|V_i(j) - V(j)\| \leq \varepsilon(\Delta x)$
 then add j to vector w_i
 end
 the index set of the i -subset is w_i
end
-

Some Observations

(IRA) builds an approximation of the **independent sub-domains**.
Moreover we have:

- ▶ the approximation **exceeds the desired** set, i.e. $\Sigma_i \subseteq \bar{\Sigma}_i$.
- ▶ The same relation holds for the **projection of the approximation** on a finer grid, i.e. for $\Delta x_1 \geq \Delta x_2$,

$$Proj_{\Delta x_2} \bar{\Sigma}_i^{\Delta x_1} \supseteq \Sigma_i$$

Example of Reconstruction (I)

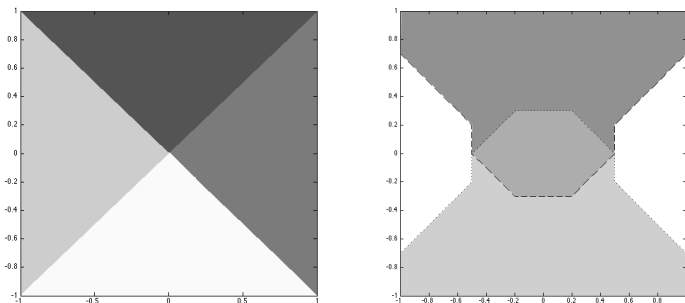


Figure: Distance function: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of 15^2 points.

Example of Reconstruction (II)

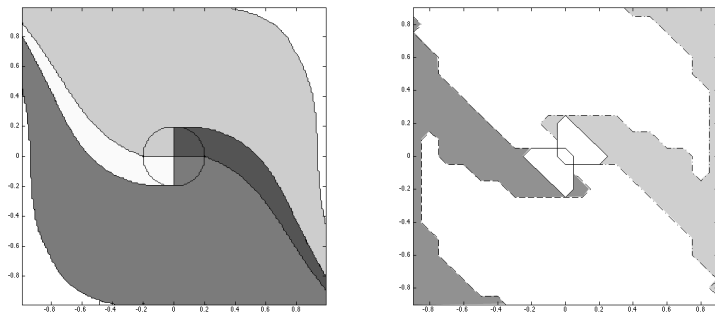


Figure: Van Der Pol: Exact decomposition and two (of the four) approximated independent subsets with a course grind of 15^2 points.

Use for Parallel Computing

Consider:

- ▶ a **coarse grid** K of discretization step Δx_K
- ▶ a **fine grid** G of step Δx_G

on the domain Ω .

The two triangulations are chosen such that $\Delta x_G \ll \Delta x_K$.

Independent Sub-Domain Parallelization

INDEPENDENT-SETS ALGORITHM (ISA).

- Given a collection of indices vectors such that $\text{union}(z_i, i = 1, \dots, M) = y$
 - 1) **(Reconstruction of the approximated independent sub-domains)**
Using IRA get a collection $w_i, i = 1, \dots, M$ of indexes sets.
 - 2) **(Projection on the fine grid)**
Extend indexes sets w_i on the fine grid G_j getting W_j
 - 3) **(Resolution on the fine grid)**
for $i = 1, \dots, M$ solve iteratively the problem
$$V_i = T(V_i)$$
end
 - 4) **(Assembly of the final Solution)**
for $j = 1, \dots, N_2$
$$\bar{V}(x_j) = \min\{V_i \text{ s.t. } j \in w_i\}$$
end
-

Convergence Result:

Proposition

Called \bar{V} the **exact discrete solution** of the (ISA) algorithm (i.e. all $V_i = T_i(V_i)$ are verified exactly) and v the exact solution of the **original continuous problem**, then, for a $C > 0$, independent from Δx , the following estimate holds

$$\|\bar{V} - v\|_{\infty} \leq C(\Delta x_G)^q.$$

Convergence Result:

Proposition

Called \bar{V} the **exact discrete solution** of the (ISA) algorithm (i.e. all $V_i = T_i(V_i)$ are verified exactly) and v the exact solution of the **original continuous problem**, then, for a $C > 0$, independent from Δx , the following estimate holds

$$\|\bar{V} - v\|_{\infty} \leq C(\Delta x_G)^q.$$

Proof.

We know that the inclusion property holds. Then for $x_j \in G$, it is assured the existence of $i \in \{1, \dots, M\}$ s.t. $v(x_j) = v_i(x_j)$, and $|\bar{V}(x_j) - V_i(x_j)| = 0$. Then

$$|\bar{V}(x_j) - v(x_j)| \leq |\bar{V}(x_j) - V_i(x_j)| + |V_i(x_j) - v(x_j)| \leq \|V_i - v_i\|_{\infty} \leq C(\Delta x_G)^q$$

for the arbitrariness of the choice of x_j we have the thesis. \square

A convergence result for a domain partition:

Corollary

Let us consider *a partition of the grid K* in subsets B_i , $i = 1, \dots, M$ such that for every $i = 1, \dots, M$

- $Z_i \subseteq B_i$
- $\forall z_j \in B_i, |V_i(j) - V(j)| \leq 2(C(\Delta x_K)^q + M\Delta x_K)$
- each B_i is connected (for each $z_j \in B_i$, there exists a direction d s.t. for $r \in [0, \Delta x_K]$, $z_j + dr = z_j \in B_i$ with $j \neq \bar{j}$)

then the following estimate for $\hat{V} := \min\{V_i(j) | x_j \in B_i\}$, where the V_i is calculated on the projection on G of the set B_i :

$$\|\hat{V} - v\|_\infty \leq C\Delta x_G + 2(C(\Delta x_K)^q + M\Delta x_K)$$

Tests: Distance function

Eikonal equation on the set $\Omega := (-1, 1)^2$ with the **boundary value** fixed to zero on $\Gamma := \partial\Omega$.

$\lambda = 1$ (non linear monotone scaling of the solution);

the correct viscosity solution is

$$v(x) + \max_{a \in B(0,1)} \{a \cdot Dv(x)\} = 1, \quad v(x) = 1 - \frac{\min\{e^{|x_1|}, e^{|x_2|}\}}{e}.$$

N. of variables	Δx	Time elapsed	$\max_i \bar{\Sigma}_i / \Omega $	$\max_i \Sigma_i / \Omega $
5^2	0.4	$1 \cdot 10^{-3}$	50%	25%
7^2	0.28	$2 \cdot 10^{-3}$	43%	
10^2	0.2	$4 \cdot 10^{-3}$	38%	
15^2	0.133	$2 \cdot 10^{-2}$	35%	
20^2	0.1	$5 \cdot 10^{-2}$	33%	
40^2	0.05	3	29%	
50^2	0.04	11	28.3%	

Tests: Distance function

Table: Time necessary for the resolution on the original domain and the reduced one with various discretization steps (15^2 (IRA))

N. of variables	Δx	Time on Ω	Time on $\max_j \bar{\Sigma}_j $
25^2	0.08	0.13	0.035
50^2	0.04	7.02	0.68
75^2	0.026	57.5	4.8
100^2	0.02	$1.5 \cdot 10^3$	16.6
200^2	0.01	—	$3 \cdot 10^3$

Table: Approximation error Error in norm Δ_∞ (and Δ_1)

	50^2	100^2	200^2
original	$1.2 \cdot 10^{-2} (1.1 \cdot 10^{-2})$	$6.5 \cdot 10^{-3} (3.6 \cdot 10^{-3})$	$2.5 \cdot 10^{-3} (1.6 \cdot 10^{-3})$
2-subs.	$1.2 \cdot 10^{-2} (7.2 \cdot 10^{-3})$	$6.5 \cdot 10^{-3} (3.7 \cdot 10^{-3})$	$2.5 \cdot 10^{-3} (1.4 \cdot 10^{-3})$
4-subs.	$9 \cdot 10^{-3} (7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3} (3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3} (1.3 \cdot 10^{-3})$
8-subs.	$9 \cdot 10^{-3} (7.2 \cdot 10^{-3})$	$4.6 \cdot 10^{-3} (3.6 \cdot 10^{-3})$	$1.4 \cdot 10^{-3} (1.3 \cdot 10^{-3})$

Tests: Van Der Pool Oscillator

We consider $\Gamma := \partial B(0, \rho)$ (in this case $\rho = 0.2$) and $\Omega := (-1, 1)^2 \setminus \bar{B}(0, \rho)$. The **nonlinear system** will be:

$$f(x, a) = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 + a \end{pmatrix}.$$

The others parameters of the system are:

$$A = [-1, 1], \quad \lambda = 1, \quad l(x, y, a) = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad g(x) \equiv 0.$$

Table: Van Der Pol: Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	Δx	Time elapsed	$\max_i \bar{\Sigma}_i / \Omega $	$\max_i \Sigma_i / \Omega $
5^2	0.4	$1.4 \cdot 10^{-3}$	62%	42.2%
10^2	0.2	0.011	55%	
20^2	0.1	0.103	47%	
30^2	0.06	1.47	45%	
40^2	0.05	5.6	44.6%	
50^2	0.04	16.3	44.1%	

Tests: Van Der Pool Oscillator

Table: Van Der Pol: Approximation error Error in norm Δ_∞ (and Δ_1)

	50^2	100^2	200^2
original	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
2-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
4-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$
8-subsets	0.09(0.07)	0.03(0.01)	$0.01(6 \cdot 10^{-3})$

Tests: A Pursuit-Evasion game

We will consider a **Pursuit evasion game**, where two agents have the opposite goal to reduce/postpone the time of capture.

The dynamics considered are the following:

$$f(x, a, b) := \begin{pmatrix} f_1(x)(a_1 - b_1) \\ f_2(x)(a_2 - b_2) \end{pmatrix}$$

where the functions f_1, f_2 are $f_1(x) := x_2 + 1$ and $f_2(x) := 1$.
The running cost $l(x, a, b) := x_1 + 0.1$

Controls are $A = B(0, 1)$ for the pursuer and $B = B(0, 1/2)$ for the evader.

Capture happens trajectory is driven to touch $B(0, 0.2)$, then
 $\Gamma := \partial B(0, 0.2)$

Tests: A Pursuit-Evasion game

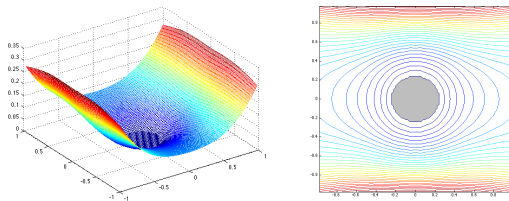


Figure: Approximated value function

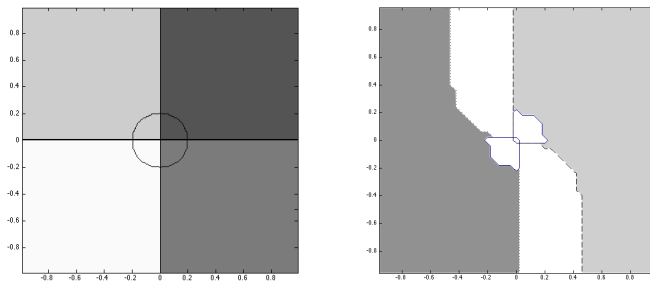


Figure: Exact decomposition and two (of the four) approximated independent subsets found with a course grind of 40^2 points.

Tests: A Pursuit-Evasion game

Table: Comparison of the accuracy of the decomposition with various discretization steps

N. of variables	Δx	Time elapsed	$\max_i \bar{\Sigma}_i / \Omega $	$\max_i \Sigma_i / \Omega $
5^2	0.4	10^{-3}	60%	
10^2	0.2	0.008	46%	
30^2	0.06	1.38	38%	25%
50^2	0.04	15.9	36.1%	

The HJ equation associated verifies the **decomposability condition (E)**; because considering the norm (it is possible because $|f_i(x)| > 0$ for $i = 1, 2$)

$$\|p\|_* := \max_{a \in B(0,1)} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} a^T \cdot p$$

the Hamiltonian associated is equivalent to

$$H(x, p) := \|p\|_* - \frac{\|p\|_*^2}{2} - (x_1^2 + 0.1)$$

evidently convex everywhere with respect to p .




Independent Domain Parallelization: Summary

- ▶ The gain is variable, but can be **considerable high** (no communication between the sub-domains)
- ▶ There is a direct dependence of the technique on the **initial decomposition** of the boundary
- ▶ Convergence for $\Delta_G \rightarrow 0^+$ is guaranteed and the error in the approximation **is bounded** by an estimate
- ▶ It is proved also **convergence for a partition decomposition** (as the *Patchy Dec.*) but it is guaranteed a convergence only for $\Delta_K \rightarrow 0^+$, and relative estimation of the error.



Conclusions and Perspectives

- ▶ We have shown a **constructive manner to obtain a decomposition of the domain** of a HJ eq. verifying (E) , in **independent subdomains**, subsets which have the properties of being computed, independently from each others.
- ▶ A critical point is **the estimation of the function $\varepsilon(\cdot)$** .
- ▶ The critical occurrence about the **balance of the dimension of the subsets** can be solved with a recursive refinement of the division of Γ .
- ▶ A more serious limit appears in **presence of flat regions**: in that case it is, for the moment, impossible to obtain a satisfactory reduction of the dimension of the decomposed domains without solving the problem on a sufficiently fine grid.

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Thank you.