From discrete microscopic models to macroscopic models and applications to traffic flow

Nicolas Forcadel, Wilfredo Salazar

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Joint work with W. Salazar

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Plan

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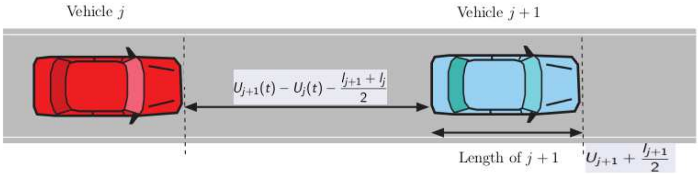
Bando model

- Second order discrete model with $n_0 \in \mathbb{N}$ types of vehicles :

$$\ddot{U}_j(t) = a_j \left(V_j \left(U_{j+1}(t) - U_j(t) - \frac{l_{j+1} + l_j}{2} \right) - \dot{U}_j(t) \right). \tag{1}$$



- U_j: position of the vehicle j.
- a_j: sensibility of the driver j ($a_{j+n_0} = a_j$).
- V_j: Optimal velocity function (OVF) of the driver j ($V_{j+n_0} = V_j$).

Passing from micro to macro

- Goal : Describe the traffic in term of density of vehicles, i.e. passing from the microscopic model to a macroscopic one.
- LWR macroscopic model:

$$\rho_t + (\rho v(\rho))_x = 0$$

where v is the average speed of vehicles.

Existing results for the Frenkel-Kontorova model

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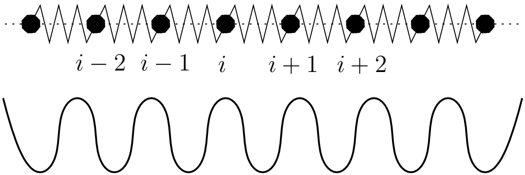
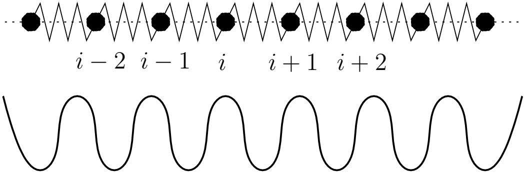


$$m\frac{d^2U_j}{dt^2}+\gamma\frac{dU_j}{dt}=(U_{j+1}-U_j)+(U_{j-1}-U_j)+\sin(2\pi U_j)+f$$

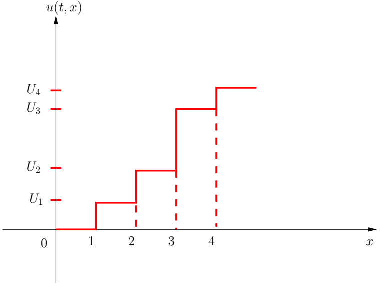
Existing results for the Frenkel-Kontorova model



$$m\frac{d^2U_j}{dt^2} + \gamma\frac{dU_j}{dt} = F_j(U_{j-m}, \dots, U_{j+m})$$

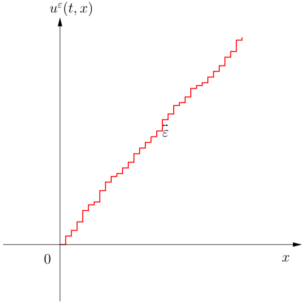
Homogenization

$$u(t,x) = U_{\lfloor x \rfloor}(t)$$



Rescalling

$$u^{\varepsilon}(t,x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$



$$u^{\varepsilon} \to ?$$

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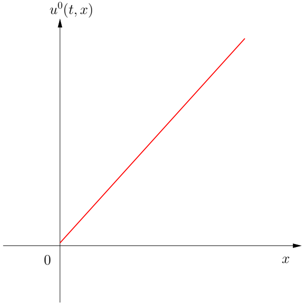

$$u^\varepsilon \to u^0$$

Homogenization results

For $\varepsilon > 0$, we define

$$\begin{cases} u^\varepsilon(t,x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \\ u^\varepsilon(0,x) = u_0(x) \end{cases}$$

Theorem (F., Imbert, Monneau)

Under certain assumptions on u_0, we have $u^\varepsilon \to u^0$ with

$$\begin{cases} u^0_t = \bar{F}(u^0_x), \\ u^0(0,x) = u_0(x) \end{cases}$$

- The density of particles $\rho = \frac{1}{u^0_x}$ satisfies formally a conservation law

$$\rho_t = (\bar{H}(\rho))_x \quad \text{with} \quad \bar{H}(\rho) = -\rho\bar{F}(1/\rho)$$

Ingredients of the proof

- Main idea: inject the system of particles into a (system of) PDE.
- If m is small, the system of PDE satisfies a comparison principle
- Order of the particles (link between the F_j).
- Notion of hull functions

Plan

Change of variables

$$\ddot{U}_j(t) = a_j\left(V_j\left(U_{j+1}(t) - U_j(t)\right) - \dot{U}_j(t)\right).$$

- We set

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha}\dot{U}_j(t) \quad \text{with} \quad \alpha = \frac{1}{2}\min_{j\in\{1,\dots,n_0\}}(a_j).$$

Then

$$\begin{cases} \dot{U}_j(t) = \alpha\left(\Xi_j(t) - U_j(t)\right) \\ \dot{\Xi}_j(t) = (a_j - \alpha)(U_j(t) - \Xi_j(t)) + \dfrac{a_j}{\alpha}V_j\left(U_{j+1}(t) - U_j(t)\right). \end{cases}$$

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- This system is monotone if $a_j \geq 4\|V'\|_\infty$

Injection in a system of PDE

For $j \in \mathbb{Z}$, pour tout $(t,x) \in (0,+\infty) \times \mathbb{R}$,

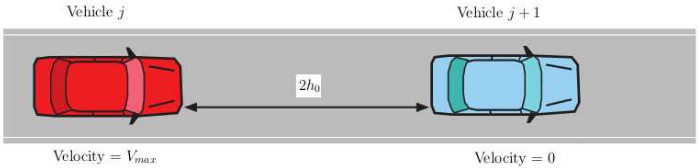
$$u_j(t,x) = U_{j+n_0\lfloor x \rfloor}(t) \quad \text{and} \quad \xi_j(t,x) = \Xi_{j+n_0\lfloor x \rfloor}(t).$$

Then

$$\begin{cases} \dfrac{\partial u_j}{\partial t} = \alpha(\xi_j(t,x) - u_j(t,x)) \\[2ex] \dfrac{\partial \xi_j}{\partial t} = (a_j - \alpha)(u_j - \xi_j) + \dfrac{a_j}{\alpha} V_j\left(u_{j+1} - u_j\right) \\[2ex] u_{j+n_0}(t,x) = u_j(t,x+1) \\ \xi_{j+n_0}(t,x) = \xi_j(t,x+1). \end{cases} \tag{2}$$

Ordering the vehicles

- We assume that $V_j(h) = 0$ for $h \leq 2h_0$ and we consider the worst case



- Using the solution of this ode as barrier, we get $U_{j+1} \geq U_j + h_0$ for a good choice of h_0
- This implies that $\Xi_{j+1} \geq \Xi_j$

Notion of hull function

Hull function h

$$\begin{cases} \dfrac{\partial u_j}{\partial t} = \alpha(\xi_j(t,x) - u_j(t,x)) \\ \dfrac{\partial \xi_j}{\partial t} = (a_j - \alpha)(u_j - \xi_j) + \dfrac{a_j}{\alpha} V_j\left(u_{j+1} - u_j\right) \end{cases}$$

- We search particular solutions of the form

$$u_j(t,x) = h_j(\lambda t + px), \quad \xi_j(t,x) = g_j(\lambda t + px)$$

$(h_j, g_j) =$ hull functions
$\lambda =$ mean velocity
$\frac{n_0}{p} =$ mean density

$$\frac{U_{i+n_0 l} - U_i}{l} \to p \quad \text{as} \quad l \to +\infty$$

Equation of the hull functions

$$\begin{cases} \lambda = \alpha(g_j(z) - h_j(z)) \\ \lambda = (a_j - \alpha)(h_j(z) - g_j(z)) + \dfrac{a_j}{\alpha} V_j \left(h_{j+1}(z) - h_j(z)\right) \end{cases}$$

Existence of hull functions

Theorem (F., Salazar)

For every $p \in (0, +\infty)$, there exists a unique $\lambda := \overline{F}(p)$ for which there exists hull functions (h_j, g_j). Moreover, (h_j, g_j) can be constructed such that

$$h_j(z) = z + h_j(0), \quad g_j(z) = z + g_j(0).$$

Properties of λ

Theorem (F., Salazar)

- *Monotonicity :* λ *is non-decreasing*
- *Upper boundary :* $\lim_{p\to+\infty} \lambda(p) = \min_{j\in\{1,\ldots,n_0\}} ||V_j||_\infty$
- *zero velocity : If* $p \leq 2h_0n_0$, *then* $\lambda = 0$

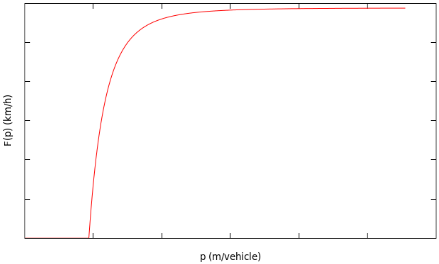


Figure : Schematic representation of the effective Hamiltonian.

Construction of the hull functions

Existence of hull functions

Theorem (F., Salazar)

For every $p > 0$, there exists some $(\bar{u}_j^\infty, \bar{\xi}_j^\infty)$ such that there exists a unique $\lambda =: \bar{F}(q)$ such that

$$\left(\left(u_j^\infty(t,x)\right)_j, \left(\xi_j^\infty(t,x)\right)_j\right) = \left(\left(px + \lambda t + \bar{u}_j^\infty\right)_j, \left(px + \lambda t + \bar{\xi}_j^\infty\right)_j\right),$$

is a solution of (2). Moreover, if we define (h_j, g_j) such that

$$u_j^\infty(t,x) = h_j(\lambda t + qx), \quad \text{and} \quad \xi_j^\infty(t,x) = g_j(\lambda t + qx)$$

then (h_j, g_j) is a hull function and satisfies

$$h_j(z) = z + h_j(0) \quad \text{and} \quad g_j(z) = z + g_j(0).$$

Ideas to construct h

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$$|u_j(t,0) - \lambda t| \leq C \quad \text{and} \quad |\xi_j(t,0) - \lambda t| \leq C$$

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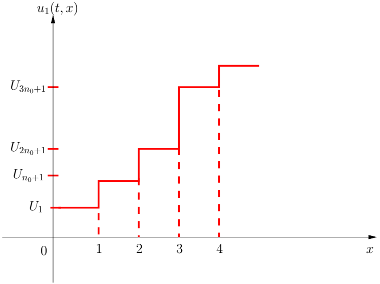
5. **(Translation at infinity)** By considering $(u_j(t+n,y) - \lambda n, \xi_j(t+n,y) - \lambda n)$ as $n \longrightarrow +\infty$, one can construct global in time solution $(u_j^\infty, \xi_j^\infty)$

Homogenization results

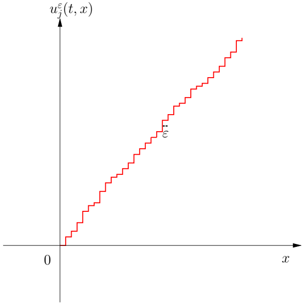
Homogenization

$$u_j(t,x) = U_{j+n_0\lfloor x \rfloor}(t)$$



Rescalling

$$u_j^\varepsilon(t,x) = \varepsilon u_j\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$



$$u_j^\varepsilon \to ?$$

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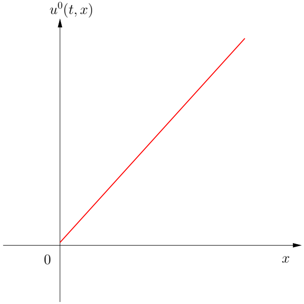

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Theorem (F., Imbert, Monneau)

Under certain assumptions on u_0, we have $u_j^\varepsilon \to u^0$ and $\xi_j^\varepsilon \to u^0$ with

$$\begin{cases} u_t^0 = \bar{F}(u_x^0), \\ u^0(0,x) = u_0(x) \end{cases}$$

- Ansatz for the proof :

$$u_j^\varepsilon(t,x) \simeq \varepsilon h_j\left(\frac{u^0(t,x)}{\varepsilon}\right), \quad \xi_j^\varepsilon(t,x) \simeq \varepsilon g_j\left(\frac{u^0(t,x)}{\varepsilon}\right)$$

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Formal idea of the proof

- $v_j^\varepsilon(t,x) := u^0(t,x) + \varepsilon h_j(0), \quad w_j^\varepsilon(t,x) := u^0(t,x) + \varepsilon g_j(0)$

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- We then have very formally

$$(v_j^\varepsilon)_t(t,x) = u_t^0 = \bar{F}(u_x^0) = \alpha(g_j(0) - h_j(0)) = \frac{\alpha}{\varepsilon}(v_j^\varepsilon - w_j^\varepsilon)$$

$$\begin{aligned}(w_j^\varepsilon)_t(t,x) =& u_t^0 = \bar{F}(u_x^0)\\ =& (a_j - \alpha)(h_j(0) - g_j(0)) + \frac{a_j}{\alpha} V_j(h_{j+1}(0) - h_j(0))\\ =& (a_j - \alpha)\frac{v_j^\varepsilon - w_j^\varepsilon}{\varepsilon} + \frac{a_j}{\alpha} V_j\left(\frac{v_{j+1}^\varepsilon - v_j^\varepsilon}{\varepsilon}\right)\end{aligned}$$

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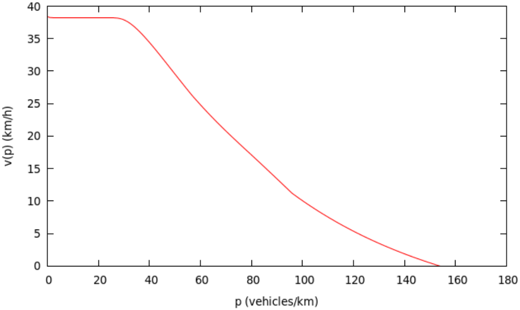
- $(v_j^\varepsilon, w_j^\varepsilon)$ and $(u_j^\varepsilon, \xi_j^\varepsilon)$ satisfy (almost) the same equation, so $u_j^\varepsilon \simeq v_j^\varepsilon$ and $\xi_j^\varepsilon \simeq w_j^\varepsilon$.

An example of computation of $\overline{F}$ (W. Salazar)



Conclusions and Perspectives

- Conclusions :
  - Homogenization results for discrete traffic flow models
  - This allows to add microscopic phenomena in the modeling (red light, car crashes,....)
- Perspectives :
  - Study of car crashes, red light
  - Homogenization on networks
  - Numerical computation of $\overline{F}$
  - Homogenization in random media