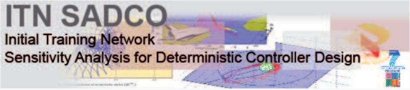
OPTEC



Indefinite Linear MPC and Approximated Economic MPC for Nonlinear Systems

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NetCo conference, Tours 2014

Outline

1. Economic MPC and Stability Analysis

2. The Linear Quadratic Case

3. Approximated EMPC with stability guarantees

MPC

Optimal Control Problem

$$\min_{x_0, u_0, \ldots, x_N}$$

$$\text{s.t.}$$

MPC

Optimal Control Problem

$$\min_{x_0, u_0, \ldots, x_N}$$

$$\text{s.t.} \quad x_0 - \bar{x}_i = 0, \qquad \text{Initial condition}$$

MPC

Optimal Control Problem

$$\begin{aligned} \min_{x_0, u_0, \ldots, x_N} \quad & \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, && \text{Initial condition} \\ & x_{k+1} - f(x_k, u_k) = 0, && \text{System dynamics} \end{aligned}$$

MPC

Optimal Control Problem

$$
\begin{aligned}
\min_{x_0, u_0, \ldots, x_N} & \\
\text{s.t.} \quad & x_0 - \bar{x}_i = 0, && \text{Initial condition} \\
& x_{k+1} - f(x_k, u_k) = 0, && \text{System dynamics} \\
& x_N - x_N^s = 0. && \text{Terminal constraint}
\end{aligned}
$$

MPC

Optimal Control Problem

$$\begin{aligned}
\min_{x_0, u_0, \ldots, x_N} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) && \text{(Quadratic) stage cost} \\
\text{s.t.} \quad & x_0 - \bar{x}_i = 0, && \text{Initial condition} \\
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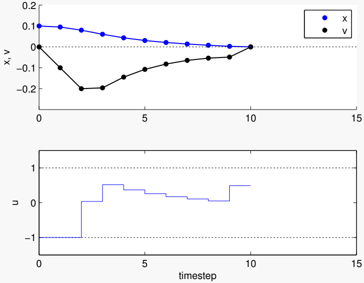


MPC

Optimal Control Problem

$$\min_{x_0, u_0, \ldots, x_N} \sum_{k=0}^{N-1} l(x_k, u_k)$$ (Quadratic) stage cost

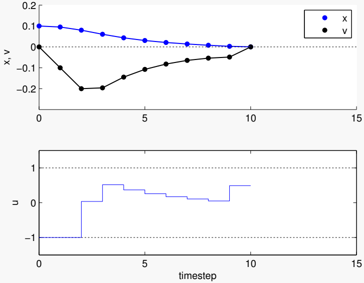
$$\text{s.t.} \quad x_0 - \bar{x}_i = 0,$$ Initial condition

$$x_{k+1} - f(x_k, u_k) = 0,$$ System dynamics

$$x_N - x_N^s = 0.$$ Terminal constraint

At each sampling time i:

- get the initial state $\bar{x}_i$
- solve the MPC OCP
- apply the first control $u_0^\star$



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$$\text{s.t.} \quad x_0 - \bar{x}_i = 0, \qquad \text{Initial condition}$$

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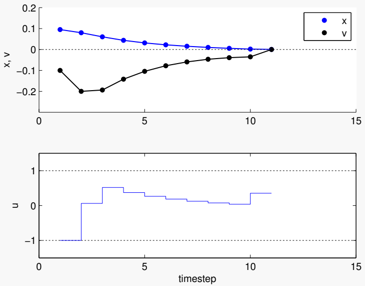

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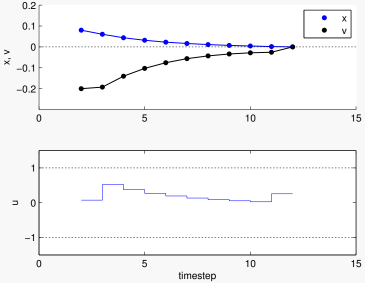

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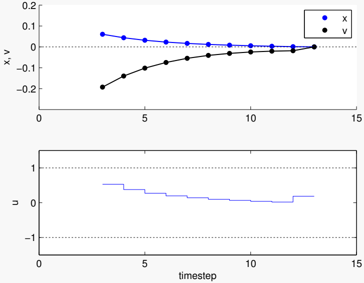

MPC

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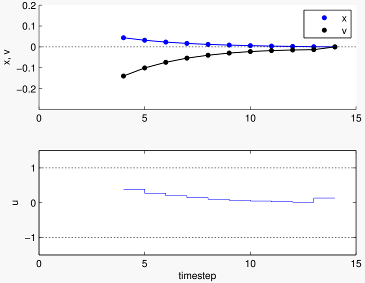
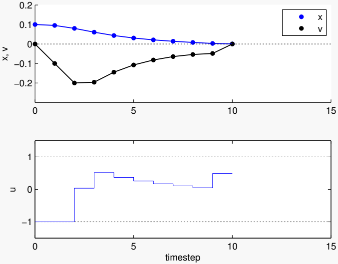


Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0 \quad \text{and} \quad \exists\ \alpha \in \mathcal{K}$ s.t. $\alpha(x - x_s) \leq l(x, u),\ \forall u \in \mathbb{U}$

Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\, \alpha \in \mathcal{K}$ s.t. $\alpha(x - x_s) \leq l(x, u),\ \forall u \in \mathbb{U}$



$V_N(\bar{x}_i)$

Lyapunov stability

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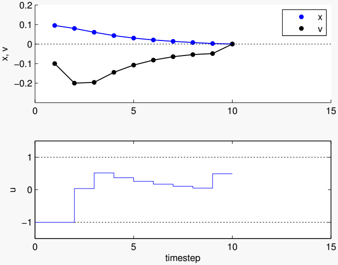

$$V_{N-1}(f(\bar{x}_i, u_0^\star))$$

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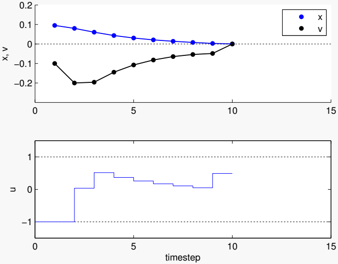

$$V_{N-1}(f(\bar{x}_i, u_0^\star))$$

$$V_{N-1}(f(\bar{x}_i, u_0^\star)) = V_N(\bar{x}_i) - l(\bar{x}_i, u_0^\star)$$

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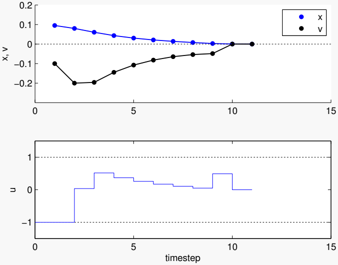
Tracking MPC: $l(x_s, u_s) = 0$ and $\exists\ \alpha \in \mathcal{K}$ s.t. $\alpha(x - x_s) \leq l(x, u),\ \forall u \in \mathbb{U}$



$$V_{N-1}(f(\bar{x}_i, u_0^\star)) + l(x_s, u_s)$$

$$\begin{aligned} V_{N-1}(f(\bar{x}_i, u_0^\star)) &= V_N(\bar{x}_i) - l(\bar{x}_i, u_0^\star) \\ & V_N(\bar{x}_i) - l(\bar{x}_i, u_0^\star) + \underbrace{l(x_s, u_s)}_{=0} \end{aligned}$$

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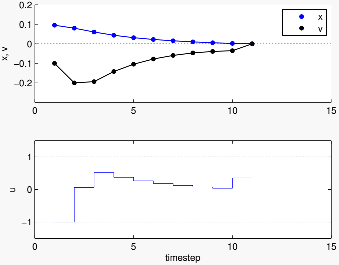

$$V_N(f(\bar{x}_i, u_0^\star))$$

$$V_{N-1}(f(\bar{x}_i, u_0^\star)) = V_N(\bar{x}_i) - l(\bar{x}_i, u_0^\star)$$
$$V_N(f(\bar{x}_i, u_0^\star)) \leq V_N(\bar{x}_i) - l(\bar{x}_i, u_0^\star) + \underbrace{l(x_s, u_s)}_{=0}$$

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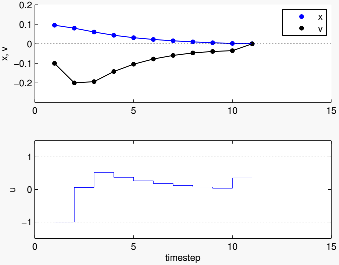

$$V_N(f(\bar{x}_i, u_0^\star))$$

$$\begin{aligned} V_{N-1}(f(\bar{x}_i, u_0^\star)) &= V_N(\bar{x}_i) - l(\bar{x}_i, u_0^\star) \\ V_N(f(\bar{x}_i, u_0^\star)) &\leq V_N(\bar{x}_i) - l(\bar{x}_i, u_0^\star) + \underbrace{l(x_s, u_s)}_{=0} \end{aligned}$$

$$V_N(f(\bar{x}_i, u_0^\star)) - V_N(\bar{x}_i) \leq -l(\bar{x}_i, u_0^\star)$$

Do we always want to track?

Do we always want to track?

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No!

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No!

Then why do we track?

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- It works

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- We have been doing it since the 80s

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What about Economic MPC?

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What about Economic MPC?

- Difficult to prove stability (2008-)

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Then why do we track?

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What about Economic MPC?

- Difficult to prove stability (2008-)
- Increased “economic” gain

Economic vs Tracking

Economic vs Tracking

- Steady state: $x_\mathrm{s} = 0 = f(0,0)$

Economic vs Tracking

- Steady state: $x_s = 0 = f(0, 0)$
- Stage cost: **Tracking** vs **Economic**

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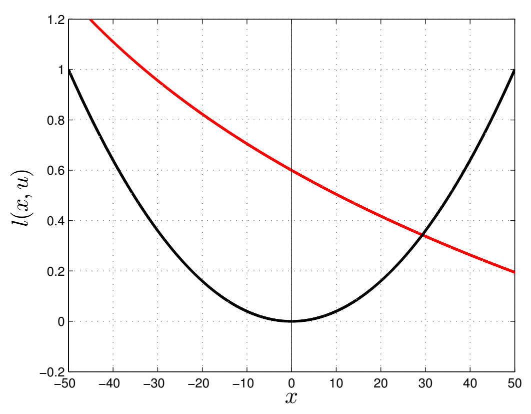

Economic vs Tracking

- Steady state: $x_\mathrm{s} = 0 = f(0,0)$
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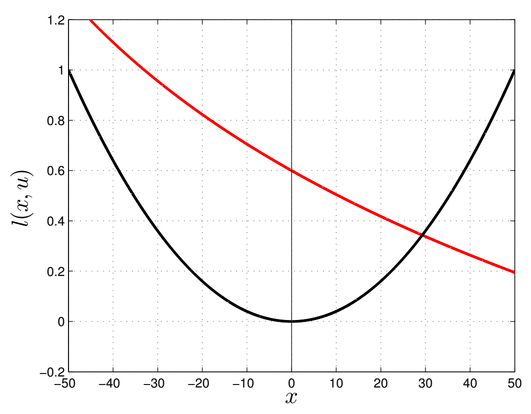


The classical stability theory does not apply!

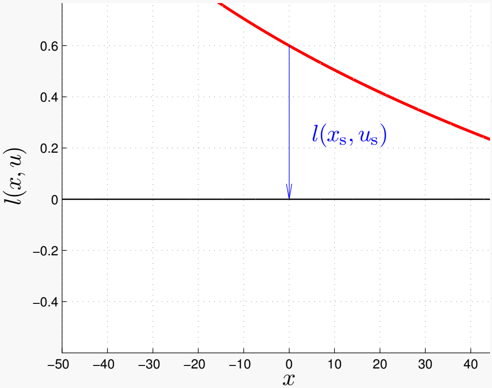
Economic Stage Cost

Steady state: $$(x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} l(x,u) \text{ s.t. } x = f(x,u)$$

Economic Stage Cost

$$\text{Steady state:} \quad (x_s, u_s) = \min_{x,u} l(x,u) \text{ s.t. } x = f(x,u)$$

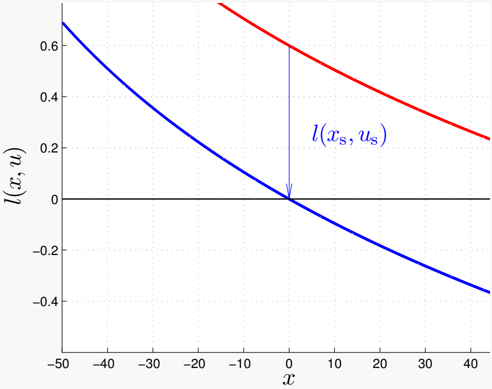
$l(x,u)$ $\quad$ $l(x_s,u_s)$



Economic Stage Cost

$$\text{Steady state:} \quad (x_{\mathrm{s}}, u_{\mathrm{s}}) = \min_{x,u} l(x,u) \text{ s.t. } x = f(x,u)$$

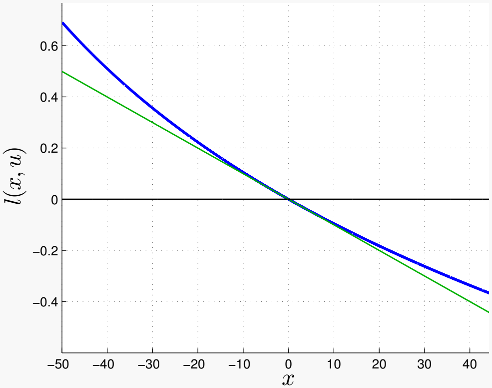
$$l(x,u) - l(x_{\mathrm{s}}, u_{\mathrm{s}})$$



Economic Stage Cost

Steady state: $(x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} l(x,u)$ s.t. $x = f(x,u)$

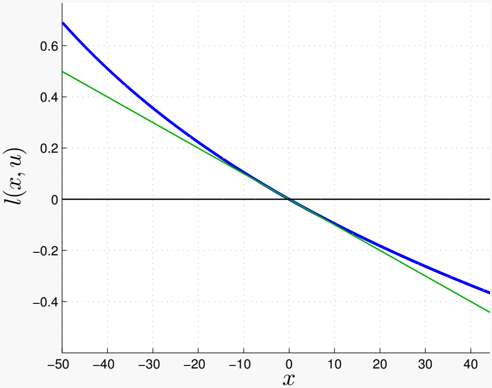
$l(x,u) - l(x_\mathrm{s}, u_\mathrm{s})$



Economic Stage Cost

$$\text{Steady state:} \quad (x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} l(x,u) \text{ s.t. } x = f(x,u)$$

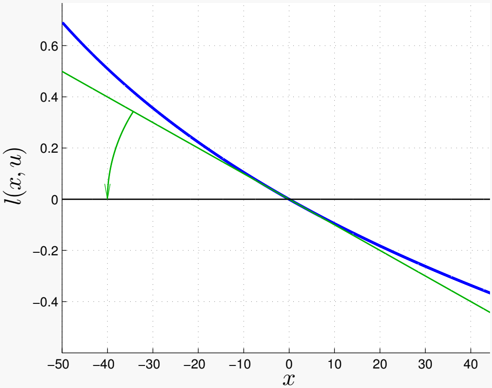
$$l(x,u) - l(x_\mathrm{s}, u_\mathrm{s}) \qquad \underbrace{\lambda_\mathrm{s}^T}_{\text{Lagrange multiplier}} \qquad (x - f(x,u))$$



Economic Stage Cost

$$\text{Steady state:} \quad (x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} l(x,u) \text{ s.t. } x = f(x,u)$$

$$l(x,u) - l(x_\mathrm{s}, u_\mathrm{s}) + \underbrace{\lambda_\mathrm{s}^T}_{\text{Lagrange multiplier}} (x - f(x,u))$$



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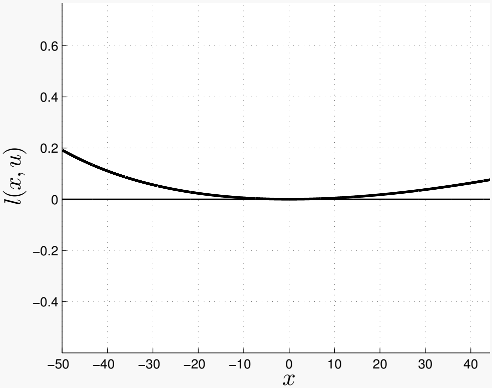
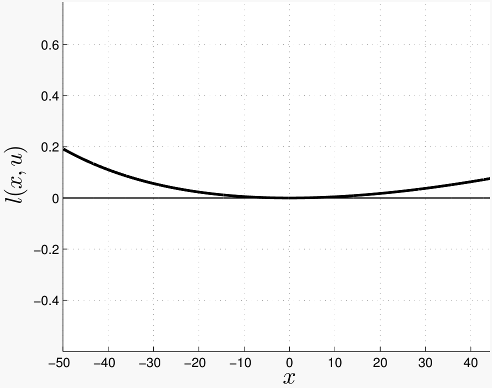


Economic Stage Cost

Steady state: $(x_\mathrm{s}, u_\mathrm{s}) = \min_{x,u} l(x,u)$ s.t. $x = f(x,u)$

$\underbrace{\textbf{Rotated cost}}_{\text{[Diehl et al. 2011]}}$: $L(x,u) = l(x,u) - l(x_\mathrm{s}, u_\mathrm{s}) + \underbrace{\lambda_\mathrm{s}^T}_{\text{Lagrange multiplier}} (x - f(x,u))$



Rotated MPC Problem

$$
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\end{aligned}
$$

Rotated Problem $\equiv$ Original Problem

$$
\sum_{k=0}^{N-1} L(x, u) = \sum_{k=0}^{N-1} l(x, u) + \underbrace{\lambda_{\mathrm{s}}^T x_0 - \lambda_{\mathrm{s}}^T x_N - (N-1)\, l(x_{\mathrm{s}}, u_{\mathrm{s}})}_{\text{constant}}
$$

Rotated MPC Problem

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Rotated Problem $\equiv$ Original Problem

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If

$$\alpha(x) \leq L(x, u), \qquad \alpha \in \mathcal{K}$$

the previous stability proof holds! [Diehl et al. 2011]

Generalization [Amrit et al. 2011]

Nonlinear rotating function: $\lambda(x)$

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New rotated cost: $L(x, u) = l(x, u) - l(x_{\mathrm{s}}, u_{\mathrm{s}}) + \lambda(x) - \lambda(f(x, u))$

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What conditions on $\lambda(x)$?

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What conditions on $\lambda(x)$?

Strict Dissipativity

System strictly dissipative wrt the supply rate $s(x,u)$ if $\exists\ \lambda(x) : \mathbb{X} \to \mathbb{R}$

$$\lambda(f(x,u)) - \lambda(x) \le -\rho(x - x_{\mathrm{s}}) + s(x,u), \qquad \forall\ (x,u) \in \mathbb{X} \times \mathbb{U}$$

We are interested in $s(x,u) = l(x,u) - l(x_{\mathrm{s}}, u_{\mathrm{s}})$

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We are interested in $s(x,u) = l(x,u) - l(x_{\mathrm{s}}, u_{\mathrm{s}})$

This entails

$$L(x,u) = l(x,u) - l(x_{\mathrm{s}}, u_{\mathrm{s}}) + \lambda(x) - \lambda(f(x,u)) \geq \rho(x - x_{\mathrm{s}})$$

LQ-EMPC

Consider a linear MPC problem

$$\begin{aligned}\mathcal{P}_N(A,B,Q,R,S,P_N) = \underset{x_0,u_0,\ldots,x_N}{\mathrm{argmin}} \quad & \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_N x_N \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, \\ & x_{k+1} - A\,x_k - B\,u_k = 0.\end{aligned}$$

with

$$H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}$$

LQ-EMPC

Consider a linear MPC problem

$$\begin{aligned}\mathcal{P}_N(A,B,Q,R,S,P_N) = \underset{x_0,u_0,\dots,x_N}{\operatorname{argmin}} \quad & \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_N x_N \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, \\ & x_{k+1} - A\,x_k - B\,u_k = 0.\end{aligned}$$

with

$$H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}$$

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- If $H \nsucc 0$ this is an **Economic MPC** problem
- When is it **stabilizing?**

Inifinite Horizon Case

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This is an (indefinite) LQR

$$\begin{aligned}\mathcal{D}(A, B, Q, R, S) := \{(P, K) \mid\, & Q + A^T PA - P - (S^T + A^T PB)K = 0, \\ & K = (R + B^T PB)^{-1}(S + B^T PA), \\ & \rho(A - BK) < 1\}\end{aligned}$$

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Stability if (strict dissipativity for the LQ case):

$$\exists\, \bar{P} \text{ s.t.} \qquad M = \begin{bmatrix} Q + A^T \bar{P} A - \bar{P} & S^T + A^T \bar{P} B \\ S + B^T \bar{P} A & R + B^T \bar{P} B \end{bmatrix} \succ 0$$

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One can enforce stability by adding $\begin{bmatrix} x \\ u \end{bmatrix}^T T \begin{bmatrix} x \\ u \end{bmatrix}$ to the cost and solving

$$\min_{P,T} \; \|T\|^2 \qquad \text{s.t. } M + T \succeq 0,$$

One can modify the cost-to-go without changing the problem

$$\mathcal{P}_\infty(A, B, Q, R, S) = \mathcal{P}_\infty(A, B, Q_{\bar{P}}, R_{\bar{P}}, S_{\bar{P}}),$$

with

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Q_{\bar{P}} &= Q + A^T \bar{P} A - \bar{P} \\
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- $K_{\bar{P}} = K$, i.e. the feedback matrix is unchanged!
- $P_P = 0$, i.e. the cost-to-go can be zero!

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- $K_K = 0$
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- R can be replaced by any $R_K (\succ 0)$

Positive Definite LQR Formulation

Lyapunov Stability Theorem

If $\rho(A_K) < 1$, then

$$\forall\ Q_{\mathrm{L}} \succ 0 \qquad \exists\, P_{\mathrm{L}} \succ 0 \text{ s.t.} \qquad Q_{\mathrm{L}} + A_K^T P_{\mathrm{L}} A_K - P_{\mathrm{L}} = 0$$

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Pos. def. is preserved when transforming back to (A, B)

What About MPC?

In the unconstrained case, the solution is given by the Discrete Riccati Equation (DRE)

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DRE equivalence

$$\mathcal{R}_N(A, B, Q, R, S, P_N) = \mathcal{R}_N(A, B, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{P}_N)$$

if

$$\begin{aligned}\tilde{P}_N - \tilde{P} &= P_N - P, \\ \tilde{R} + B^T \tilde{P} B &= R + B^T P B, \\ \tilde{S} + B^T \tilde{P} A &= S + B^T P A\end{aligned}$$

with P and $\tilde{P}$ computed from $\mathcal{D}(A, B, Q, R, S)$ and $\mathcal{D}(A, B, \tilde{Q}, \tilde{R}, \tilde{S})$.

A Practical View on the Problem

Solve the following SDP

$$
\begin{aligned}
\min_{\tilde{P},\tilde{Q},\tilde{R},\tilde{S},\tilde{H}} \quad & \|\tilde{P} - I\|^2 + \|\tilde{H} - I\|^2 \\
\text{s.t.} \quad & \tilde{H} = \begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} \\
& \tilde{H} \succeq 0 \\
& \tilde{P} \succeq 0 \\
& \tilde{Q} + A^T \tilde{P} A - \tilde{P} - (\tilde{S}^T + A^T \tilde{P} B) K = 0, \\
& (\tilde{R} + B^T \tilde{P} B) K - (\tilde{S} + B^T \tilde{P} A) = 0, \\
& \tilde{R} + B^T \tilde{P} B = R + B^T P B, \\
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\end{aligned}
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This problem is convex!

Look at the Steady State Properties

Lagrangian of steady state problem

$$\mathcal{L} = l(x, u) - \lambda^T (x - f(x, u))$$

Lagrangian Hessian:

$$H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} = \left. \frac{\partial^2 \mathcal{L}}{\partial (x, u)^2} \right|_{x_\mathrm{s}, u_\mathrm{s}}$$

Local linearization

$$A = \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_\mathrm{s}, u_\mathrm{s}} \qquad B = \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_\mathrm{s}, u_\mathrm{s}}$$

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- Formulate a tracking NMPC scheme using $\tilde{H}$

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- System dynamics: $\dot{x} = \frac{u}{10}(1 - x) - 0.4x$, (discr.: 50 RK4 steps, $\Delta = 0.5$)

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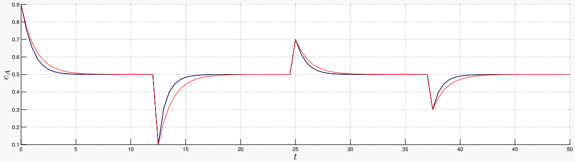

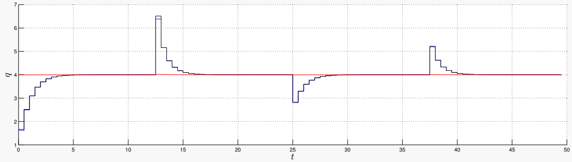

Thank you for your attention!