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Indefinite Linear MPC and Approximated Economic MPC for Nonlinear Systems

Mario Zanon¹, Sébastien Gros², Moritz Diehl¹

1. KU Leuven & Univ. of Freiburg, 2. Chalmers Univ.

NetCo conference, Tours 2014

- 1 Economic MPC and Stability Analysis
- 2 The Linear Quadratic Case
- 3 Approximated EMPC with stability guarantees

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MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N}$$

s.t.

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$$\text{s.t. } x_0 - \bar{x}_i = 0,$$

Initial condition

MPC

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$$\min_{x_0, u_0, \dots, x_N}$$

$$\text{s.t.} \quad x_0 - \bar{x}_i = 0,$$

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$$x_{k+1} - f(x_k, u_k) = 0,$$

System dynamics

MPC

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Initial condition

System dynamics

Terminal constraint

MPC

Optimal Control Problem

$$\min_{x_0, u_0, \dots, x_N} \sum_{k=0}^{N-1} l(x_k, u_k)$$

(Quadratic) stage cost

$$\text{s.t. } x_0 - \bar{x}_i = 0,$$

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At each sampling time i :

- get the initial state $\bar{x}_i$
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Lyapunov stability

Tracking MPC: $l(x_s, u_s) = 0$ and $\exists \alpha \in \mathcal{K}$ s.t. $\alpha(x - x_s) \leq l(x, u), \forall u \in \mathbb{U}$

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$$V_{N-1}(f(\bar{x}_i, u_0^*)) = V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*)$$

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$$V_{N-1}(f(\bar{x}_i, u_0^*)) + l(x_s, u_s)$$

$$\begin{aligned}
 V_{N-1}(f(\bar{x}_i, u_0^*)) &= V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) \\
 &\quad V_N(\bar{x}_i) - l(\bar{x}_i, u_0^*) + \underbrace{l(x_s, u_s)}_{=0}
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$$V_N(f(\bar{x}_i, u_0^*)) - V_N(\bar{x}_i) \leq -l(\bar{x}_i, u_0^*)$$

Do we always want to track?

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No!

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Then why do we track?

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- Difficult to prove stability (2008-)

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Then why do we track?

- It works
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What about Economic MPC?

- Difficult to prove stability (2008-)
- Increased “economic” gain

Economic vs Tracking

Economic vs Tracking

- Steady state: $x_s = 0 = f(0, 0)$

Economic vs Tracking

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- Stage cost: **Tracking** vs **Economic**

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Economic vs Tracking

- Steady state: $x_s = 0 = f(0,0)$
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The classical stability theory does not apply!

Economic Stage Cost

Steady state: $(x_s, u_s) = \min_{x, u} l(x, u) \text{ s.t. } x = f(x, u)$

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$$l(x, u) \quad l(x_s, u_s)$$

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$$l(x, u) - l(x_s, u_s) \quad \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} \quad (x - f(x, u))$$

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Steady state: $(x_s, u_s) = \min_{x,u} l(x, u) \text{ s.t. } x = f(x, u)$

$$l(x, u) - l(x_s, u_s) + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$$

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Steady state: $(x_s, u_s) = \min_{x, u} l(x, u) \text{ s.t. } x = f(x, u)$

$$L(x, u) = \underbrace{l(x, u)}_{\text{Lagrange multiplier}} - \underbrace{l(x_s, u_s)}_{\text{Lagrange multiplier}} + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$$

Economic Stage Cost

Steady state: $(x_s, u_s) = \min_{x, u} l(x, u) \text{ s.t. } x = f(x, u)$

Rotated cost: $L(x, u) = \underbrace{l(x, u)}_{\text{[Diehl et al. 2011]}} - \underbrace{l(x_s, u_s)}_{\text{Lagrange multiplier}} + \underbrace{\lambda_s^T}_{\text{Lagrange multiplier}} (x - f(x, u))$

Rotated MPC Problem

$$\begin{aligned} \min_{x_0, u_0, \dots, x_N} \quad & \sum_{k=0}^{N-1} L(x_k, u_k) \\ \text{s.t.} \quad & x_0 - \bar{x}_i = 0, \\ & x_{k+1} - f(x_k, u_k) = 0, \\ & x_N - x_N^s = 0. \end{aligned}$$

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Rotated Problem $\equiv$ Original Problem

$$\sum_{k=0}^{N-1} L(x, u) = \sum_{k=0}^{N-1} l(x, u) + \underbrace{\lambda_s^T x_0 - \lambda_s^T x_N - (N-1) l(x_s, u_s)}_{\text{constant}}$$

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If

$$\alpha(x) \leq L(x, u), \quad \alpha \in \mathcal{K}$$

the previous stability proof holds! [Diehl et al. 2011]

Generalization [Amrit et al. 2011]

Nonlinear rotating function: $\lambda(x)$

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New rotated cost: $L(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u))$

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What conditions on $\lambda(x)$?

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New rotated cost: $L(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u))$

What conditions on $\lambda(x)$?**Strict Dissipativity**

System strictly dissipative wrt the supply rate $s(x, u)$ if $\exists \lambda(x) : \mathbb{X} \rightarrow \mathbb{R}$

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(x - x_s) + s(x, u),$$

$$\forall (x, u) \in \mathbb{X} \times \mathbb{U}$$

We are interested in $s(x, u) = l(x, u) - l(x_s, u_s)$

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We are interested in $s(x, u) = l(x, u) - l(x_s, u_s)$

This entails

$$L(x, u) = l(x, u) - l(x_s, u_s) + \lambda(x) - \lambda(f(x, u)) \geq \rho(x - x_s)$$

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LQ-EMPC

Consider a linear MPC problem

$$\begin{aligned} \mathcal{P}_N(A, B, Q, R, S, P_N) = \operatorname{argmin}_{x_0, u_0, \dots, x_N} & \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P_N x_N \\ \text{s.t.} & \quad x_0 - \bar{x}_i = 0, \\ & \quad x_{k+1} - A x_k - B u_k = 0. \end{aligned}$$

with

$$H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix}$$

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- If $H \not\succeq 0$ this is an **Economic MPC** problem

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with

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- If $H \not\prec 0$ this is an **Economic MPC** problem
- When is it **stabilizing**?

Inifinite Horizon Case

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This is an (indefinite) LQR

$$\begin{aligned}\mathcal{D}(A, B, Q, R, S) &:= \{(P, K) \mid Q + A^T P A - P - (S^T + A^T P B)K = 0, \\ &\quad K = (R + B^T P B)^{-1}(S + B^T P A), \\ &\quad \rho(A - BK) < 1\}\end{aligned}$$

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Stability if (strict dissipativity for the LQ case):

$$\exists \bar{P} \text{ s.t. } M = \begin{bmatrix} Q + A^T \bar{P} A - \bar{P} & S^T + A^T \bar{P} B \\ S + B^T \bar{P} A & R + B^T \bar{P} B \end{bmatrix} \succ 0$$

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One can enforce stability by adding $\begin{bmatrix} x \\ u \end{bmatrix}^T T \begin{bmatrix} x \\ u \end{bmatrix}$ to the cost and solving

$$\min_{P, T} \|T\|^2 \quad \text{s.t. } M + T \succeq 0,$$

One can modify the cost-to-go without changing the problem

$$\mathcal{P}_\infty(A, B, Q, R, S) = \mathcal{P}_\infty(A, B, Q_{\bar{P}}, R_{\bar{P}}, S_{\bar{P}}),$$

with

$$Q_{\bar{P}} = Q + A^T \bar{P} A - \bar{P}$$

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NOTE:

- $K_{\bar{P}} = K$, i.e. the feedback matrix is unchanged!

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NOTE:

- $K_{\bar{P}} = K$, i.e. the feedback matrix is unchanged!
- $P_{\bar{P}} = 0$, i.e. the cost-to-go can be zero!

One Can Pre-stabilize the System

i.e. use $u_k = v_k - \bar{K}x_k$

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If one uses the optimal feedback gain $\bar{K} = K$:

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If one uses the optimal feedback gain $\bar{K} = K$:

- $K_K = 0$

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If one uses the optimal feedback gain $\bar{K} = K$:

- $K_K = 0$
- The DARE becomes a Lyapunov Equation: $Q_K + A_K^T P A_K - P = 0$

One Can Pre-stabilize the System

i.e. use $u_k = v_k - \bar{K}x_k$

Then

$$\mathcal{P}_\infty(A, B, Q, R, S) = \mathcal{P}_\infty(A_{\bar{K}}, B, Q_{\bar{K}}, R, S_{\bar{K}}),$$

with

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- R can be replaced by any $R_K (\succ 0)$

Positive Definite LQR Formulation

Lyapunov Stability Theorem

If $\rho(A_K) < 1$, then

$$\forall Q_L \succ 0 \quad \exists P_L \succ 0 \text{ s.t.} \quad Q_L + A_K^T P_L A_K - P_L = 0$$

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Pos. def. is preserved when transforming back to (A, B)

What About MPC?

In the unconstrained case, the solution is given by the Discrete Riccati Equation (DRE)

$$\begin{aligned}\mathcal{R}_N(A, B, Q, R, S, P_N) = \{ & (P_0, P_1, \dots, P_N, K_0, \dots, K_{N-1}) \mid \\ & P_{k-1} = Q + A^T P_k A - (S^T + A^T P_k B) K_{k-1} \\ & K_{k-1} = (R + B^T P_k B)^{-1} (S + B^T P_k A)\}\end{aligned}$$

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DRE equivalence

$$\mathcal{R}_N(A, B, Q, R, S, P_N) = \mathcal{R}_N(A, B, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{P}_N)$$

if

$$\begin{aligned}\tilde{P}_N - \tilde{P} &= P_N - P, \\ \tilde{R} + B^T \tilde{P} B &= R + B^T P B, \\ \tilde{S} + B^T \tilde{P} A &= S + B^T P A\end{aligned}$$

with P and $\tilde{P}$ computed from $\mathcal{D}(A, B, Q, R, S)$ and $\mathcal{D}(A, B, \tilde{Q}, \tilde{R}, \tilde{S})$.

A Practical View on the Problem

Solve the following SDP

$$\begin{aligned}
 & \min_{\tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}, \tilde{H}} \quad \|\tilde{P} - I\|^2 + \|\tilde{H} - I\|^2 \\
 & \text{s.t.} \quad \tilde{H} = \begin{bmatrix} \tilde{Q} & \tilde{S}^T \\ \tilde{S} & \tilde{R} \end{bmatrix} \\
 & \quad \tilde{H} \succeq 0 \\
 & \quad \tilde{P} \succeq 0 \\
 & \quad \tilde{Q} + A^T \tilde{P} A - \tilde{P} - (\tilde{S}^T + A^T \tilde{P} B) K = 0, \\
 & \quad (\tilde{R} + B^T \tilde{P} B) K - (\tilde{S} + B^T \tilde{P} A) = 0, \\
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 \end{aligned}$$

This problem is convex!

- 1 Economic MPC and Stability Analysis
- 2 The Linear Quadratic Case
- 3 Approximated EMPC with stability guarantees

Look at the Steady State Properties

Lagrangian of steady state problem

$$\mathcal{L} = l(x, u) - \lambda^T (x - f(x, u))$$

Lagrangian Hessian:

$$H = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} = \frac{\partial^2 \mathcal{L}}{\partial (x, u)^2} \Big|_{x_S, u_S}$$

Local linearization

$$A = \frac{\partial f(x, u)}{\partial x} \Big|_{x_S, u_S} \qquad B = \frac{\partial f(x, u)}{\partial u} \Big|_{x_S, u_S}$$

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- Formulate a tracking NMPC scheme using $\tilde{H}$

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Thank you for your attention!