

# Transport equation with source and generalized Wasserstein distance

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# Transport equation with source and generalized Wasserstein distance

Benedetto Piccoli

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joint work with  
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Tours, June 23<sup>rd</sup> 2014

# Outline

- 1 Wasserstein distance and optimal transport
- 2 Generalized Wasserstein distance

# Optimal transport

Given  $\mu$  measure and measurable  $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ :

$$(\gamma\#\mu)(E) = \mu(\gamma^{-1}(E)).$$

Given  $\mu, \nu$  probability measures, look for  $\gamma$  such that  $\gamma\#\mu = \nu$  and minimizes the cost

$$\int c(x, \gamma(x)) d\mu.$$

Optimal transport first proposed by Monge problem in 1781.  
Particular case is given by the cost  $c(x, y) = |x - y|^p$  with  $p \geq 1$ ,  
defining the Wasserstein distance:

$$W_p(\mu, \nu) = \inf_{\gamma\#\mu=\nu} \left( \int_{\mathbb{R}^n} |\gamma(x) - x|^p d\mu \right)^{1/p}.$$

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# Wasserstein distance

Take  $\mu = \delta_1$ ,  $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ : but there exists no  $\gamma$  with  $\nu = \gamma\#\mu$ , since  $\gamma$  cannot separate masses.

A probability measure  $\pi$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , one can interpret it as a method to transfer a measure.  $\mu$  is sent to  $\nu$  if:

$$|\mu| \int_{\mathbb{R}^d} d\pi(x, \cdot) = d\mu(x), \quad |\nu| \int_{\mathbb{R}^d} d\pi(\cdot, y) = d\nu(y).$$

Monge-Kantorovich problem. Wasserstein distance:

$$W_p(\mu, \nu) = \left( |\mu| \min_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \right)^{1/p}.$$

where  $\Pi(\mu, \nu)$  is the set of transference plans from  $\mu$  to  $\nu$ .

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The function  $v : \mathcal{M} \rightarrow C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  satisfies

- $v[\mu]$  is uniformly Lipschitz and uniformly bounded, i.e. there exist  $L, M$  not depending on  $\mu$ , such that for all  $\mu \in \mathcal{M}, x, y \in \mathbb{R}^n$ ,

$$|v[\mu](x) - v[\mu](y)| \leq L|x - y| \quad |v[\mu](x)| \leq M.$$

- $v$  is a Lipschitz function, i.e. there exists  $K$  such that

$$\|v[\mu] - v[\nu]\|_{C^0} \leq KW_p(\mu, \nu).$$

Under such assumption there exists a unique solution to the transport equation:

$$\begin{cases} \partial_t \mu + \nabla \cdot (v[\mu] \mu) = 0 \\ \mu(0) = \mu_0 \end{cases}.$$

Replacing  $W$  with  $L^1$  (total variation) we lose uniqueness.

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# Generalized Wasserstein distance

Three different admissible actions on  $\mu, \nu$ : either add/remove mass to  $\mu$ , or add/remove mass to  $\nu$  or transport mass from  $\mu$  to  $\nu$ .

$\mathcal{M}$  space of Borel measures with finite mass.

Given  $a, b \in (0, \infty)$  and  $p \geq 1$ , the generalized Wasserstein distance is given by

$$W_p^{a,b}(\mu, \nu) = \inf_{\substack{\tilde{\mu}, \tilde{\nu} \in \mathcal{M}^p \\ |\tilde{\mu}| = |\tilde{\nu}|}} (a^p |\mu - \tilde{\mu}| + a |\nu - \tilde{\nu}|^p + b^p W_p(\tilde{\mu}, \tilde{\nu}))^{1/p}.$$

## Proposition

*The operator  $W_p^{a,b}$  is a distance. Moreover, one can restrict the computation to  $\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu$  and the infimum is always attained.*

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# Properties of the generalized Wasserstein distance

## Theorem

Let  $\mu_n, \mu \in \mathcal{M}$ . Then

$W_p^{a,b}(\mu_n, \mu) \rightarrow 0$  is equivalent to  $\mu_n \rightharpoonup \mu$  and  $\mu_n$  is tight.

Moreover,  $M$  is complete with respect to  $W_p^{a,b}$ .

## Proposition

$W_p^{a,b}$  is equivalent to the Levy-Prokhorov distance for probability measures:

$$d_{LP}(\mu, \nu) := \inf \{ \alpha > 0 : \text{for any closed } A, \mu(A) \leq \nu(A^\alpha) + \alpha \}.$$

## Proposition

We have (see Kantorovich-Rubinstein duality):

$$W_1^{1,1}(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^d} f d(\mu - \nu) \mid \|f\|_{C^0} \leq 1, \|f\|_{Lip} \leq 1 \right\}.$$



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# Benamou-Brenier formula

Consider the transport equation with source:

$$\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = h_t. \quad (1)$$

Define:

$$\mathcal{B}[\mu, v, h] := a^2 \left( \int_0^1 dt \left( \int_{\mathbb{R}^d} d|h_t| \right) \right)^2 + b^2 \int_0^1 dt \left( \int_{\mathbb{R}^d} d\mu_t |v_t|^2 \right).$$

## Proposition

We have

$$\left( W_2^{a,b} \right)^2 = \inf \left\{ \mathcal{B}[\mu, v, h] \mid \begin{array}{l} \mu \text{ is a solution of (1) for } v, h \\ \text{and } \mu(0) = \mu_0, \mu(1) = \mu_1 \end{array} \right\}.$$

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