A routing game in networks with lossy links

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***Abstract*—Standard assumptions in the theory of routing games are that costs are additive over links and that there is flow conservation. The assumptions typically hold when the costs represent delays. We introduce here a routing game where losses occur on links in a way that may depend on the congestion. In that case both assumptions fail. We study a load balancing network and identify a Kameda type paradox in which by adding capacity, all players suffer larger loss rates.**

I. Introduction

Routing games have been studied within various contexts and within various communities for a long time. Some of the pioneers have been Wardrop [12] and Beckmann et al [3] in the road traffic community, Rosenthal [10, 11] in the mathematics community and Orda et al [9] in the telecommunications and networking community. The above references have all in common a cost framework which is additive over links, such as delays or tolls, and is flow conserving (the amount entering a node equals the amount leaving it). Link cost densities are further assumed to be convex in this framework. Little is known in the case where some of these assumptions fail to hold. Nonadditive costs in the shortest path problem were studied in [6, 5] (the cost here depends on the path but not on the congestion). Important classes of nonadditive cost problems occurring in telecommunication networks, are those that deal with losses. The two main frameworks to consider losses are in (1) circuit switching networks, in which calls that do not find sufficient resources on each link on a path from the source to the destination are rejected. Routing games with this type of cost were studied in [2] and in [1]; (2) packet switching networks, in which packet losses may occur either due to buffer overflows (these are congestion losses of packets) or random non-congestion losses that are due to the transmission channel (e.g., a radio channel).

We study a problem in packet switching networks, where losses occur due to buffer overflow. As we show in this paper, this gives rise to a cost which is nonadditive, for which the flows are not conserved and for which the link cost densities are non convex. We are able, however, to identify a unique symmetric equilibrium and to compute it in a three node problem, **assuming that packet service times are exponentially distributed**.

The network that we analyze exhibits interesting paradoxical behavior at equilibrium, that of a Kameda type paradox [7]. It is similar to the standard Braess type paradox but in contrast to it, the Kameda-type paradox does not occur when the number of players is sufficiently large.

The structure of the paper is as follows. We first introduce the model, then compute the Nash equilibrium and identify the paradoxical behavior. We end with the computation of the price of anarchy.

II. The Model

We consider the following symmetric network. There are $2N$ sources of traffic (players). N of the sources, 1, 2, . . ., N are connected to a node S1, while sources $(N+1)$, $(N+2)$, . . ., $2N$ are connected to a node S2. The traffic of each player is an independent Poisson point process with total rate ϕ. Each source has to ship the whole amount ϕ to a destination node D. A player connected to S_i, $i = 1, 2$, can route packets either through a direct path to the destination D or use an indirect two-hop path. It first sends it to the other node S_j, $j \neq i$, and then the packet uses the link from S_j to D to arrive at the destination. Each player i chooses a probability p_i and then routes any arriving packet from source i to the direct route with probability p_i and to the indirect one with the complementary probability $1 - p_i$. The routing decisions for each packet are assumed to be independent.

The transmission time of a packet that goes from S_i to D, $i = 1, 2$ is a random variable σ. Transmission times of different packets are assumed to be independent. We further assume that there is no buffering on these links, so that a packet that arrives during the transmission of another packet is lost. A transmission of a packet from a source S_i to S_j (as the first hop of an indirect path) is lost independently of any other loss with a fixed probability q. Let α_i denote $p_i\phi$. Then a player i connected to a node S_j, $j = 1, 2$, sends an independent Poisson process of packets with rate α_i to the direct path S_jD and another independent Poisson process of packets with rate $\phi - \alpha_i$ to the indirect path. As a result of the random losses between the link $S_j - S_k$, $j \neq k$, the flow of packets from player i that arrive to node S_k is also Poisson and its rate is $(1-q)(\phi - \alpha_i)$.

A similar network was introduced in [7] to model load balancing, but there were no losses of packets. The network represented there a load balancing problem between two processors. A link between S_k, $k = 1, 2$ and D represented a processor and the link between S_1 and S_2 represented a common bus as is shown in Figure 1. Our network has the same topology.

We conclude that the flow over link S_1D is Poisson with a total rate T_L given by

$$T_L = \sum_{i=1}^{N} \alpha_i + (1-q)\left(N\phi - \sum_{j=N+1}^{2N} \alpha_j\right)$$

Similarly, the total traffic T_R on the right link is given by

$$\sum_{j=N+1}^{2N} \alpha_j + (1-q)\left(N\phi - \sum_{i=1}^{N} \alpha_i\right)$$

The probability of no (congestion) loss on the link S_1D equals the probability that there is no arrival during a service

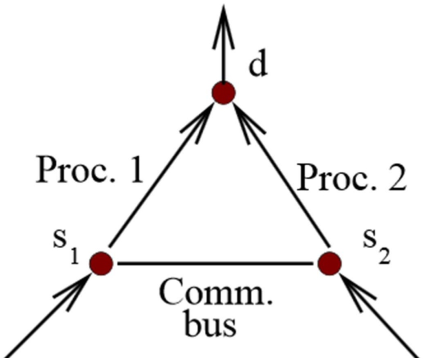

Fig. 1: The system.

time σ, which is given by

$$E[\exp(-T_L\sigma)] = L_\sigma(T_L)$$

where E denotes the expectation with respect to the probability distribution of the random variable σ. $L_\sigma(T_L)$ is the Laplace Stieltjes Transform of the service time evaluated at T_L.

Similarly, the loss probability on the link S_2D equals

$$E[\exp(-T_R\sigma)] = L_\sigma(T_R)$$

which is the Laplace Stieltjes Transform of the service time evaluated at T_R.

Henceforth, we assume that service times are *exponentially* distributed with parameter μ. With this, we have

$$L_\sigma(T_L) = \frac{T_L}{\mu + T_L}, \quad and \quad L_\sigma(T_R) = \frac{T_R}{\mu + T_R}.$$

The loss rate seen by user i in the left group

$$\begin{aligned} & LR^{(i)}(\alpha_i, \alpha_{-i}) \\ = \; & \alpha_i L_\sigma(T_L) + (\phi - \alpha_i)q + (\phi - \alpha_i)(1-q)L_\sigma(T_R) \end{aligned} \quad (1)$$

For the case of exponentially distributed service time with parameter μ, this gives

$$LR^{(i)}(\alpha_i, \alpha_{-i}) = \alpha_i(1 - \frac{\mu}{T_L + \mu})$$
$$+(\phi - \alpha_i)q + (\phi - \alpha_i)(1-q)(1 - \frac{\mu}{T_R + \mu})$$

III. Computing the symmetric equilibrium

In the sequel, we shall consider exponentially distributed service time with parameter μ. We differentiate (2) with respect to α_i, noting that both T_L and T_R are functions of α_i:

$$\begin{aligned} & \left(LR^{(i)}\right)'(\alpha_i, \alpha_{-i}) \\ = \; & (1 - \frac{\mu}{T_L+\mu}) + \frac{\alpha_i \mu}{(T_L+\mu)^2} - q \\ & +(1-q)\left\{-(1 - \frac{\mu}{T_R+\mu}) - (\phi - \alpha_i)\mu\frac{1-q}{(T_R+\mu)^2}\right\} \end{aligned}$$

We note that

$$\begin{aligned} T_L(\boldsymbol{\alpha}) &= T_R(\boldsymbol{\alpha}) \\ &= N\alpha + (1-q)(N\phi - N\alpha) \\ &= N\alpha + (1-q)N\phi - (1-q)N\alpha \\ &= qN\alpha + (1-q)N\phi \\ &= N\left(q\alpha + (1-q)\phi\right) \end{aligned}$$

We seek a vector $\boldsymbol{\alpha} := (\alpha, \alpha, \ldots, \alpha)$ such that

$$\boldsymbol{\alpha} \in BR(\boldsymbol{\alpha}) \quad (2)$$

where BR stands for the best response set. We are interested in finding a symmetric Nash equilibrium, in which, given that each user except i sends traffic on the direct link at rate α, the best response (miniming the loss rate) of user i is also α.

Such an α is given by a solution to the following equation provided that it lies between 0 and ϕ:

$$\begin{aligned} & \left(1 - \frac{\mu}{N\alpha + (1-q)N(\phi-\alpha) + \mu}\right) \\ & \quad + \frac{\alpha\mu}{(N\alpha + (1-q)N(\phi-\alpha)+\mu)^2} - q \\ & - (1-q)\Big\{(1 - \frac{\mu}{N\alpha + (1-q)N(\phi-\alpha)+\mu}) \\ & \quad + \frac{(\phi-\alpha)\mu(1-q)}{(N\alpha + (1-q)N(\phi-\alpha)+\mu)^2}\Big\} = 0 \end{aligned}$$

After a bit of simplification, this reduces to

$$\frac{\alpha\mu\{1 + (1-q)^2\} - \mu(1-q)^2\phi}{N\left(q\alpha + (1-q)\phi\right) + \mu} = q\mu$$

This is a linear equation in α, whose solution is

$$\alpha = \frac{(1-q)^2\phi + Nq(1-q)\phi + q\mu}{1 + (1-q)^2 - Nq^2} \quad (3)$$

Conditions

A valid best response α must be such that $0 \le \alpha \le \phi$. We consider the implications of this.

First, consider $\alpha \ge 0$. The numerator is positive for any value of q — even for $q = 0$ and $q = 1$. So, for $\alpha \ge 0$, the denominator must be nonnegative. Thus,

$$\begin{aligned} 1 + (1-q)^2 - Nq^2 &\ge 0 \\ \text{or,} \;\; N &\le \frac{1 + (1-q)^2}{q^2} \end{aligned} \quad (4)$$

Next, consider $\alpha \le \phi$. Assuming that N satisfies (4), we have

$$\begin{aligned} (1-q)^2\phi + Nq(1-q)\phi + q\mu &\le \phi\{1 + (1-q)^2 - Nq^2\} \\ \text{or,} \;\; Nq\phi + q\mu &\le \phi \\ \therefore \;\; N &\le \frac{\phi - q\mu}{q\phi} \\ &= \frac{1}{q} - \frac{\mu}{\phi} =: K \end{aligned} \quad (5)$$

If N satisfies (5), then it satisfies (4) also. This is because

$$\frac{1}{q} - \frac{\mu}{\phi} < \frac{1}{q} < \frac{1}{q^2} < \frac{1+(1-q)^2}{q^2}$$

We conclude the following.

Theorem III.1. *(i) If $\phi \leq \mu q$, then the symmetric equilibrium is $\alpha_i = \phi$, for any number N of players. This equilibrium is then globally optimal.*
(ii) If $1 \leq K$ where $K = 1/q - \mu/\phi$, then for all $N > K$, the symmetric equilibrium is as in (i), whereas for all $2 \leq N \leq K$ it is given by eq (3).

Proof. If $\phi \leq \mu q$ then for any strictly positive N, N does not satisfy (5) and thus there is no interior solution α to the fixed point equation (2). It is then easy to see that $\alpha = \phi$ is the unique fixed point. This establishes (i). If $N > K$ then again N does not satisfy (5) and thus there is no interior solution α to the fixed point equation (2). Note that for this to be a game, we need $N \geq 2$. Finally, for $N \leq K$, we saw that (3) is indeed a valid solution to (2), and is thus the equilibrium.

Remark: Suppose that μ, q and ϕ are fixed. If we fix N to be larger than the bound K, then there is no flow at the symmetric equilibrium through the indirect paths and then the equilibrium is globally optimal. Now, by decreasing q, K increases, so that for a sufficiently small q, we shall have now $N < K$, and thus $\alpha < \phi$. This is a Braess type paradox since decreasing q means that we improve the quality of the radio link (as it has less losses), but the performance at equilibrium becomes worse as the indirect paths start to be used.

The original Braess paradox [4] was shown to hold in a framework of a very large number of players (Wardrop equilibrium) and later on it was shown to occur also in the case of any $N > 1$ players in [8]. The paradox we introduced, known as the Kameda paradox, does not occur in the case of a very large number of players. This was shown for standard delay type cost functions in [7]. The contribution of this paper is to show that this type of behavior also extends to a routing game where the loss rate of packets is the optimization criterion.

IV. Price of Anarchy

If there were a central entity that could enforce *system optimal* behaviour, what would be amount of traffic that a user would send on the direct link? Let this be denoted by α, $0 \leq \alpha \leq \phi$. By symmetry, α is the same for all users.

The total traffic on either link is given by

$$T = N\alpha + N(\phi - \alpha)(1-q) = N\{(1-q)\phi + q\alpha\}$$

Then, the loss rate seen by an individual user on either side is

$$\begin{aligned} & \alpha\tfrac{T}{T+\mu} + q(\phi-\alpha) + (1-q)(\phi-\alpha)\tfrac{T}{T+\mu} \\ = \quad & \alpha(1 - \tfrac{\mu}{T+\mu}) + q(\phi-\alpha) + (1-q)(\phi-\alpha)(1-\tfrac{\mu}{T+\mu}) \end{aligned}$$

The derivative of the loss rate with respect to α is given by

$$\begin{aligned} LR^{'}(\alpha) &= (1 - \frac{\mu}{T+\mu}) + \alpha\frac{\mu N q}{(T+\mu)^2} - q \\ & \quad +(1-q)\{-(1-\frac{\mu}{T+\mu}) + (\phi-\alpha)\frac{\mu N q}{(T+\mu)^2}\} \\ &= q(1-\frac{\mu}{T+\mu}) + \frac{\mu N q}{(T+\mu)^2}((1-q)\phi + q\alpha) - q \\ &= \frac{qT(\mu+T) + \mu N q((1-q)\phi + q\alpha)}{(T+\mu)^2} - q \end{aligned}$$

Thus, the numerator of the derivative is given by

$$\begin{aligned} & qT(\mu+T) + \mu N q((1-q)\phi + q\alpha) - q(T+\mu)^2 \\ = \quad & -q\mu(T+\mu) + \mu N q((1-q)\phi + q\alpha) \end{aligned}$$

Now $T+\mu = \mu + N(1-q)\phi + Nq\alpha$. So, the above expression reduces to

$$\begin{aligned} = \quad & -q\mu^2 - Nq(1-q)\mu\phi - Nq^2\mu\alpha + \mu N q(1-q)\phi + \mu N q^2\alpha \\ = \quad & -q\mu^2 \\ < \quad & 0 \end{aligned}$$

Thus, the derivative is *negative* for *any* $\alpha \in [0,1]$. So, the loss rate decreases with α, and hence, the system optimal is achieved when $\alpha = \phi$, *i.e.*, each user sends *nothing* on the wireless link.

So, the loss rate experienced by a user at the system optimal is given by

$$LR(\phi) = \phi\frac{T(\phi)}{\mu + T(\phi)}$$

As $T(\phi) = N\phi$, the loss rate is

$$LR(\phi) = \frac{N\phi^2}{\mu + N\phi}$$

We compare this with the loss rate seen by an individual user at the Nash Equilibrium (NE). Let $\tilde{\alpha}$ denote the rate on the direct link at the NE. From (3), we know that

$$\tilde{\alpha} = \frac{(1-q)^2\phi + Nq(1-q)\phi + q\mu}{1 + (1-q)^2 - Nq^2}$$

The loss rate at the NE is given by

$$\begin{aligned} & \tilde{\alpha}\tfrac{T(\tilde{\alpha})}{T(\tilde{\alpha})+\mu} + (\phi-\tilde{\alpha})q + (\phi-\tilde{\alpha})(1-q)\tfrac{T(\tilde{\alpha})}{T(\tilde{\alpha})+\mu} \\ = \quad & \tfrac{\tilde{\alpha}T(\tilde{\alpha}) + q(\phi-\tilde{\alpha})(T(\tilde{\alpha})+\mu) + (1-q)(\phi-\tilde{\alpha})T(\tilde{\alpha})}{T(\tilde{\alpha})+\mu} \end{aligned}$$

The numerator simplifies to

$$\begin{aligned} & \tilde{\alpha}T(\tilde{\alpha}) + (\phi - \tilde(\alpha))\{q(T(\tilde{\alpha})+\mu) + (1-q)T(\tilde{\alpha})\} \\ = \quad & \tilde{\alpha}T(\tilde{\alpha}) + (\phi - \tilde(\alpha))(T(\tilde{\alpha}) + q\mu) \\ = \quad & \phi T(\tilde{\alpha}) + (\phi - \tilde(\alpha))q\mu \end{aligned}$$

Recalling that $T(\tilde{\alpha}) = N(q\tilde(\alpha) + (1-q)\phi)$, the loss rate seen be a single user becomes

$$\frac{N\phi q\tilde{\alpha} + N\phi^2(1-q) + \phi q\mu - \tilde{\alpha}q\mu}{Nq\tilde{\alpha} + N(1-q)\phi + \mu}$$

Substituting for $\tilde{\alpha}$, and after some algebraic simplifications, we get the following expressions:

$$\text{Denominator} \quad = \quad \frac{N(1-q)^2\phi + \mu\{1 + (1-q)^2\}}{1 + (1-q)^2 - Nq^2}$$

$$\text{Numerator} \quad = \quad \frac{N\phi^2(1-q)(2-q)+q\mu(\phi-q\mu)}{1+(1-q)^2-Nq^2}$$

With these, the loss rate experienced by a single user becomes

$$\text{Loss rate at NE } = \frac{N\phi^2(1-q)(2-q)+q\mu(\phi-q\mu)}{N(1-q)^2\phi+\mu\{1+(1-q)^2\}} \quad (6)$$

Remark: With $q \approx 0$, the loss rate at NE becomes

$$\frac{2N\phi^2}{N\phi+2\mu}$$

The system optimal solution does not depend on q, as the wireless link is not used at the system optimal. So, the *Price of Anarchy* in the limit of a very good wireless link ($q \to 0$) is

$$\frac{2N\phi^2}{N\phi+2\mu} \times \frac{\mu+N\phi}{N\phi^2} = \frac{2\mu+2N\phi}{2\mu+N\phi}$$

V. Conclusions

We have studied in this paper a routing game within a cost framework that has not yet been studied—that of networks with random and with congestion losses. Although many of the standard conditions appearing in standard routing games [9] were not satisfied, we were able to compute the symmetrical equilibrium and to identify a Kameda-type paradox.

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