Mean-field Game Approach to Admission Control of an M/M/∞ Queue with Decreasing Congestion Cost

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Abstract— **We study a mean field approximation of the M/M/∞ queuing system. The problem we deal with is quite different from standard games of congestion as we consider the case in which higher congestion results in smaller costs per user. This is motivated by a situation in which some TV show is broadcast so that the same cost is needed no matter how many users follow the show. Using a mean-field approximation, we show that this results in multiple equilibria of threshold type which we explicitly compute. We show that the mean-field approximation becomes tight as the workload increases, thus the results obtained for the mean-field model well approximate the discrete one.**

I. Introduction

This paper is devoted to the problem of whether an arrival should queue or not in an M/M/∞ queue. It is assumed that the cost per customer decreases in the number of customers.

In a wireless context, the M/M/∞ queue may model the number of calls in a cell with a large capacity. The asumption that the cost per call decreases in the number of calls is typical for multicast in which the same content is broadcast to all mobiles, so that the cost of the transmission can be shared among the number of calls present.

Our objective in this paper is to study the structure of individually optimal policies. We obtain threshold policies in which an individual is admitted if the number of ongoing calls exceeds some threshold. This is exactly the opposite to what is optimal in most of the queuing models.

The reason behind it is that the assumption on the cost in this problem is quite different than that in standard congestion control problem, since the larger the number of customers, the lower the cost per customer is. The structure of individually optimal policies can thus be expected to be quite different than those obtained when congestion costs per customer increase. These have been studied for over half a century starting with the seminal paper of Pinhas Naor [8]. Naor had considered an M/M/1 queue, in which a controller has to decide whether arrivals should enter a queue or not. The objective was to minimize a weighted difference between the average expected waiting time of those that enter, and the acceptance rate of customers. Naor then considers the individual optimal threshold (which can be viewed as a Nash equilibrium in a non-cooperative game between the players) and shows that it is also of a threshold type with a threshold larger than in a centralized model. Under individual optimality, arrivals that join the queue wait longer in average than in the global optimization case. Finally, he showed that there exists some toll such that if it is imposed on arrivals for joining the queue, then the threshold value of the individually optimal policy can be made to agree with the socially optimal one. Since this seminal work of Naor there has been a huge amount of research that extend the model. More general interarrival and service times have been considered, more general networks, other objective functions and other queueing disciplines, see e.g. [14], [12], [11], [6], [7], [3], [5], [1], [10], [2] and references therein.

We study here a simplified mean field limit of the M/M/∞ queuing system rather than the actual discrete model since, on one hand it is much simpler to handle and solve than the original discrete problem (we obtain closed-form formulas for all the equilibria) and, on the other the approximation becomes tight as the workload increases. To show that, in the paper we establish the convergence of the game to its mean field limit under appropriate conditions.

The organization of the paper is as follows: in Section 2 we introduce both the discrete model and its mean-field approximation. In Section 3 we find equilibria of the mean-field model. In Section 4 we establish the convergence of discrete models to the mean-field one as the workload increases. In Section 5 we numerically illustrate our results. We end the paper with concluding remarks in Section 6.

II. The Model

A. Discrete Model

We consider a service facility in which an arriving customer can observe the length of the queue upon arrival X_t (the system state). The value of service is γ and the cost of spending time in service can be computed as an integral of the cost function c over the service time with c – a continuous decreasing function of the number of users in the queue. An arriving customer can either join the queue or leave without being served. The decision is made upon arrival. The situation is modeled as a M/M/∞ system with incoming rate λ and service rate μ.

A customer k arriving at time t_k chooses whether to enter the queue (E) or not (N). It follows that the set of pure actions for any customer is $V = \{E, N\}$. Since the decision that he makes is based on the length of the queue, a policy (or a strategy) of any customer will be a mapping[1] π_k :

[1]For any finite set A, $\Delta(A)$ denotes the set of all probability measures on A.

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$S \to \Delta(V)$ (since the set V is only a two-point set, we will identify π_k with a function from $\mathbb{N}$ to $[0,1]$, describing the probability it assigns to action E), where S denotes the set of possible system states (in the discrete model $S = \mathbb{N}$). In what will follow we will assume that the users limit their policies to the sets of so-called *impulse* or *threshold* policies, defined below.

Definition 1 *A policy π_k of a user is called an* impulse *policy if*

(a) it is discontinuous in finitely many points;

(b) it is constant on each interval between two discontuity points[2].

A subclass of the set of impulse policies with very simple structure are threshold policies.

Definition 2 *A policy π_k of a user is called an $[\Theta, q]$-*threshold *policy if*

$$\pi_k(x) = \begin{cases} 0 & \text{if } x < \Theta \\ q & \text{if } x = \Theta \\ 1 & \text{if } x > \Theta \end{cases} \tag{1}$$

At time t, an incoming client who employs this policy joins the system if the queue length, X_t, is bigger than Θ, while if $X_t = \Theta$ he does so with probability q. Otherwise he never joins the queue.

The cost of a user k arriving at time t_k is defined as follows:

$$C_k(X_{t_k}) = \int_{t_k}^{t_k+\sigma_k} c(X_t)\, dt - \gamma,$$

where σ_k is user k's service time.

For each multi-policy $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots)$, let $[\boldsymbol{\pi}^{-k} \mid \pi'_k]$ be the policy which replaces π_k by π'_k in $\boldsymbol{\pi}$. Now we are ready to define the solution we will be looking for:

Definition 3 *A policy π_k is an* optimal response *for user k against a multi-policy $\boldsymbol{\pi}$ if*

$$\mathbb{E}\left[C_k(X_t([\pi_k, \boldsymbol{\pi}^{-k}]))\right] \le \mathbb{E}\left[C_k(X_t([\pi'_k, \boldsymbol{\pi}^{-k}]))\right] \tag{2}$$

for every policy π'_k of player k. A multi-policy $\boldsymbol{\pi}^* = (\pi^*_1, \pi^*_2, \ldots)$ is a Nash equilibrium *if policy of every user k is the optimal response for user k against $\boldsymbol{\pi}^*$, for every k. If inequalities (2) are true up to some $\varepsilon > 0$, we say that $\boldsymbol{\pi}^*$ is an ε-Nash equilibrium.*

B. Fluid Model

In what follows we will mostly analyze the fluid approximation, which can be viewed as the weak limit of the system (scaled in a proper way) as the arrival rate of players goes to infinity (see e.g. [13]). Below we describe the fluid model.

The system state (the length of the queue) belongs to $S = \mathbb{R}^+$. Consequently, the policies of the players are defined on $\mathbb{R}^+$. The customers arrive at the queue according to a fluid process with rate λ. As each of them uses some policy π_k, the real incoming rate at time t is $\overline{\boldsymbol{\pi}}(X_t)\lambda$ where $\overline{\boldsymbol{\pi}}(X_t)$ is the average strategy of the arriving users. They stay in the queue an exponentially distributed time with parameter μ, and so the outflow is according to a fluid process with rate μX_t. This can be described as the following ODE:

$$\begin{cases} \dot{X}_t(\boldsymbol{\pi}) = \overline{\boldsymbol{\pi}}(X_t)\lambda - \mu X_t(\boldsymbol{\pi}), \forall t \ge 0 \\ X_0 = x_0 \end{cases} \tag{3}$$

Since there are infinitely many players in the game, we encounter problems with defining the multi-policies now. For that reason we assume that in multi-policy $\boldsymbol{\pi}$ all the players use the same policy π. If we want to write that only one player, say player k changes his policy to some π'_k, we write that players apply policy $[\boldsymbol{\pi}^{-k}, \pi'_k]$, meaning that each player uses policy π except player k. The assumption that all the players except possibly one use the same policy may look unrealistic, but it is enough to define and find an equilibrium in such a symmetric model. Moreover, $\overline{\boldsymbol{\pi}}$ is not always well defined, but in such a case $\overline{\boldsymbol{\pi}}(X_t) \equiv \pi(X_t)$. Also with these assumptions, both the cost and the equilibrium can be defined as in the discrete model.

III. Equilibria of the Fluid Model

In this section we characterize the equilibrium points of our game. We begin by characterizing the evolution of the system state in case all the users apply the same impulse policy.

Lemma 1 *Suppose all the players (except maybe one) apply the same impulse policy π. Then if the initial state of the system is x_0, then X_t is continuous in t for any x_0 and is nondecreasing in x_0.*

Proof: It is clear that for $\overline{\boldsymbol{\pi}} \equiv \pi$ having finitely many discontinuity points, the (non-classical) solution to the equation (3) is well-defined a.e. and continuous in t.

Next, suppose that $x_0 < x'_0$ and there exists a s such that[3] $X_s[x_0] > X_s[x'_0]$. X_t is continuous in t, thus by the intermediate value property there exists a $t^* < s$ such that $X_{t^*}[x_0] = X_{t^*}[x'_0]$. But in both cases and at each time all users apply the same policy π, depending only on the current state of the system, thus for any $t > t^*$ $X_t[x_0] = X_t[x'_0]$, which is a contradiction, as we assumed that $X_s[x_0] > X_s[x'_0]$. ■

We have one immediate corollary to this lemma.

Corollary 1 *The expected cost of a player joining the queue at time t_k, when all the other players apply policy π*

$$\mathbb{E}\left[\int_{t_k}^{t_k+\sigma_k} c(X_t(\boldsymbol{\pi}))\, dt\right] - \gamma \tag{4}$$

[2] In case $S = \mathbb{N}$, assumption (a) is trivially satisfied, but together with (b) it implies that the policy changes its value only in finitely many points.

[3] We shall write $X_t[x_0]$ for the value at t of the solution to (3) when $X_0 = x_0$.

when $\sigma_k \sim Exp(\mu)$, is decreasing in X_{t_k}.[4]

Note that in the above corollary we have replaced x_0 with X_{t_k}. This is justified, as the coefficients of (3) depend on t only through X_t. Corollary 1 has an important consequence which is stated in the lemma below:

Lemma 2 *Any best response to a symmetric impulse multi-strategy $\boldsymbol{\pi}$ is a threshold strategy. Moreover, the best response is unique up to the value of q (see (1)).*

Proof: A player k arriving at time t_k has only two pure actions: to enter the queue (E) or not to enter the queue (N). When he uses the former, his cost is

$$\mathbb{E}\left[\int_{t_k}^{t_k+\sigma_k} c(X_t(\boldsymbol{\pi}))\,dt\right] - \gamma,$$

with $\sigma_k \sim \mathrm{Exp}(\mu)$, which is by Corollary 1 decreasing in X_{t_k}. On the other hand, when k uses action N, his cost is 0. Thus, if k prefers to use action E for $X_{t_k} = x_1$, he will also prefer it for $X_{t_k} = x_2 > x_1$. Similarly, if he prefers to use N for $X_{t_k} = x_2$, he will also prefer it for $X_{t_k} = x_1 < x_2$. Finally, as the cost of using E is strictly decreasing in X_{t_k}, there may only exist one point where k is indifferent between E and N and so he may choose to randomize. Moreover, in any other point the best response is uniquely determined. ■

An immediate, but very important consequence of Lemma 2 is the following:

Corollary 2 *In any symmetric*[5] *equilibrium to our queuing game any player uses a threshold policy.*

Remark 1 *Note that the equilibrium specifies the action to take at any state, including states that are in practice never reached. If a state x is never visited then any variation of the equilibrium at states larger than x will not change the performance of any player. Yet since we allow for any initial state, there may be customers that will find the system at states that are transient and will not be visited again. Therefore specifying the equililbrium in such states is considered to be important in game theory. Equilibria that are specified in all states including transient ones, are known as perfect equilibria. It can also be shown that such equilibria are good approximations of those that we obtain in case that there is some sufficiently small constant uncontrolled inflow. This follows from [4].*

Assuming that all (except maybe one) users apply the same $[\Theta, q]$-threshold strategy, we may write explicitly what is the evolution of the system state X_t:

[4]The fact that we have strong monotonicity here, even though we had weak monotonicity in Lemma 1, is a consequence of the continuity of X_t, which implies that a trajectory starting at time t_k in a larger X_{t_k} stays above the one starting in a smaller X'_{t_k} on some interval, which affects the integral in (4).

[5]The result can be generalized to the nonsymmetric case, but it would require some technical assumptions to make sure $\overline{\boldsymbol{\pi}}$ is well-defined.

Lemma 3 *Suppose the initial state of the system is x_0 and that all the users (except maybe one) apply the $[\Theta, q]$-threshold policy. Then the system state at time t can be explicitly written as:*

(a) If $x_0 > \Theta$ and $\Theta \leq \frac{\lambda}{\mu}$ or $x_0 = \Theta < \frac{q\lambda}{\mu}$ then

$$X_t = \frac{\lambda}{\mu} + (x_0 - \frac{\lambda}{\mu})e^{-\mu t}.$$

(b) If $x_0 > \Theta > \frac{\lambda}{\mu}$ then

$$X_t = \begin{cases} \frac{\lambda}{\mu} + (x_0 - \frac{\lambda}{\mu})e^{-\mu t} \text{ if } t \in [0, \bar{t}_{(x_0,\mu)}] \\ \Theta e^{\mu(\bar{t}_{(x_0,\Theta)} - t)} \text{ if } t \geq \bar{t}_{(x_0,\Theta)}, \end{cases}$$

where $\bar{t}_{(x_0,\Theta)} = \frac{1}{\mu}\log\frac{x_0 - \frac{\lambda}{\mu}}{\Theta - \frac{\lambda}{\mu}}$.

(c) If $x_0 = \Theta = \frac{q\lambda}{\mu}$ then

$$X_t = \frac{q\lambda}{\mu}$$

(d) If $x_0 < \Theta$ or $x_0 = \Theta > \frac{q\lambda}{\mu}$ then

$$X_t = x_0 e^{-\mu t}.$$

Proof: We know that when p is a constant, the solution of the equation

$$\begin{cases} \dot{X}_t = p\lambda - \mu X_t, \forall t \geq 0 \\ X_0 = x_0 \end{cases}$$

is

$$X_t = \frac{p\lambda}{\mu} + (x_0 - \frac{p\lambda}{\mu})e^{-\mu t}. \tag{5}$$

Note that, when $p = 1$, this means that $X_t \to \frac{\lambda}{\mu}$ monotonically when $t \to \infty$. Thus, if $x_0 > \Theta$ and $\frac{\lambda}{\mu} > \Theta$, X_t never leaves the region where policy π prescribes to use action E, and so

$$X_t = \frac{\lambda}{\mu} + (x_0 - \frac{\lambda}{\mu})e^{-\mu t}.$$

Similarly, note that for $p = 0$, X_t decreases monotonically to 0, thus when $x_0 < \Theta$, X_t never leaves the region where policy π prescribes to use action N, and so (5) reduces to

$$X_t = x_0 e^{-\mu t}.$$

Now suppose that $x_0 > \Theta > \frac{\lambda}{\mu}$. Then X_t starts in the region where π prescribes to use action E with probability one, which implies that its trajectory decreases towards $\frac{\lambda}{\mu}$ until time $\bar{t}_{(x_0,\Theta)}$ when it reaches the threshold Θ. From then on π prescribes to use action N with probability 1. It is easy to compute that for $t \leq \bar{t}_{(x_0,\Theta)}$,

$$X_t = \frac{\lambda}{\mu} + (x_0 - \frac{\lambda}{\mu})e^{-\mu t}.$$

Since by definition $\bar{t}_{(x_0,\Theta)}$ is such that $X_{\bar{t}_{(x_0,\Theta)}} = \Theta$, we easily obtain that $\bar{t}_{(x_0,\Theta)} = \frac{1}{\mu}\log\frac{x_0 - \frac{\lambda}{\mu}}{\Theta - \frac{\lambda}{\mu}}$. Then, for $t \geq \bar{t}_{(x_0,\Theta)}$, X_t has to satisfy (5) with $p = 0$ and $t_0 = \bar{t}_{(x_0,\Theta)}$ instead of 0, which gives

$$X_t = \Theta e^{\mu(\bar{t}_{(x_0,\Theta)} - t)}.$$

Finally, when $x_0 = \Theta$, X_t satisfies at $t = 0$ (5) with $p = q$. If $x_0 = \frac{q\lambda}{\mu}$, by (5) $X_t \equiv \frac{q\lambda}{\mu}$. Otherwise if $x_0 < \frac{q\lambda}{\mu}$, X_t moves upwards and for $t > 0$ behaves like when $x_0 > \Theta$, while if $x_0 > \frac{q\lambda}{\mu}$, X_t moves downwards and for $t > 0$ behaves like when $x_0 < \Theta$. ∎

Now, to simplify the notation, we will make use of the fact that all the players use threshold policies. Let us define

$$\widehat{C}_k(x, (\Theta_{-k}, q_{-k}))$$

to be the expected service cost for player k if he enters the queue when its state is x and all the players except k apply a $[\Theta_{-k}, q_{-k}]$-threshold policy. $\widehat{C}_k$ can be written as

$$\begin{aligned}
&\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) \\
&= \mathbb{E}^{\sigma_k} \sim \mathrm{Exp}(\mu), X_0 = x \left[\int_{t_k}^{t_k+\sigma_k} c(X_t(\boldsymbol{\pi})) \right] \\
&= \int_0^\infty \int_0^\tau c(X_t) \mu e^{-\mu\tau} \, dt \, d\tau.
\end{aligned}$$

The following lemma gives exact ways to compute $\widehat{C}_k$ in each of the cases of Lemma 3.

Lemma 4 *$\widehat{C}_k$ can be computed using following formulas:*

(a) If $x > \Theta_{-k}$ and $\Theta_{-k} \leq \frac{\lambda}{\mu}$ or $x = \Theta_{-k} < \frac{q_{-k}\lambda}{\mu}$ then[6]

$$\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) = \frac{1}{\lambda - x\mu} \int_x^{\frac{\lambda}{\mu}} c(u) \, du.$$

(b) If $x > \Theta_{-k} > \frac{\lambda}{\mu}$ then

$$\begin{aligned}
\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) &= \frac{1}{x\mu - \lambda} \int_{\Theta_{-k}}^x c(u) \, du \\
&+ \frac{\Theta_{-k}\mu - \lambda}{\Theta_{-k}\mu(x\mu - \lambda)} \int_0^{\Theta_{-k}} c(u) \, du.
\end{aligned}$$

(c) If $x = \Theta_{-k} = \frac{q_{-k}\lambda}{\mu}$ then

$$\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) = \frac{1}{\mu} c\left(\frac{q_{-k}\lambda}{\mu}\right) = \frac{1}{\mu} c(\Theta_{-k}).$$

(d) If $x < \Theta_{-k}$ or $x = \Theta_{-k} > \frac{q_{-k}\lambda}{\mu}$ then

$$\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) = \frac{1}{x\mu} \int_0^x c(u) \, du.$$

In next two lemmas we characterize the best responses to any given threshold strategies.

Lemma 5 *$[\Theta_k, q_k]$-threshold policy is a best response of player k to a $[\Theta_{-k}, q_{-k}]$-threshold policy used by all the others if Θ_k is obtained by finding the unique solution to the equation*

$$\widehat{C}_k(\Theta_k, (\Theta_{-k}, q_{-k})) = \gamma. \tag{6}$$

and taking any q_k. If equation (6) has no solutions then Θ_k is taken as the only value such that

$$\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) < \gamma \text{ for } x > \Theta_k \text{ and} \tag{7}$$

$$\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) > \gamma \text{ for } x < \Theta_k \tag{8}$$

and $q_k = 1$ if the first inequality is satisfied for $x = \Theta_k$, while $q_k = 0$ if the second one is satisfied for $x = \Theta_k$.

Proof: First note that $\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) - \gamma$ is exactly the expected cost of player k if he joins the queue when its state is x, while his cost when he does not join is 0. Moreover, the expected cost of player joining the queue is by Corollary 1 monotone decreasing function of x. Thus, equation (6) may have at most one solution, and the cost of joining the queue for $x > \Theta_k$ is negative, that for $x < \Theta_k$ is positive, while that for $x = \Theta_k$ is 0, regardless of q_k. Thus $[\Theta_k, q_k]$-threshold policy always gives player k the smallest cost available.

Similarly, when (6) has no solutions, from the monotonicity of the cost $\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) - \gamma$, there must exist exactly one Θ_k such that $\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) - \gamma > 0$ for $x < \Theta_k$ and $\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) - \gamma < 0$ for $x > \Theta_k$. Now we can repeat the arguments from the proof of the first part of the lemma, to show that $[\Theta_k, 1]$- or $[\Theta_k, 0]$-threshold policy is the best response to $[\Theta_{-k}, q_{-k}]$-threshold policy of the others in this case. ∎

Lemma 6 *Let $[\Theta_k, q_k]$-threshold policy be a best response of player k to a $[\Theta_{-k}, q_{-k}]$-threshold policy used by all the others and define $\underline{\Theta}$ and $\overline{\Theta}$ as the unique solutions to the following equations*[7]*:*

$$\frac{1}{\lambda - \overline{\Theta}\mu} \int_{\overline{\Theta}}^{\frac{\lambda}{\mu}} c(u) \, du = \gamma, \quad \frac{1}{\underline{\Theta}\mu} \int_0^{\underline{\Theta}} c(u) \, du = \gamma. \tag{9}$$

Then Θ_k and q_k satisfy the following:

(a) If $\gamma \in \left(0, \frac{1}{\mu} \lim_{u\to\infty} c(u)\right]$ then $\Theta_k = \infty$ and q_k is arbitrary (which means that the best response is a policy never prescribing to enter the queue).

(b) If $\gamma \in \left(\frac{1}{\mu} \lim_{u\to\infty} c(u), \frac{1}{\lambda} \int_0^{\frac{\lambda}{\mu}} c(u) \, du\right)$ then

$$\Theta_k(\Theta_{-k}) = \begin{cases} \overline{\Theta}, & \text{for } \Theta_{-k} < \min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\} \\ \Theta_{-k}, & \text{for } \min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\} \leq \Theta_{-k} \leq \frac{\lambda}{\mu} \\ \widetilde{\Theta}(\Theta_{-k}), & \text{for } \frac{\lambda}{\mu} < \Theta_{-k} < \underline{\Theta} \\ \underline{\Theta}, & \text{for } \Theta_{-k} \geq \underline{\Theta} \end{cases}$$

where $\widetilde{\Theta}$ is some function defined on $(\frac{\lambda}{\mu}, \underline{\Theta})$ satisfying $\widetilde{\Theta}(x) > x$. q_k is arbitrary for $\Theta_{-k} \notin \left[\min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\}, \frac{\lambda}{\mu}\right]$, while for $\Theta_{-k} \in \left[\min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\}, \frac{\lambda}{\mu}\right]$

$$q_k(\Theta_{-k}, q_{-k}) =$$

$$\begin{cases} 0, & \text{if } \Theta_{-k} > \frac{q_{-k}\lambda}{\mu} \text{ or } \Theta_{-k} = \frac{q_{-k}\lambda}{\mu} \text{ and } c(\frac{q_{-k}\lambda}{\mu}) > \mu\gamma \\ \text{arbitrary}, & \text{if } \Theta_{-k} = \frac{q_{-k}\lambda}{\mu} \text{ and } c(\frac{q_{-k}\lambda}{\mu}) = \mu\gamma \text{ or } \Theta_{-k} = \overline{\Theta} < \frac{q_{-k}\lambda}{\mu} \\ 1, & \text{if } \Theta_{-k} < \frac{q_{-k}\lambda}{\mu} \text{ or } \Theta_{-k} = \frac{q_{-k}\lambda}{\mu} \text{ and } c(\frac{q_{-k}\lambda}{\mu}) < \mu\gamma. \end{cases}$$

[6]In the degenerate case when $x = \frac{\lambda}{\mu}$, $\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) = \frac{1}{\mu} c\left(\frac{\lambda}{\mu}\right)$, which is the limit of the expression in (a) when $x \to \frac{\lambda}{\mu}$. We will use similiar convention throughout the paper, putting $\frac{1}{a-a} \int_a^a f(u) \, du = f(a)$, if needed. This will reduce the number of cases considered in subsequent results, without affecting the validity of any of them.

[7]If there are any solutions.

(c) If $\gamma \in \left[\frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du, \frac{1}{\mu}c(0)\right)$ then

$$\Theta_k(\Theta_{-k}) = \begin{cases} \Theta_{-k}, & \text{for } \Theta_{-k} \le \underline{\Theta} \\ \underline{\Theta}, & \text{for } \Theta_{-k} \ge \underline{\Theta}. \end{cases}$$

q_k is arbitrary for $\Theta_{-k} > \underline{\Theta}$, while for $\Theta_{-k} \le \underline{\Theta}$, $q_k(\Theta_{-k}, q_{-k}) =$

$$\begin{cases} 0, & \text{if } \Theta_{-k} > \frac{q_{-k}\lambda}{\mu} \text{ or } \Theta_{-k} = \frac{q_{-k}\lambda}{\mu} \text{ and } c(\frac{q_{-k}\lambda}{\mu}) > \mu\gamma \\ \text{arbitrary}, & \text{if } \Theta_{-k} = \frac{q_{-k}\lambda}{\mu} \text{ and } c(\frac{q_{-k}\lambda}{\mu}) = \mu\gamma \\ 1, & \text{if } \Theta_{-k} < \frac{q_{-k}\lambda}{\mu} \text{ or } \Theta_{-k} = \frac{q_{-k}\lambda}{\mu} \text{ and } c(\frac{q_{-k}\lambda}{\mu}) < \mu\gamma. \end{cases}$$

(d) If $\gamma \ge \frac{1}{\mu}c(0)$ then $\Theta_k = 0$ and $q_k = 1$ (which means that the best response is a policy always prescribing to enter the queue).

Proof: To show (a) first note that any form of $\widehat{C}_k$ described in Lemma 4 is bounded below by $\frac{1}{\mu}\inf_{u\ge 0} c(u)$, which equals $\frac{1}{\mu}\lim_{u\to\infty} c(u)$, as c is a strictly decreasing function. Thus, in case $\gamma \le \frac{1}{\mu}\lim_{u\to\infty} c(u)$, also $\gamma < \widehat{C}_k(x, (\Theta_{-k}, q_{-k}))$ for any value of x, thus (7) is satisfied for $\Theta_k = \infty$. This means that the strategy never prescribing player k to enter the queue is his best response to the $[\Theta_{-k}, q_{-k}]$-threshold policy used by all the others.

Now suppose the assumptions of part (b) of the lemma are satisfied. Note that the function

$$\overline{C}(x) := \frac{1}{\lambda - x\mu}\int_x^{\frac{\lambda}{\mu}} c(u)\,du$$

is continuous on $\mathbb{R}^+$ and satisfies $\lim_{x\to 0^+}\overline{C}(x) = \frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du$ and $\lim_{x\to\infty}\overline{C}(x) = \frac{1}{\mu}\lim_{u\to\infty} c(u)$. Thus, by the intermediate value property, and since $\gamma \in \left(\frac{1}{\mu}\lim_{u\to\infty} c(u), \frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du\right)$, there exists an x such that $\overline{C}(x) = \gamma$, but this is exactly how $\overline{\Theta}$ is defined.

Similarly, the function

$$\underline{C}(x) := \frac{1}{x\mu}\int_0^x c(u)\,du$$

is continuous on $\mathbb{R}^+$ and satisfies $\underline{C}(\frac{\lambda}{\mu}) = \frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du$ and $\lim_{x\to\infty}\underline{C}(x) = \lim_{u\to\infty} c(u)$. Thus, again by the intermediate value property, and since $\gamma \in \left(\frac{1}{\mu}\lim_{u\to\infty} c(u), \frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du\right)$, there exists an x such that $\underline{C}(x) = \gamma$, which is how $\underline{\Theta}$ is defined. Moreover, $\underline{\Theta}$ is always larger than $\frac{\lambda}{\mu}$.

Next note that by Lemma 4, $\widehat{C}_k(x, (\Theta_{-k}, q_{-k}))$ equals $\overline{C}(x)$ if $x > \Theta_{-k}$ and $\Theta_{-k} \le \frac{\lambda}{\mu}$ or $x = \Theta_{-k} < \frac{q_{-k}\lambda}{\mu}$, and $\underline{C}(x)$ if $x < \Theta_{-k}$ or $x = \Theta_{-k} > \frac{q_{-k}\lambda}{\mu}$. Thus by Lemma 5, $\Theta_k = \overline{\Theta}$ for $\Theta_{-k} \le \min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\}$, $\Theta_k = \Theta_{-k}$ for $\min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\} \le \Theta_{-k} \le \frac{\lambda}{\mu}$, $\Theta_k \ge \Theta_{-k}$ for $\frac{\lambda}{\mu} < \Theta_{-k} \le \underline{\Theta}$ and $\Theta_k = \underline{\Theta}$ for $\Theta_{-k} \ge \underline{\Theta}$. The values of q_k for $\Theta_{-k} \le \frac{\lambda}{\mu}$ depend on the relation between Θ_{-k} and $\frac{q_{-k}\lambda}{\mu}$: If the former is smaller, for $\Theta_k = \Theta_{-k}$ we are in the set where $\widehat{C}_k(\Theta_k, (\Theta_{-k}, q_{-k})) = \overline{C}(\Theta_k) < \gamma$, and so $q_k = 1$. If $\Theta_{-k} = \frac{q_{-k}\lambda}{\mu}$, we are in the set where $\widehat{C}_k(\Theta_k, (\Theta_{-k}, q_{-k})) = \frac{1}{\mu}c(\Theta_{-k})$, thus according to Lemma 5 the value of q_k depends on the relation between $\frac{1}{\mu}c(\Theta_{-k})$ and γ, exactly as it is written in Lemma 6. Finally if $\Theta_{-k} > \frac{q_{-k}\lambda}{\mu}$, $\widehat{C}_k(\Theta_k, (\Theta_{-k}, q_{-k})) = \underline{C}(\Theta_k) > \gamma$, and so $q_k = 0$.

To finish the proof of part (b) of the Lemma, we need to show that for $\Theta_{-k} \in \left(\frac{\lambda}{\mu}, \underline{\Theta}\right)$, $\Theta_k > \Theta_{-k}$. To do that, it is enough to prove that for any fixed $\Theta_{-k} \in \left(\frac{\lambda}{\mu}, \underline{\Theta}\right)$, $\widehat{C}_k(x, (\Theta_{-k}, q_{-k}))$ is continuous at $x = \Theta_{-k}$ as a function of x. If it is, then from the fact that for $x = \Theta_{-k}$, $\widehat{C}_k(x, (\Theta_{-k}, q_{-k})) = \underline{C}(x) > \gamma$, also $\widehat{C}_k(x + \varepsilon, (\Theta_{-k}, q_{-k})) > \gamma$ for some $\varepsilon > 0$, and thus Θ_k defined by Lemma 5 is not smaller than $\Theta_{-k} + \varepsilon$. Thus fix $\Theta_{-k} \in \left(\frac{\lambda}{\mu}, \underline{\Theta}\right)$ and take $x_n \to \Theta_{-k}^+$. For such x_n, $\widehat{C}_k(x_n, (\Theta_{-k}, q_{-k}))$

$$\begin{aligned} & \frac{1}{x_n\mu - \lambda}\int_{\Theta_{-k}}^{x_n} c(u)\,du + \frac{\Theta_{-k}\mu - \lambda}{\Theta_{-k}\mu(x_n\mu - \lambda)}\int_0^{\Theta_{-k}} c(u)\,du \\ \to\ & \frac{1}{\Theta_{-k}\mu - \lambda}0 + \frac{1}{\Theta_{-k}\mu}\int_0^{\Theta_{-k}} c(u)\,du \\ & = \underline{C}(\Theta_{-k}) = \widehat{C}_k(\Theta_{-k}, (\Theta_{-k}, q_{-k})), \end{aligned}$$

which proves the desired property.

To prove part (c) of the lemma note that since now $\gamma \in \left[\frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du, \frac{1}{\mu}c(0)\right)$, the equation $\overline{C}(x) = \gamma$ has no solutions. Moreover, its LHS is always smaller than its RHS. On the other hand, since $\lim_{x\to 0^+}\underline{C}(x) = \frac{1}{\mu}c(0)$ and $\underline{C}(\frac{\lambda}{\mu}) = \frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du$, by the intermediate value property the equation $\underline{C}(x) = \gamma$ has a unique solution $\underline{\Theta} \in \left(0, \frac{\lambda}{\gamma}\right)$. Next, again by Lemma 4, $\widehat{C}_k(x, (\Theta_{-k}, q_{-k}))$ equals $\overline{C}(x)$ if $x > \Theta_{-k}$ and $\Theta_{-k} \le \frac{\lambda}{\mu}$ or $x = \Theta_{-k} < \frac{q_{-k}\lambda}{\mu}$, and $\underline{C}(x)$ if $x < \Theta_{-k}$ or $x = \Theta_{-k} > \frac{q_{-k}\lambda}{\mu}$, and thus by Lemma 5, $\Theta_k = \Theta_{-k}$ for $\Theta_{-k} \le \underline{\Theta}$ and $\Theta_k = \underline{\Theta}$ for $\Theta_{-k} > \underline{\Theta}$. The choice of q_k is made exactly as in part (b) of the lemma.

Finally, suppose that $\gamma \ge \frac{1}{\mu}c(0)$. Then for any value of x, $\widehat{C}(x, (\Theta_{-k}, q_{-k})) < \gamma$, and thus the optimal response of player k to the $[\Theta_{-k}, q_{-k}]$-threshold strategy of all the others is always to join the queue. ■

Now we are ready to state the main result of this section.

Theorem 1 *The game under consideration always has a symmetric equilibrium where each of the players uses the same $[\Theta, q]$-threshold strategy. Moreover:*

(a) If $\gamma \in \left(0, \frac{1}{\mu}\lim_{u\to\infty} c(u)\right]$ then the equilibrium is unique, with $\Theta = \infty$, which means that the equilibrium policies prescribe every user never to enter the queue.

(b) If $\gamma \in \left(\frac{1}{\mu}\lim_{u\to\infty} c(u), \frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du\right)$ then there are infinitely many equilibria, whose forms depend on the relation between $\overline{\Theta}$ and $\frac{\lambda}{\mu}$:

(b1) If $\overline{\Theta} < \frac{\lambda}{\mu}$ then there are equilibria of five types: $\Theta = \overline{\Theta}$ and any $q > \frac{\overline{\Theta}\mu}{\lambda}$; $\Theta = \Theta^*$, with Θ^* satisfying

$c(\Theta^*) = \mu\gamma$ and $q = \frac{\Theta^*\mu}{\lambda}$; $\Theta = \underline{\Theta}$ and any $q \in [0,1]$; any $\Theta \in \left[\overline{\Theta}, \frac{\lambda}{\mu}\right]$ and $q = 0$; any $\Theta \in \left[\overline{\Theta}, \frac{\lambda}{\mu}\right]$ and $q = 1$.

(b2) If $\overline{\Theta} = \frac{\lambda}{\mu}$ then either $\Theta = \overline{\Theta}$ and $q \in \{0,1\}$ or $\Theta = \underline{\Theta}$ and q is any number from $[0,1]$.

(b3) If $\overline{\Theta} > \frac{\lambda}{\mu}$ then either $\Theta = \underline{\Theta}$ and q is an arbitrary number from $[0,1]$ or $\Theta = \frac{\lambda}{\mu}$ and $q = 0$.

(c) *If $\gamma \in \left[\frac{1}{\lambda}\int_0^{\frac{\lambda}{\mu}} c(u)\,du, \frac{1}{\mu}c(0)\right)$ then there are infinitely many equilibria of three types: with $\Theta \in [0,\underline{\Theta}]$ and $q = 0$; with $\Theta \in [0,\underline{\Theta}]$ and $q = 1$; with $\Theta = \Theta^*$ satisfying $c(\Theta^*) = \mu\gamma$ and $q = \frac{\Theta^*\mu}{\lambda}$.*

(d) *If $\gamma \geq \frac{1}{\mu}c(0)$ then the equilibrium is unique, with $\Theta = 0$ and $q = 1$, which means that the equilibrium policies prescribe every user to always enter the queue.*

Proof: A strategy for any player k will induce a symmetric equilibrium if it is a best response to itself. Below we analyze which strategies may satisfy this condition.

In case (a) it is obvious by (a) of Lemma 6 that the policy prescribing never to join the queue is always the best response to itself, and since this is the only best response to any policy, this is the only possible equilibrium.

In case (b) Θ has to be either in interval $\left[\min\{\overline{\Theta}, \frac{\lambda}{\mu}\}, \frac{\lambda}{\mu}\right]$ or equal to $\underline{\Theta}$. In the latter case it is clear that for any q the $[\Theta, q]$-threshold policy will be the best response to itself. In the former one, q and Θ must satisfy one of the following conditions:

$q = 0$ and $\Theta > \frac{q\lambda}{\mu} = 0$, which is always true for $\Theta \geq \min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\}$, so $[\Theta, 0]$-threshold policies form equilibria in this case.

$q = 1$ and $\Theta < \frac{q\lambda}{\mu} = \frac{\lambda}{\mu}$, and so $[\Theta, 1]$-threshold policies form equilibria for any $\Theta \in \left[\min\left\{\overline{\Theta}, \frac{\lambda}{\mu}\right\}, \frac{\lambda}{\mu}\right)$.

$q = 1$ and $\Theta = \frac{q\lambda}{\mu} = \frac{\lambda}{\mu}$ with $c(\frac{\lambda}{\mu}) < \gamma\mu$, which is always true as long as $\overline{\Theta} < \frac{\lambda}{\mu}$.

$\Theta = \overline{\Theta} < \frac{q\lambda}{\mu}$, which implies that $q > \frac{\overline{\Theta}\mu}{\lambda}$. It can always be satisfied when $\overline{\Theta}$ is as assumed, so $[\overline{\Theta}, q]$-threshold policies form an equilibrium in this case.

$\Theta = \frac{q\lambda}{\mu} \in \left[\overline{\Theta}, \frac{\lambda}{\mu}\right]$ and $c(\Theta) = \mu\gamma$. Note however that by the definition of $\overline{\Theta}$ and continuity of c, if $\overline{\Theta} < \frac{\lambda}{\mu}$ then there must exist a solution Θ^* to the equation $c(\Theta) = \mu\gamma$ in $\left[\overline{\Theta}, \frac{\lambda}{\mu}\right]$, so Θ^* and $q = \frac{\Theta^*\mu}{\lambda}$ is an equilibrium. In particular, if $\overline{\Theta} = \frac{\lambda}{\mu}$, then also $\Theta^* = \frac{\lambda}{\mu}$ and $q = 1$ is one.

In case (c) Θ has to be in interval $[0, \underline{\Theta}]$ and needs to be related to q in one of the following ways:

$q = 0$ and $\Theta > \frac{q\lambda}{\mu} = 0$ or $\Theta = 0$ with $c(0) > \mu\gamma$, which is always true in case (c).

$q = 1$ and $\Theta < \frac{q\lambda}{\mu} = \frac{\lambda}{\mu}$, which is always true, as $\Theta \leq \underline{\Theta} < \frac{\lambda}{\mu}$ in this case, which was shown in the proof of Lemma 6.

$\Theta = \frac{q\lambda}{\mu} \leq \underline{\Theta}$ and $c(\Theta) = \mu\gamma$. Note however that by the definition of $\underline{\Theta}$ and continuity of c, there must exist some Θ^* in the interval $(0, \underline{\Theta})$ such that $c(\Theta^*) = \mu\gamma$, so Θ^* and $q = \frac{\Theta^*\mu}{\lambda}$ is the only equilibrium in this case.

Finally, in case (d), by (d) of Lemma 6 it is obvious that the policy always prescribing to join the queue is the best response to itself. Since this is the only best response to any policy in this case, this is the only possible equilibrium. This ends the proof of the theorem. ■

Remark 2 *It should be noted here that there are multiple equilibria in certain situations. In that case, it is normally not clear which one would prevail. Nevertheless, as the cost of being served is a decreasing function of Θ and of q for a fixed value of Θ, we may assume that the customers will naturally choose the equilibrium strategies with the biggest values of Θ and q.*

IV. Approximation of the Discrete Model

In the section below we present a result which joins the equilibria of the fluid model with ϵ-equilibria of the discrete model when the incoming rate is high. To formulate it, we need to introduce some additional notation, differentiating between the discrete and the fluid model. Let us start with fixing that the function c and parameters λ and μ define the fluid model, whose state will be denoted by X_t. Then let M_n be a discrete model with service cost $c^n(x) = c\left(\frac{x}{n}\right)$, incoming rate $\lambda^n = n\lambda$ and service rate μ. The state in model M_n will be denoted by X_t^n, while

$$\widetilde{X}_t^n := \frac{1}{n}X_t^n$$

will be its normalized state. Using this notation we can formulate the main result of this section and its proof.

Theorem 2 *Suppose that the initial (normalized) state of the queue $x_0 \in [0, x_{max}]$ for some fixed x_{max} and that the user k plays against $[\Theta, q]$-threshold policies of all the others (denoted shortly as π policies) in the fluid model with service cost c, incoming rate λ and service rate μ. Then for any $\varepsilon > 0$ there exists an N such that for any $n \geq N$ his expected cost from entering the queue in the discrete model M_n,*

$$\mathbb{E}\left[C_k^n(X_t^n(\pi^n))\right] = \mathbb{E}\left[\int_{t_k}^{t_k+\sigma_k} c^n(X_t^n(\pi^n))\,dt\right] - \gamma,$$

where π^n denotes a $[n\Theta, q]$-threshold policy (which is a proper rescaling of policy π to fit M_n), differs from the expected cost $\mathbb{E}\left[C_k(X_t(\pi))\right]$ in the fluid model by at most ε.

Proof: Let us consider two policies for the discrete model M_n:

$$\overline{\pi}^{\beta,n}(x) = \begin{cases} 0, & \text{when } x < n(\Theta - \beta) \\ \frac{x - n(\Theta-\beta)}{n\beta}, & \text{when } x \in [n(\Theta-\beta), n\Theta] \\ 1, & \text{when } x > n\Theta \end{cases}$$

and

$$\underline{\pi}^{\beta,n}(x) = \begin{cases} 0, & \text{when } x < n\Theta \\ \frac{x - n\Theta}{n\beta}, & \text{when } x \in [n\Theta, n(\Theta+\beta)] \\ 1, & \text{when } x > n(\Theta+\beta) \end{cases}.$$

They are rescalings of the following policies for the fluid model:

$$\overline{\pi}^\beta(x) = \begin{cases} 0, & \text{when } x < \Theta - \beta \\ \frac{x-\Theta+\beta}{\beta}, & \text{when } x \in [\Theta-\beta, \Theta] \\ 1, & \text{when } x > \Theta \end{cases}.$$

and

$$\underline{\pi}^\beta(x) = \begin{cases} 0, & \text{when } x < \Theta \\ \frac{x-\Theta}{\beta}, & \text{when } x \in [\Theta, \Theta+\beta] \\ 1, & \text{when } x > \Theta+\beta \end{cases}.$$

These policies differ from $[\Theta, q]$-threshold policy π only on sets $(\Theta-\beta, \Theta)$ or $(\Theta, \Theta+\beta)$ respectively. Next, consider equation (3) when all the players apply policy $\overline{\pi}^\beta$. It can be directly computed that the solution $X_t(\overline{\pi}^\beta)$ has the following properties:

1) $X_t(\overline{\pi}^\beta) \to X_t(\pi)$ pointwise as $\beta \to 0$.
2) Whenever $X_t(\overline{\pi}^\beta) \notin (\Theta-\beta, \Theta)$, it is of the form $X_t(\overline{\pi}^\beta) = D_1 e^{-\mu t} + D_2$ for some constants D_1, D_2 with $|D_1| \le \max\left\{x_0, \frac{\lambda}{\mu}\right\} \le \max\left\{x_{max}, \frac{\lambda}{\mu}\right\}$.
3) When $X_t(\overline{\pi}^\beta) \in (\Theta-\beta, \Theta)$, it satisfies equation

$$\dot{X}_t(\overline{\pi}^\beta) = \left(\frac{X_t(\overline{\pi}^\beta) - \Theta + \beta}{\beta}\right)\lambda - \mu X_t(\overline{\pi}^\beta), \forall t \ge 0$$

and consequently

$$\left|\dot{X}_t(\overline{\pi}^\beta)\right| \le \max_{x \in (\Theta-\beta,\Theta)} \left|\frac{x-\Theta+\beta}{\beta}\lambda - \mu x\right| \le \lambda + \mu\Theta.$$

Properties (ii) and (iii) clearly imply that $X_t(\overline{\pi}^\beta)$ is Lipschitz-continuous with constant $\max\left\{x_{max}, \frac{\lambda}{\mu}, \lambda+\mu\Theta\right\}$, independent of β. Thus all the functions $X_t(\overline{\pi}^\beta)$ are equicontinuous (as functions of t).

Next, we can find T_ε such that

$$\mathbb{E}\left[\sigma_k \mid \sigma_k > T_\varepsilon\right] \mathbb{P}[\sigma_k > T_\varepsilon] < \frac{\varepsilon}{8c(0)}. \tag{10}$$

Clearly, as $X_t(\overline{\pi}^\beta)$ are equicontinuous and converging to $X_t(\pi)$, by the Arzelà-Ascoli theorem $X_t(\overline{\pi}^\beta)$ converges to $X_t(\pi)$ uniformly on interval $[0, T_\varepsilon]$. On the other hand, c is continuous, decreasing and bounded, thus it is uniformly continuous, which means that there exists a $\delta > 0$ such that for any x, y such that $|x-y| < \delta$ we have $|c(x)-c(y)| < \frac{\varepsilon}{8T_\varepsilon}$. Using uniform convergence of $X_t(\overline{\pi}^\beta)$ we can further conclude that there exists a $\beta > 0$ such that

$$\sup_{t \in [0,T_\varepsilon]} \left|c(X_t(\overline{\pi}^\beta)) - c(X_t(\pi))\right| < \frac{\varepsilon}{8T_\varepsilon}. \tag{11}$$

Now note that by the Kurtz theorem (see Theorem 5.3 in [9]),

$$\mathbb{P}[\sup_{0 \le t \le T_\varepsilon} |\widetilde{X}_t^n(\overline{\pi}^{\beta,n}) - X_t(\overline{\pi}^\beta)| \ge \delta] \le De^{-nF(\delta)}$$

for some positive constant D and a function F satisfying $\lim_{\eta \searrow 0} \frac{F(\eta)}{\eta^2} \in (0, \infty)$. By this last property, the probability bounded above converges to zero as n goes to infinity at rate of e^{-n}, so for n large enough this probability is not larger than $\frac{\varepsilon}{8T_\varepsilon c(0)}$.

Next, using uniform continuity of c we get

$$\begin{aligned} &|\widetilde{X}_t^n(\overline{\pi}^{\beta,n}) - X_t(\overline{\pi}^\beta)| < \delta \\ \Longrightarrow\ & |c^n(X_t^n(\overline{\pi}_k^{\beta,n})) - c(X_t(\overline{\pi}_k^\beta))| < \frac{\varepsilon}{8T_\varepsilon}. \end{aligned} \tag{12}$$

Finally we get after some Algebra

$$\left|\mathbb{E}\left[C_k^n(X_t^n(\overline{\pi}^{\beta,n}))\right] - \mathbb{E}\left[C_k(X_t(\pi))\right]\right| \tag{13}$$

$$< T_\varepsilon \frac{\varepsilon}{8T_\varepsilon} + \frac{\varepsilon}{8T_\varepsilon} + c(0)T_\varepsilon \frac{\varepsilon}{8T_\varepsilon c(0)} + c(0)\frac{\varepsilon}{8c(0)} = \frac{\varepsilon}{2}, \tag{14}$$

where the last inequality is a consequence of (10), (11), (12) and the bound on the probability that $\widetilde{X}_t^n(\overline{\pi}^{\beta,n})$ and $X_t(\overline{\pi}^\beta)$ differ by more than δ (recall that $c(0)$ is the biggest value of c)

Now we can repeat all the above considerations for policies $\underline{\pi}^\beta$ and $\underline{\pi}^{\beta,n}$, obtaining similiar inequality

$$\left|\mathbb{E}\left[C_k^n(X_t^n(\underline{\pi}^{\beta,n}))\right] - \mathbb{E}\left[C_k(X_t(\pi))\right]\right| < \frac{\varepsilon}{2}. \tag{15}$$

To complete the proof note that $X_t^n(\underline{\pi}^{\beta,n})$, $X_t^n(\pi)$ and $X_t^n(\overline{\pi}^{\beta,n})$ are birth-death processes starting at the same x_0, with the same death rate, but with increasing birth rates. As a consquence $X_t^n(\underline{\pi}^{\beta,n})$ is for any $t \ge 0$ stochastically dominated by $X_t^n(\pi)$, which in turn is stochastically dominated by $X_t^n(\overline{\pi}^{\beta,n})$. This however implies that

$$\mathbb{E}\left[\int_{t_k}^{t_k+\sigma_k} c^n(X_t^n(\underline{\pi}^{\beta,n}))\,dt\right] \ge \mathbb{E}\left[\int_{t_k}^{t_k+\sigma_k} c^n(X_t^n(\pi^n))\,dt\right]$$

$$\ge \mathbb{E}\left[\int_{t_k}^{t_k+\sigma_k} c^n(X_t^n(\overline{\pi}^{\beta,n}))\,dt\right],$$

which is equivalent to

$$\mathbb{E}\left[C_k^n(X_t^n(\underline{\pi}^{\beta,n}))\right] \ge \mathbb{E}\left[C_k^n(X_t^n(\overline{\pi}^{\beta,n}))\right].$$

This, together with (14) and (15) implies the theorem. ■

Using Theorem 2 we can immediately show that all the results proved for the mean-field model can be viewed as good approximations of what happens in the discrete case when service rates go to infinity.

V. Numerical Analysis

Here we numerically evaluate NE strategy profile in our setting for some special class of cost functions $c(\cdot)$. We consider the following cost function

$$c(u) = \frac{1}{a+u} \tag{16}$$

where $a > 0$. It is easy to discern that $c(\cdot)$ is strictly decreasing with u.

To avoid cumbersomeness, henceforth we denote $\dfrac{\lambda}{\mu}$ as ρ.

From (9), $\bar{\Theta}$ is the solution of $\dfrac{1}{\lambda - \bar{\Theta}\mu}\int_{\bar{\Theta}}^{\rho} c(u)du = \gamma$ which gives after some Algebra

$$\bar{\Theta} = -\frac{\text{LambertW}(-\gamma\mu(a+\rho)e^{-\gamma\mu(a+\rho)}) + \gamma\mu a}{\gamma\mu} \tag{17}$$

Since from (17) the minimum value of the argument of LambertW function can be $-e^{-1}$, thus $\bar{\Theta}$ is always real valued.

Again from (9) $\underline{\Theta}$ is the solution of $\frac{1}{\underline{\Theta}}\int_0^{\underline{\Theta}} c(u)du = \gamma$ or equibalently

$$\frac{1}{\underline{\Theta}\mu}\log(\frac{a+\underline{\Theta}}{a}) = \gamma$$

In order to solve the above equation, we use the matlab function *fsolve*.

We consider $a = 0.2, \lambda = 5, \mu = 10$. Thus, $\rho = 0.5$. Also, note that

$$\frac{1}{\lambda}\int_0^{\rho} c(u)du = \frac{1}{\lambda}\log(\frac{a+\rho}{a}) = 0.2503 \qquad (18)$$

Now, we are ready to state formally all the possible NE strategy profile using theorem 1

1) $\gamma \geq \frac{1}{a\mu} = 0.5$, then $\Theta = 0$ and $q = 1$. Thus, players will always enter the queue.
2) $\gamma \in [0.2503, 0.5)$, then there are infinitely many equilibria which are of the following types:
   a) $\Theta \in [0, \underline{\Theta}], q = 0$.
   b) $\Theta \in [0, \underline{\Theta}], q = 1$.
   c) $\Theta = \frac{1-a\mu\gamma}{\mu\gamma}, q = \frac{1-a\mu\gamma}{\lambda\gamma}$
3) $\gamma \in (0, 0.2503)$, then there are infinitely many equilibria, which are of the following types:
   a) If $\bar{\Theta} < \rho$, (which occurs when $\gamma > \frac{1}{7}$) then there are five types of equilibria:
      - $\Theta = \bar{\Theta}, q > \frac{\bar{\Theta}}{\rho}$
      - $\Theta = \frac{1-a\mu\gamma}{\mu\gamma}, q = \frac{1-a\mu\gamma}{\lambda\gamma}$
      - $\underline{\Theta}, q \in [0, 1]$
      - $\Theta \in [\bar{\Theta}, \rho], q = 0$.
      - $\Theta \in [\bar{\Theta}, \rho], q = 1$.
   b) If $\bar{\Theta} = \rho$ (which occurs when $\gamma = \frac{1}{7}$), then either $\Theta = \bar{\Theta}$ and $q \in \{0, 1\}$ or $\underline{\Theta}$ and $q \in [0, 1]$[8]
   c) If $\bar{\Theta} > \rho$ (which occurs when $\gamma < \frac{1}{7}$), then either $\Theta = \underline{\Theta}, q \in [0, 1]$ or $\Theta = \rho$ and $q = 0$.

Figure 1 shows the variation of $\underline{\Theta}$ and the lowest possible threshold value of NE strategy profile with γ. From the above characterization of NE, it is easy to discern that the lower threshold value is $\max\{\min\{\bar{\Theta}, \rho\}, 0\}$ i.e. there is no NE with the threshold value lower than the above value. On the upper threshold is always given by $\underline{\Theta}$ i.e. there is no NE with the threshold value higher than $\underline{\Theta}$. Note that lower threshold value goes to 0 at $\gamma = 0.2503$ and the upper threshold value goes to 0 at $\gamma = 0.5$.

VI. Conclusions

We have studied a mean field limit of the M/M/∞ queuing system. It proved to be much simpler to handle and solve than the original discrete problem. We obtained closed-form formulas for all equilibria in the model. Finally, we have shown that the mean-field approximation becomes tight as the workload increases, thus the results obtained can be viewed as good approximations of what happens in the original discrete model.

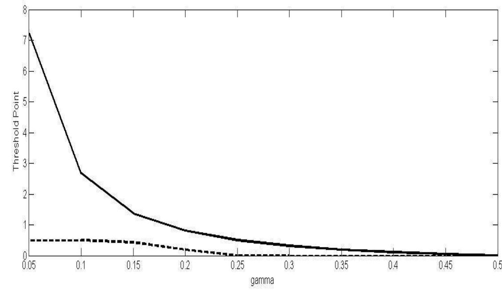

Fig. 1. Variation of $\underline{\Theta}$ and Lower Threshold value with γ

[8] In this case $\underline{\Theta} = 1.4966$

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