Homogeneous Resonance Asymptotic Stability for Homogeneous Systems

Henry Hermes, Matthias Kawski, Fabio Ancona

:

HOMOGENEOUS RESONANCE & ASYMPTOTIC STABILITY FOR HOMOGENEOUS SYSTEMS

HENRY HERMES

Department of Mathematics, Box 395

University of Colorado

Boulder, CO 80309–0395, USA

hermes@euclid.colorado.edu

MATTHIAS KAWSKI†

School of Mathematical and Statistical Sciences

Arizona State University

Tempe, AZ 85287–1804, USA

kawski@asu.edu

FABIO ANCONA

Dipartimento di Matematica

Università degli Studi di Padova

Via Trieste n.63, 35121 Padova, Italy

ancona@math.unipd.it

ABSTRACT. This paper surveys existing necessary conditions, and gives new conditions based on homogeneous resonance, for a homogeneous system to admit a homogeneous (of correct order) continuous, asymptotically stabilizing, state, feedback control. Such conditions are basic in utilizing high order, homogeneous, approximations of non-

†This research was supported by the National Science Foundation grant DMS 09–08204
The third author was partially supported by the European Union 7th Framework Programme [FP7-PEOPLE-2010-ITN] under grant agreement n.264735-SADCO and by Fondazione CaRiPaRo Project "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems"

linear systems to construct asymptotically stabilizing feedback controls when linear approximations of the nonlinear system are not locally controllable. For a class of three dimensional, homogeneous, locally controllable, systems, we determine at which resonance values they can be stabilized.

INTRODUCTION. If a linear approximation of a nonlinear control system admits a linear asymptotically stabilizing feedback control (hereafter ASFC) this control is also a local ASFC for the nonlinear system. It is well known that if a linear approximation is controllable, it admits a linear ASFC and the construction of such is known. If a linear approximation is not controllable, it is possible that a (higher order) homogeneous, nonlinear, approximation is small time locally controllable (STLC), (see [Su],[K4],[AL]), and if such an approximation admits a continuous homogeneous ASFC (of appropriate order) then it also is a local ASFC for the original system [BS],[H2],[S]. This is the major motivation for developing necessary and sufficient conditions that a homogeneous system admit a continuous, homogeneous ASFC. In [G], Grüne studies the existence of a discontinuous, homogeneous, ASFC. The main result of this paper is a new necessary and sufficient condition, based on homogeneous resonance [A2], for a class of three dimensional, nonlinear, systems. Some modern control systems have linearizations which are not controllable, thereby giving the system "quick response" but still rely on control for stabilization. The high order approximation of such systems is often a high order, homogeneous approximation [H2]. Kawski [K3] has shown that even if the homogeneous approximating system is STLC the system need not be stabilizable by a continuous, homogeneous ASFC. We extend his ideas here. Sepulchre [Sep] has shown that a STLC homogeneous system may admit a continuous ASFC (but not a continuous, homogeneous ASFC) but this control need not asymptotically stabilize a nonlinear system having this homogeneous approximation.

For uncontrolled systems the interplay between resonance and instability is well illustrated by the Tacoma Narrows bridge. This paper deals with the problem of determining at which resonance values an STLC nonlinear control system can be stabilized. This is answered for a class of three dimensional systems.

For $\varepsilon > 0$ a dilation is a mapping $\delta_\varepsilon^r : R^n \to R^n$, $\delta_\varepsilon^r(x) = (\varepsilon^{r_1}x_1, \dots \varepsilon^{r_n}x_n)$, with $r_1, \dots, r_n$ positive integers. A real valued function f is said to be homogeneous of degree k with respect to a dilation δ_ε^r if $f(\delta_\varepsilon^r x) = \varepsilon^k f(x)$. We let $\mathcal{H}_k^r(\delta_\varepsilon^r)$, or just $\mathcal{H}_k^r$ if the dilation is understood, denote **continuous** functions homogeneous of degree k. Note that $\mathcal{H}_k^r$ need not be finite dimensional, e.g. we will consider controls $u \in \mathcal{H}_k^r$ and for, say, $r = (1, 3)$, $u(x) = (x_1^3 + x_2)^{1/3}$ is homogeneous of degree one. The notation $\mathcal{H}^{r,l}(\delta_\varepsilon^r)$ or $\mathcal{H}^{r,l}$ when the dilation is obvious, will be used to denote **analytic** vector fields, homogeneous of degree l with respect to δ_ε^r. A *vector field* $X \in \mathcal{H}^{r,k}$ if for all smooth $f \in \mathcal{H}_l^r, \quad Xf \in \mathcal{H}_{l-k}^r, \quad l \geq k$. In local coordinates, if $X(x) = \sum_{i=1}^n a_i(x)\partial/\partial x_i$ then X is homogeneous of degree k if a_i is analytic and in $\mathcal{H}_{r_i-k}^r, \quad i = 1, \dots, n$. $\mathcal{H}^{1,l}$ will denote analytic vector fields, homogeneous of degree l with respect to the standard dilation δ_ε^1 having $r_1, \dots, r_n = 1$. For fixed $r, l, \quad \mathcal{H}^{r,l}$ is a finite dimensional vector space.

In what follows we examine four three-dimensional STLC control systems. Each contains a parameter $k > 0$: each is homogeneous with respect to the dilation $\delta_\varepsilon^r x = (\varepsilon x_1, \varepsilon^3 x_2, \varepsilon^9 x_3)$,and each has an uncontrollable linearization and unstable uncontrolled dynamics. The problem is to determine values of k for which there exists a continuous, asymptotically stabilizing, state feedback control u which is homogeneous of degree one with respect to δ_ε^r i.e. $u \in \mathcal{H}_1^r$.

A main result (see theorem 1) is that if k is a homogeneous resonance value, [A2], for which the system can be transformed into a homogeneous normal form, an extension of the Poincaré normal form, (hereafter Ancona normal form), there exists a free constant in the normal form which can be used to show that a homogeneous ASFC does not exist. If k is not a resonance value the system does transform to Ancona normal form and the existence of a continuous, homogeneous, ASFC is proved. For some such (limited values) k, Sepulchre, [SEP], gives an elegant geometrical construction based on work of Coron-Praly, [CP], and Kawski, [K3], to show existence of a homogeneous ASFC. This is extended in theorem 1 to all nonresonance values by constructing a system containing a free parameter which is diffeomorphic to the original system. The parameter can then be chosen to extend the Sepulchre construction. We also include open problems and conjectures. Some basic results for the four systems depend on two theorems, one due to Coron–Praly [CP],(discussed in Appendix A1-1), the other due to Kawski, [K2], and material found in the doctoral thesis of R. Sepulchre [Sep]. The use of the Coron–Praly theorem, as given in R. Sepulchre's doctoral thesis, is also given in Appendix A1-1.

The four systems have the form

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \qquad \dot{x}_3 = kx_3 + P_9(x_1, x_2) \tag{0.1}$$

where P_9 is a homogeneous polynomial, of degree 9, with respect to δ_ε^r. In general, we denote by $r = (r_1, \ldots, r_n)$ the dilation exponents. For our above systems the dilation is $r = (1, 3, 9)$.

Kawski Resonance, Homogeneous Resonance, and Kawski's theorem.[K3],[A1].

Homogeneous Resonance [A1]. Let X be an analytic vector field homogeneous of degree zero with respect to a dilation δ_ε^r. Expand $X(x) = Ax+$(higher order terms with respect to the standard dilation). Let $\lambda_1, \cdots, \lambda_n$, denote the eigenvalues of A. Then A is said to be *homogeneous resonant of order l with respect to δ_ε^r* if there exist relations of the form

$$\lambda_i = \sum_{s=1}^{n} \nu_s \lambda_s, \qquad r_i = \sum_{s=1}^{n} \nu_s r_s$$

with $\nu_1, \cdots, \nu_n$ non negative integers, $\sum_{s=1}^{n} \nu_s = l$, with $l \geq 2$. If there is no resonance of any order, A is said to be non resonant.

REMARK 0-1. It is basic to note that changing the first equation in (0.1) to be $\dot{x}_1 = ax_1 + v$, which can always be done by introduction of the new control $v(x) = u(x) - ax_1$, does not change the existence, or lack of existence, of an ASFC. However, the term ax_1 can change the resonance values! We will always absorb terms in the first equation which are homogeneous of degree one into the control and thereby have the first equation $\dot{x}_1 = u(x)$.

Theorem. (Ancona [A1]). *Let X be a real analytic vector field, homogeneous of degree zero with respect to the dilation δ_ε^r. If the linear part A of X is non resonant with respect to δ_ε^r there exists a real analytic, homogeneous, coordinate change (a diffeomorphism of R^n) which preserves homogeneity and transforms X to its linear part.*

Remark 0-2. As we shall see, non resonance is sufficient, but not necessary, to transform X to its linear part, preserving homogeneity.

Let $\mathcal{A}(x)$ denote the vector field Ax and ad$\mathcal{A}$ be defined by (ad$\mathcal{A}$,v)=[$\mathcal{A}$,v], the Lie product of the vector fields. A key step in the proof of Ancona's theorem is that if A has resonance of order l then the linear map $\mathrm{ad}\mathcal{A} : \mathcal{H}^{r,0} \cap \mathcal{H}^{1,l-1} \to \mathcal{H}^{r,0} \cap \mathcal{H}^{1,l-1}$ is not onto, i.e. has a nontrivial kernel.

Definition Let X, Y be real analytic vector fields homogeneous of degrees zero and minus one, respectively, with respect to the dilation δ_ε^r. If the system $\dot{x} = X(x) + uY(x)$ can be transformed to $\dot{y} = Ay + uZ(y)$, where A is the linear part of X, and $Z \in \mathcal{H}^{r,1}$ we call the latter a *homogeneous Ancona normal form* of the former. **This normal**

form need not be unique.

Suppose the nonlinear systems (0.1) can be transformed to "Ancona normal form", i.e. there exists a change of variable which transforms system (0.1) to the form

$$\dot{y} = Ay + uY(y), \qquad A = A(k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}. \tag{0.2}$$

Let $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$. The vector field Ry is called the *Euler field* associated with the dilation δ_ε^r.

Kawski's Theorem [K1] [K2]. *If there exists a $c \geq 0$ such that rank $(A - cR) \leq 1$, then system (0.2) has Kawski resonance (for the given value of k) and system (0.2), hence also system (0.1), does not admit an asymptotically stabilizing, feedback control (hereafter ASFC) in $\mathcal{H}_1^r$.*

Coron-Rosier Lemma [C1]. *Assume the n dimensional system (i) $\dot{x} = F(x)$ is homogeneous with respect to a dilation δ_ε^r, $r = (r_1, \ldots, r_n)$. Let R be the diagonal matrix R=diag $(r_1, \ldots, r_n)$and assume the null solution of (i) is asymptotically stable. Then:*

(a) For $m \geq r_n + 1$, system (i) admits a Lyapunov function V(x) which is homogeneous of degree m with respect to δ_ε^r and is C^1 on $\mathbf{R}^n - \{0\}$.

(b) Any homogeneous Lyapunov function V for system (i) is also a Lyapunov function for $\dot{x} = F(x) - cRx, \quad c > 0$.

(c) The null solution of $\dot{x} = F(x) - cRx$ is also asymptotically stable for any $c \geq 0$.

In section 1 we will present four examples, each of the form (0.1) and homogeneous with respect to the dilation $\delta_\varepsilon^r x = (\varepsilon x_1, \varepsilon^3 x_2, \varepsilon^9 x_3)$ and STLC at the origin. This means that the first Brockett necessary condition, [B], for the existence of a continuous ASFC (i.e. that there exist open loop controls which drive initial points near the origin toward the origin as $t \to \infty$) is satisfied.

From (0.1) we see each example will have the same linear part with matrix A having eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = k$ and dilation exponents $r_1 = 1, r_2 = 3, r_3 = 9$. Homogeneous resonance occurs at $k = 3, 2, 1, 0$. Indeed:

$$\lambda_3 = 3 = 3\lambda_2 + 0\lambda_1, \quad r_3 = 9 = 3r_2 + 0r_1$$

showing k=3 gives resonance of order 3.

$$\lambda_3 = 2 = 2\lambda_2 + 3\lambda_1, \quad r_3 = 9 = 2r_2 + 3r_1$$

showing k=2 gives resonance of order 5.

$$\lambda_3 = 1 = 1\lambda_2 + 6\lambda_1, \quad r_3 = 9 = 1r_2 + 6r_1$$

showing k=1 gives resonance of order 7.

$$\lambda_3 = 0 = 0\lambda_2 + 9\lambda_1, \quad r_3 = 9 = 0r_2 + 9r_1$$

showing k=0 gives resonance of order 9.

We will be interested only in the case $k > 0$ for which it easily follows that, with the examples in their initial form, the following necessary condition is satisfied.

Brockett necessary condition (iii), [B]. *A necessary condition that system (0.1) admit a continuous ASFC is that*

$$\{(u, x_2 + x_1^3, kx_3 + P_9(x_1, x_2)) : x \in \mathbf{R}^n, u \in \mathbf{R}^1\}$$

covers a neighborhood of the origin in $\mathbf{R}^3$.

Note. To utilize the Coron-Rosier lemma in (0.1) one must assume the control u is homogeneous. Then the Brockett condition (iii) becomes a necessary condition for the existence of a continuous homogeneous ASFC.

Hereafter we consider $k > 0$. All examples have homogeneous resonance at $k = 1, 2, 3$.

Transforming system (0.1) to Ancona normal Form. The general transform of (0.1) to Ancona normal form proceeds as follows. Let

$$y_1 = x_1, \quad y_2 = x_2 + x_1^3, \quad y_3 = kx_3 + f(x_1, x_2) \tag{0.3}$$

or $y = \phi(x)$, where f is a δ_ε^r homogeneous polynomial of degree 9. Specifically, for our examples, we choose

$$f(x_1, x_2) = ax_2^3 + bx_1^3x_2^2 + cx_1^6x_2 + ex_1^9. \tag{0.4}$$

It is useful to leave the transformation of system (0-1) in the form

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2u, \quad \dot{y}_3 = k(kx_3 + P_9(x_1, x_2)) + f_{x_2}(x_2 + x_1^3) + f_{x_1}u.$$

Thus to obtain Ancona normal form we need

$$kP_9(x_1, x_2) + (x_2 + x_1^3)f_{x_2}(x_1, x_2) = kf(x_1, x_2) \tag{0.5}$$

which yields simple linear equations to determine the constants a, b, c, e by equating like powers of x_1, x_2. If (0.5) can be satisfied the system (0.1) transforms to

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2u, \quad \dot{y}_3 = ky_3 + f_{x_1}(\phi^{-1}(y))u.$$

At resonance values k, the transform may not exist, or if it exists, it is not unique, i.e. the constants a, b, c, e are not uniquely determined.

THEOREM 1. *(A) The systems (1.1.1),(1.2.1),(1.3.1), (1.4.1), below, do not admit an ASFC in $\mathcal{H}_1^r$ at a resonance value k for which they can be transformed to Ancona normal form.*

(B) A continuous, homogeneous, ASFC does exist for all other values $k > 0$.

SECTION 1. THE FOUR EXAMPLES & THE ROLE OF RESONANCE.

1.1. Example 1. (Kawski)

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \quad \dot{x}_3 = kx_3 + x_1^9. \tag{1.1.1}$$

For any $k \neq 0$ this can be transformed to the Ancona normal form, i.e.

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \quad \dot{y}_3 = ky_3 + 9y_1^8 u. \qquad (1.1.2).$$

By using the Coron-Praly theorem to eliminate the integrator and the construction, as given in the appendix, Theorem A-1, it is shown in [Sep, §6.2.1]:

(1-A) System (1.1.1) has an ASFC in $\mathcal{H}_1(\delta_\varepsilon^r)$ if $k < 1$ or $k > 3$.

(1-B) From (1.1.2) and Kawski's theorem, there is no ASFC in $\mathcal{H}_1(\delta_\varepsilon^r)$ if $k = 3$. (Using the Coron-Rosier lemma, subtract $(1/3)Ry$ from the right side of (1.1.2) and the resulting system is easily shown to not satisfy the Brockett necessary condition (iii). The Brockett necessary condition (iii) is a vector field index condition, hence invariant under a coordinate change, such as transforming to Ancona normal form. The index can, however, be changed by subtracting $cRy, \quad c > 0$.)

(1-C) For $k \in [1, 3)$ it was unknown if there exists an AFSC in $\mathcal{H}_1^r$. A consequence of theorem 1 is that there is no such for $k = 1, 2$ but there is such for $k \in (1, 2)$ and $k \in (2, 3)$.

For the resonance value $k = 1$, of order 7, system (1.1.1) admits the Ancona normal form

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \quad \dot{y}_3 = y_3 + (6cy_1^5 y_2 + 3cy_1^8 + 9y_1^8)u. \tag{1.1.3}$$

Here the freedom in the normal form arises from the fact that the vector field $v^7(x) = (0, 0, cx_1^6 x_2)$ is not in the range of $\mathrm{ad}\mathcal{A} : \mathcal{H}^{r,0} \cap \mathcal{H}^{1,6} \to \mathcal{H}^{r,0} \cap \mathcal{H}^{1,6}$.

For the resonance value $k = 2$, system (1-1) admits the Ancona normal form

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \quad \dot{y}_3 = 2y_3 + (3cy_1^2 y_2^2 + cby_1^5 y_2 + 9y_1^8)u. \tag{1.1.4}$$

For resonance value $k = 3$ the Ancona normal form is

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \quad \dot{y}_3 = 3y_3 + (9cy_1^2 y_2^2 + 9y_1^8)u. \tag{1.1.5}$$

One can get a *reduced order*, related, system to (1.1.1) as follows. Multiply the first equation by $3x_1^2$, define $v = 3x_1^2 u, \quad y_1 = x_1^3, \quad y_2 = x_2, \quad y_3 = x_3$ getting the system

$$\dot{y}_1 = v, \quad \dot{y}_2 = y_2 + y_1, \quad \dot{y}_3 = ky_3 + y_1^3. \tag{1.1.1a}$$

This system is homogeneous with respect to the dilation δ_ε^r having $r = (1, 1, 3)$, again $k = 1, 2, 3$ are resonance values. The Sepulchre analysis of system (1.1.1a) is exactly the same as that for system (1.1.1) and the known results (1-A), (1-B), (1-C), above, hold for system (1.1.1a) also. One should note that knowing an ASFC v for system (1.1.1a) does not mean one can recover such (or that such even exists) for system (1.1.1). System (1.1.1a) is often referred to as the *dynamic extension* of the planar system

$$\dot{x}_1 = x_1 + u, \quad \dot{x}_2 = kx_2 + u^3. \tag{1.1.1b}$$

Sepulchre, [Sep, theorem 6.3] shows system (1.1.1b) does not admit a continuous ASFC for $k \in (1, 3]$. A theorem of Coron-Praly-Rosier (see [Sep, Theorem 5.4]) states that if (1.1.1b) admits a continuous, homogeneous, ASFC then so does system (1.1.1a). The converse need not, in general, be true so the fact that (1.1.1b) does not admit a continuous ASFC for $k \in (1, 3]$ does not mean (1.1.1a) has no continuous ASFC for $k \in (1, 3]$.

1.2. Example 2.

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \quad \dot{x}_3 = kx_3 + x_1^6 x_2. \tag{1.2.1}$$

For $k \neq 1$, this can be transformed to the Ancona normal form,

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \quad \dot{y}_3 = ky_3 + \left[\frac{6ky_1^5 y_2}{(k-1)} + \frac{3(3-2k)}{(k-1)} y_1^8\right] u. \tag{1.2.2}$$

By eliminating the integrator and using the Sepulchre approach one can show (see Appendix A2-1):

(2-A) System (1.2.1) has an ASFC in $\mathcal{H}_1^r$ if $0 < k < 3/2$.

(2-B) From (1.2.2) and the Kawski theorem, there is no ASFC in $\mathcal{H}_1^r$ if $k = 3$.

(2-C) For $k \in [3/2, 3)$ or $k > 3$ it was unknown if there exists an ASFC in $\mathcal{H}_1^r$. Theorem 1 shows there is no such for $k = 2$ while for $k \in [3/2, 2) \cup (2, 3)$ and $k > 3$ such an ASFC does exist.

For the resonance value $k = 2$ system (1.2.1) admits the Ancona normal form

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \quad \dot{y}_3 = 2y_3 + [c(3y_1^2 y_2^2 + 6y_1^5 y_2) + 12y_1^5 y_2 - 3y_1^8]u. \tag{1.2.3}$$

Since Kawski's theorem gives the result for $k = 3$ we will not give the general Ancona normal form for that resonance value.

The reduced order system associated with system (1.2.1) is

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + y_1, \quad \dot{y}_3 = ky_3 + y_1^2 y_2. \qquad (1.2.1a).$$

This system is homogeneous with respect to the dilation δ_ε^r having $r = (1, 1, 3)$. Again, the Sepulchre theorem analysis for (1.2.1a) is exactly the same as that for (1.2.1) and the results (2-A), (2-B), (2-C) above are valid for system (1.2.1a).

System (1.2.1a) is the dynamic extension of the planar system

$$\dot{x}_1 = x_1 + u, \quad \dot{x}_2 = kx_2 + x_1 u^2. \tag{1.2.1b}$$

Sepulchre, [Sep, theorem 6.4], shows that for $k = 3$ system (1.2.1b) admits a continuous ASFC, but not a continuous, homogeneous ASFC. Indeed, if it had a continuous, homogeneous, ASFC it would imply the same for system (1.2.1a). But subtracting Ry from system (1.2.1a) yields a system which fails the Brockett necessary condition (iii) and hence, by the Coron-Rosier lemma, system (1.2.1a) does not have an ASFC in $\mathcal{H}_1^r$ for $k = 3$.

1.3 Example 3.

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \quad \dot{x}_3 = kx_3 + x_1^3 x_2^2. \tag{1.3.1}$$

For $k \neq 1, 2$, this can be transformed to Ancona normal form,

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \tag{1.3.2}$$

$$\dot{y}_3 = ky_3 + \left[\tfrac{3ky_1^2(y_2 - y_1^3)^2}{(k-2)} + \tfrac{6y_1^8(3-2k) + 12y_1^5 y_2}{(k-1)(k-2)} \right] u.$$

Again, by eliminating the integrator and using the Sepulchre approach on (1.3.1) one can show

(3-A) System (1.3.1) has an ASFC in $\mathcal{H}_1^r$ if $k \neq 3$.

(3-B) From (1.3.2) and Kawski's theorem, there is no ASFC in $\mathcal{H}_1^r$ for $k = 3$.

The reduced order system associated with system (1.1.3) is

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_1 + x_2, \quad \dot{x}_3 = kx_3 + x_1 x_2^2. \tag{1.3.1a}$$

and properties (3-A), (3-B) apply to system (1.3.1a) also.

Example 4. (Kawski, [K3].)

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \quad \dot{x}_3 = kx_3 + x_2^3. \tag{1.4.1}$$

Here, for $k \neq 1, 2, 3$ this can be put into Ancona normal form,

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_2 + 3y_1^2 u, \tag{1.4.2}$$

$$\dot{y}_3 = ky_3 + \left[\frac{9ky_1^2y_2^2}{(k-3)(k-2)} + \frac{18y_1^5y_2}{(k-2)(k-1)} - \frac{9y_1^8}{(k-1)}\right] u.$$

As shown by Kawski, and also easily shown by the Sepulchre analysis, system (1.4.1) has an ASFC in $\mathcal{H}_1^r$ for all $k > 0$.

1.5 THE ROLE OF HOMOGENEOUS RESONANCE.

While we will deal, here, with theorem 1 which is specific to the four examples, hopefully the proof extends to the more general result which, at this point we leave as

Conjecture. *Consider the analytic, affine, homogeneous, n-dimensional system*

$$\dot{x} = X(x) + uY(x). \tag{1.5.1}$$

where X is homogeneous of degree zero with respect to some dilation δ_ε^r and Y is homogeneous of degree one. Expand $X = Ax$ +(higher order terms) and assume the eigenvalues of A are non negative. If A has homogeneous resonance with respect to δ_ε^r but system (1.5.1) can still be transformed to Ancona normal form, then system (1.5.1) does not admit an ASFC in $\mathcal{H}_1^r$.

Remark 1.5.1. If A is allowed to have negative eigenvalues the above theorem would not be true. See example A-2 of the appendix.

The Ancona normal form of example 4 shows singularities at $k = 1, 2, 3$ (the homogeneous resonance values). This shows that if one attempts to achieve Ancona normal form at a resonance value for which a homogeneous ASFC exists, there would be need for "huge" control values near the resonance point. This occurs since if we were to write, in (1.5.1), $X(x) = Ax + f(x)$, at resonance f is not in the range of ad$\mathcal{A}$. Therefore it can have effect in yielding an ASFC when the system is in the form $\dot{x} = Ax + f(x) + uY(x)$ but the goal of the Ancona form is to remove f.

The Ancona normal form of example 3 has singularities at $k = 1, 2$ but not at $k = 3$. Again, in normal form, the control has huge effect near $k = 1, 2$ but not near 3. Here there is an ASFC in $\mathcal{H}_1^r$ except for $k = 3$. The Ancona normal form of example 2 has a singularity only at $k = 1$ but not at the other two resonance values $k = 2, 3$. Here there is an ASFC in $\mathcal{H}_1^r$ for all $k > 0$, except for $k = 2, 3$.

In example 1, there are no singularities in the Ancona normal form, i.e. no strong control effects near or at the resonance values $k = 1, 2, 3$. We know an ASFC in $\mathcal{H}_1^r$ exists for $k < 1$ or $k > 3$. There is no such at $k = 1, 2, 3$.

1.6 The proof of THEOREM 1. We will provide, here, only a proof of the statement A, with $k = 1, 2$ of THEOREM 1 relative to the system (1.1.1), the case of system (1.2.1) being entirely similar. We next show the arguments for statement B for system (1.1.1) with $k = 1.5$ and for system (1.2.1) for $k = 3/2$ and $k = 4$. The remaining cases can be shown by similar constructions and therefore are not included. In the case of system (1.3.1) the conclusions follow from Kawski's Theorem 1.2 while system (1.4.1) cannot be transformed to Ancona normal form for any resonance value and all results are known.

A. We shall begin by showing that there is no ASFC in $\mathcal{H}_1^r$ for the system (1.1.1) for resonance value $k = 1$. For the resonance value $k = 1$, which is of order 7, the Ancona normal form (1.1.3) can be obtained from the general form (1.1.2) by a coordinate transformation. Specifically:

$$z_1 = y_1, \quad z_2 = y_2, \quad z_3 = y_3 + cy_1^6 y_2 \tag{1.6.1}$$

where the term $cy_1^6 y_2$ results from the resonance of order 7. Assume (1.1.1) has a homogeneous ASFC for $k = 1$ which means (1.1.2) has a homogeneous ASFC for $k = 1$

which we denote by $u^0(y)$. Therefore, for every c, system (1.1.3) has an ASFC denoted u^c. From the variable change (1.6.1), for any c

$$u^c(y_1, y_2, y_3) = u^0(y_1, y_2, y_3 - cy_1^6 y_2) \tag{1.6.2.}$$

Thus to show (1.1.1) does not have an ASFC for $k = 1$ it suffices to show there exists a value c^* for which system (1.1.3) does not admit an ASFC. For $k = 1$ and notational ease denote the system (1.1.2) as $\dot{y} = f^0(y, u^0(y))$ and system (1.1.3) as $\dot{y} = f^c(y, u^c(y))$ where u^0, u^c are related by (1.6.2).

Construction of c^* for k=1. Following Kawski, [K3], let $\nu(x) = \sum r_i x_i \partial/\partial x_i$ denote the Euler vector field (which generates homogeneous rays). Let $m = 2r_1 r_2 r_3$. Define the (smooth) δ_ε^r unit two sphere $S_r^2 = \{x \in \mathcal{R}^3 : \sum x_i^{m/r_i} = 1\}$. Let $\pi : (\mathcal{R}^3 - 0) \to S_r^2$ be defined by $\pi(x)$ is that point $y \in S_r^2$ where the homogeneous ray through x intersects S_r^2. Since $\pi(\delta_{e^{ct}}^r x) = \pi(x)$ for all t, evaluating a t-derivative of the previous at 0 one finds $\pi_*(\nu(x)) = 0$. Also, if $F(x) = \sum a_i(x)\partial/\partial x_i$ is a smooth vector field homogeneous of degree zero, one can compute that the Lie product $[F, \nu] = 0$ and $\pi_* F$ is well defined, i.e. for $y \in S_r^2$, $\quad \pi_* F(\delta_{e^{ct}}^r y) = \pi_* F(y)$. If y is such that $\pi_* F(y) = 0$ then $F(y)$ is a scalar multiple of $\nu(y)$ and this holds along the homogeneous ray through y hence the solution of $\dot{x} = F(x), \quad x(0) = y$ lies on the homogeneous ray through y. We proceed formally for a moment. Choose a value c_0 ; define $\pi_* f^{c_0}(z, u^{c_0}(z))$ as homogeneous projection of f^{c_0} onto the unit sphere S_r^2. Then $\pi_* f^{c_0}$ must have a zero on S_r^2 (it may have several), say at $z^{c_0} = (z_1^{c_0}, z_2^{c_0}, z_3^{c_0})$ and the solution of (1.1.3) through z^{c_0} then lies on the homogeneous ray through z^{c_0}. Specifically, this ray has the form

$$z_1(t) = z_1^{c_0} e^t, \quad z_2(t) = z_2^{c_0} e^{3t}, \quad z_3(t) = z_3^{c_0} e^{9t}.$$

(a) If $z_1^{c_0} = 0$, the solution of (1.1.3) lies in the plane $z_1 = 0$ and the third component of the solution is unstable.

(b) For $z_1^{c_0} \neq 0$, the first two components of the solution $y(t)$ of (1.1.3) through z^{c_0} lie on the curve

$$y_2/y_1^3 = z_2^{c_0}/(z_1^{c_0})^3.$$

For notational ease let $e^0 = z_2^{c_0}/(z_1^{c_0})^3$ and (for later) $e^i = z_2^{c_i}/(z_1^{c_i})^3$. Then $y_2(t) - e^0 y_1^3(t) \equiv 0$. Differentiating this and combining with the second equation of (1.1.3) yields

$$y_2(t) = 3y_1^2(t)u^{c_0}(e^0 - 1). \tag{i}$$

If $e^0 \neq 1$, from (i), $3y_1^2(t)u^{c_0} = y_2(t)/(e^0 - 1)$ and the second component of (1.1.3) becomes

$$\dot{y}_2 + 3y_1^2(t)u^{c_0} = y_2(t)(1 + (1/(e^0 - 1))) = e^0 y_2(t)/(e^0 - 1). \tag{ii}$$

Notice that $e^0/(e^0 - 1) > 0$ iff $e^0 \in (-\infty, 0) \cup (1, \infty)$. Therefore, in the case $z_1^{c_0} \neq 0$, it follows that the second component of the solution $y(t)$ of (1.1.3) through z^{c_0} is unstable whenever $e^0 \in (-\infty, 0) \cup (1, \infty)$.

(c) We proceed, by induction, to prove the existence of a constant c_* and equilibrium point z^* of $\pi_* f^{c_*}$ so that the solution of (1.1.3) through z^* is unstable.

Step 1: Fix $c_0 \in R$, let $z^{c_0} = (z_1^{c_0}, z_2^z c_0, z_3^{c_0})$ be a zero of $\pi_* f^{c_0}$. If either $z_1^{c_0} = 0$ or $e^0 \in (-\infty, 0) \cup (1, \infty)$ we have instability by (a), (b) and we are done. Otherwise we have $z_1^{c_0} \neq 0, \quad e^0 \in [0, 1]$ and we may define

$$c_1 = -3/(1 + 2e^0) \in [-3, 1]. \tag{iii}$$

Notice that c_1 is defined to make the coefficient

$$[c_1(2z_2^{c_0} + (z_1^{c_0})^3 + 3(z_1^{c_0})^3]$$

of u equal to zero in the third equation of (1-3). Now let z^{c_1} be a zero of $\pi_* f^{c_1}$, c_2 be defined by (iii) with e^0 replaced by e^1, etc. Inductively this defines a sequence of constants $\{c_i\}$, with $c_i \in [-3, -1]$ and points $z^{c_{i-1}} \in S_r^2$. If ever $z_1^{c_{i-1}} = 0$ or

$e^{i-1} \in (-\infty, 0) \cup (1, \infty)$ we may stop and the solution of (1-3) through $z^{c_{i-1}}$ will be unstable. If this does not occur, we obtain sequences $\{c_i\}, \{z^{c_{i-1}}\}$ having values in the compact sets [-3,-1], S_r^2 respectively, hence there exist subsequences which converge to $c_* \in [-3, -1], z^* \in S_r^2$.

Note. We need not worry about convergence of $\{u^{c_i}\}$ since u^{c_*} is given in terms of the assumed stabilizing control u^0 by (1.6.2).

Then $\pi_* f^{c_*}(z^*, u^{c_*}) = 0, \quad c_* = -3(z_1^*)^3/((z_1^*)^3 + 2z_2^*)$ and the third component of the solution of (1.1.3) through z^* satisfies $\dot{y}_3 = y_3$ and is unstable.

Construction of c_* for Example 1, k=2. With the notation of the previous case for $k = 1$, choose a real c_0, let $z^{c_0} \in S_r^2$ be a zero of $\pi_* f^{c_0}$ where now f^{c_0} is given by (1.1.4) with c replaced by c_0. Again, if $z_1^{c_0} = 0$ the solution of (1.1.4) lies in the plane $z_1 = 0$ and the third component of (1.1.4) is $\dot{y}_3 = 2y_3$ and unstable.

If $z_1 \neq 0$ and $e^0 = z_2^{c_0}/(z_1^{c_0})^3 \in (-\infty, 0) \cup (1, \infty)$, the second component of (1.1.4) reduces to $\dot{y}_2 = e^0 y_2/(e^0 - 1)$ with $e^0/(e^0 - 1) > 0$ and we have instability.

Now to make the coefficient of u equal to zero in the third equation of (1.1.4), formally we need

$$c_1 = -3(z_1^{c_0})^6/(z_2^{c_0} + 2(z_1^{c_0})^3)z_2^{c_0}$$

or if $z_1^{c_0} \neq 0$,

$$c_1 = -3(z_1^{c_0})^3/(e^0 + 2)z_2^{c_0}. \tag{iv}$$

If $z_2^{c_0} = 0$ the homogeneous ray through z^{c_0} on which the solution of (1.1.4) through z^{c_0} lies has second component which satisfies $\dot{y}_2 = 3y_1^2 u$. But from the first equation, $3y_1^2(t)\dot{y}_1 = d/dt(y_1^3(t)) = 3y_1^2 u$ so subtracting shows $d/dt(y_2(t) - y_1^3(t)) = 0, \quad y_2(t) - y_1^3(t) = z_2^{c_0} - (z_1^{c_0})^3 = (z_1^{c_0})^3$. But $y_2(t) \equiv 0$ hence $y_1(t) = -z_1^{c_0}$ is a nonzero constant and this shows instability.

In summary, if either $z_1^{c_0} = 0$ or $z_2^{c_0} = 0$ or $e^0 \in (-\infty, 0) \cup (1, \infty)$ we may stop since there would be instability. Thus assume$z_1^{c_0} \neq 0, \quad z_2^{c_0} \neq 0, e^0 \in [0, 1]$ in which case

(iv) will define $c_1 \in [-3/2, 3/2]$ and one can continue, inductively, to obtain sequences $\{c_i\}, \{z^{c_i}\}$ which have convergent subsequences converging to a real $c_* \in [-3/2, 3/2]$ and $z^* \in S_r^2$ with the third component of the solution of (1.1.4) through z^* being $\dot{y}_3 = y_3$ and hence unstable.

The details for instability, with $k = 2$ in Example 2 are similar and will be omitted.

B. We begin with seeking the most general system of the form

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \quad \dot{x}_3 = kx_3 + P_9(x_1, x_2) \tag{1.6.3}$$

with $k > 0$, which has (1.1.2) as its Ancona normal form. *For such a system, since both it and (1.1.1) smoothly transform to the same system, they can be smoothly transformed to each other and hence one has a homogeneous ASFC if and only if the other also does.*

In (1.6.3) choose

$$P_9 = \alpha x_1^9 + \beta x_2^3 + \gamma x_2^2 x_1^3 + \delta x_2 x_1^6.$$

Let $f(x_1, x_2)$ be given by (0-4), the transform $y = \phi(x)$ be given by (0-3) and explicitly, $f_{x_1}(\phi^{-1}(y)) = 3by_1^2y_2^2 - 6by_1^5y_2 + 3by_1^8 + 6cy_1^5y_2 - 6cy_1^8 + 9ey_1^8$.
We use (0.5) to attempt to transform (1.6.3) into (1.1.2). Equating like powers, using (0.5), requires

$$a = \beta k/(k-3), \quad b = \gamma k/(k-2) + 3\beta k/(k-2)(k-3), \tag{iii}$$

$c = 2\gamma k/((k-1)(k-2)) + 6\beta k/((k-1)(k-2)(k-3)) + \delta k/(k-1)$,
$e = (k\alpha + c)/k = \alpha + (c/k)$.
We assume, hereafter, that $k \neq 1, 2, 3$, i.e. the system we obtain will transform to system (1.1.2) except at the resonance values. Substituting these into $f_{x_1}(\phi^{-1}(y))$

which we need to be equal to $9y_1^8$ in order to have the Ancona transform of (1.6.3) be (1.1.2) requires $b = 0$ or $2\gamma k(k-3) + 6\beta k = 0$, $\quad c = 0$ or $\delta k(k-2)(k-3) = 0$ which requires $\delta = 0$, $\quad e = 1$ or $\alpha = 1$. Thus, from $b = 0$, $\quad \gamma = -3\beta/(k-3)$ with β arbitrary. Therefore the most general system of the form (1.6.3) which has Ancona transform system (1.1.2) is

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \tag{1.6.4}$$

$$\dot{x}_3 = kx_3 + x_1^9 + \beta x_2^3 - 3(\beta/(k-3))x_1^3 x_2^2, \quad k \neq 1, 2, 3.$$

System (1.6.4) has the advantage of the free constant β. We now use the generalized Sepulchre method of Appendix A to study system (1.6.4). Sepulchre [Sep] showed that for $0 < k < 1$ and for $3 < k$ the system (1.1.1) admits a homogeneous ASFC. we next use THEOREM A-1 (Generalized Sepulchre construction) to show that for $1 < k < 2$ and for $2 < k < 3$ there does exist a homogeneous ASFC.

Following the construction in appendix A, let

$$P_9(v^{1/3}, \theta) = v^3 + \beta\theta^3 - ((3\beta/(k-3))v\theta^2.$$

In short, THEOREM A-1 states:

For P_9 depending on both x_1 and x_2 a sufficient condition that equation (1.6.4) has a homogeneous ASFC is that there exists a line, having nonzero slope, in the (θ, v) plane which intersects the set

$$Z = \{(\theta, v) : -\beta\theta^4 + (3\beta/(k-3))v\theta^3 - (v^3 + k - 3)\theta + 3v = 0\} \tag{1.6.5}$$

only in points (θ^*, v^*) with either $v^*/\theta^* < -1$ or

$P_9((v^*)^{1/3}, \theta^*) < -k$, if $\quad \theta^* \neq 0$.

To illustrate the idea to show such a line is possible consider:

CASE $\quad k = 3/2$, $\quad \beta = -0.5$: (Here k can be chosen in the open interval(1,2) while $\beta < 0$ is quite arbitrary and all calculations work essentially the same.) Use MATHEMATICA to examine (get a geometrical picture of) the set Z in (1.6.5), i.e. the level curves of Z. For notational ease let $x = \theta, \quad y = v$ in Z. Use ContourPlot (and be sure to leave spaces between x^3 and y and also x and y^3) specifically:

ContourPlot$[0.5x^4 + x^3y - xy^3 + 1.5x + 3y, \{x, -10, 10\}, \{y, -10, 10\}]$.

Move the cursor around to find the (obvious) zero level curves. You now want a line, say $y = c_1x + c_2$, which intersects these only in points (x^*, y^*) with $y^*/x^* < -1$ or $P_9((y^*)^{1/3}, x^*) < -1.5, \quad x^* \neq 0$. See Fig. 1.

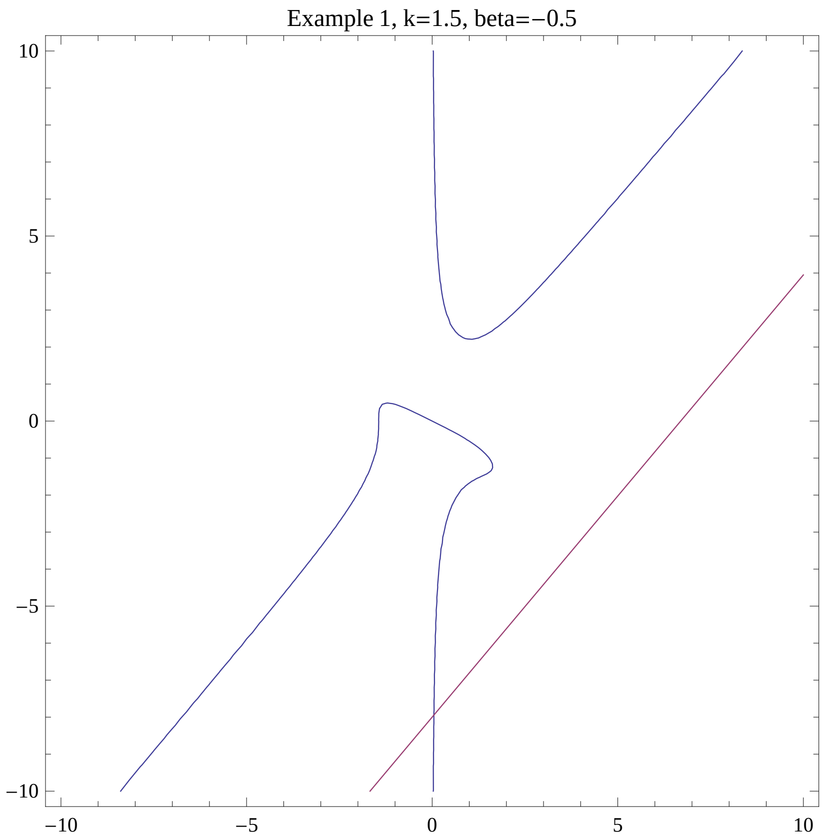

FIG 1, SYSTEM (1.6.4), $(k = 3/2, \quad \beta = -0.5)$

Substituting $y = c_1x + c_2$ one can use MATHEMATICA to seek these intersections, i.e., NSolve$[0.5x^4 + 1x^3(c_1x + c_2) - x(c_1x + c_2)^3 + 1.5x + 3(c_1x + c_2) = 0, x]$

For general c_1, this is a quartic and will have four roots (x^*, y^*) with at least one not satisfying THEOREM A.1. HOWEVER, if c_1 satisfies the cubic equation $-c_1^3 + 1c_1 + 0.5 = 0$ then instead of a quartic in the NSolve one gets a cubic, i.e. the x^4 terms vanish. The roots of this cubic are $c_1 = 1.19149$ and then two complex numbers. (It is instructional to roughly sketch the lines corresponding to c_1 with various negative values of c_2 in the MATHEMATICA output of the ContourPlot and see (roughly) where the lines intersect the zero level curves.) For $c_2 = -3$ there are three real crossings, with x values x=0.47917, x=1.16964, x=1.41774, and all are "bad" crossings, i.e. have $y^*/x^* > -1$. For $c_2 = -8$ the line $y = 1.19149x - 8$ crosses the zero curve of the ContourPlot at $x^* = 0.047416$ with corresponding $y^* = -7.9435$ (the other crossings have complex values, except for $x = 3.78063(10)^6$ which is a numerical round off root since with an exact value of c_1 the cubic in the ContourPlot equation will only have three roots.) Here $y^*/x^* < -1$. Hence (1.6.4) has a homogeneous ASFC for k=1.5 and $\beta = -0.5$ and (1.1.1) also has a homogeneous ASFC for k=1.5. (It is instructional to set $\beta = 0$ in the ContourPlot to see how the zero curves (obtained by Sepulchre [Sep])show an inability to find a line crossing them only at points satisfying Theorem A-1,and how these zero curves compare with the case $\beta = -0.5$.) With $\beta = -0.5$ and any $k \in (1, 2)$ or $k \in (2, 3)$ the above construction leads to a line satisfying THEOREM A.2.

For Example 1 with $k \in (2, 3)$, say $k = 5/2$, choose $\beta = -0.2$ in (1.6.5) and again choose a line with slope so that the coefficient of x^4 is zero. Here this requires $c_1 = -0.163054$. Choose the line $y = -0.163054x - 10$ and Mathematica shows this line crosses the curve $Z = 0$ with real root $x = 0.0299554$, two complex roots (and the numerical error root $x = 8.682(10)^7$). The crossing $(x^* = 0.0299554, y^* = 10.0048)$ shows Example 1 has a homogeneous ASFC at $k = 5/2$.

We next turn to system (1.2.1) of Example 2 and its Ancona transform system (1.2.2). The direct approach using the Generalized Sepulchre THEOREM A-1 shows the existence of a homogeneous ASFC if $0 < k < 3/2$. Details showing this direct approach will not work at $k = 3/2$ or $k > 3$ are given in appendix A2-1. We next briefly outline how to show such a control exists for $k \in [3/2, 2) \cup (2, 3)$ or $k > 3$. Again, we seek the most general system of the form (1.6.3) which has system (1.2.2) as its Ancona transform. Choose f as in (0-4), form (0-5), equate like powers to determine the constants, and one finds: For $k \neq 1, 2, 3$:

The most general system of the form (1.6.3) which has Ancona transform (1.2.2) is

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_2 + x_1^3, \quad \dot{x}_3 = kx_3 + \tag{1.6.6}$$

$$(\beta k/(k-3))x_1^9 + \beta x_2^3 - (3\beta/(k-3))x_2^2 x_1^3 + x_2 x_1^6, \quad k \neq 1, 2, 3.$$

Note that if $\beta = 0$ we get back system (2-1). For application of THEOREM A-1, here $P_9(x_1, x_2) = (\beta k/(k-3))x_1^9 + \beta x_2^3 - (3\beta/(k-3))x_2^2 x_1^3 + x_2 x_1^6$.

Again, let $x = \theta, \quad y = v^{1/3}$ in $P_9(v^{1/3}, \theta)$ and use MATHEMATICA to graph the zero curves of Z as given in the appendix by (A-6), with various values of k and β as illustrations. This leads to

ContourPlot$[\frac{-\beta k}{(k-3)}xy^3 - \beta x^4 + \frac{3\beta}{(k-3)}x^3 y - x^2 y^2 + (3-k)x + 3y, \{x, -10, 10\}, \{y, -10, 10\}]$

In APPENDIX A2-2 it is shown that the direct use ot THEOREM A-1 on system (1.2.1) for $k > 3$ will not give the existence of an ASFC in $\mathcal{H}_1^r$ but will for $k < 3/2$.

We next illustrate the use of the modified system (1.6.6) to show the existence of a homogeneous ASFC for $k = 3/2$ by choosing $\beta = -0.2$. (This construction works simililarly for $k \in (3/2, 2) \cup (2, 3)$.) We, again, use MATHEMATICA to plot level curves of Z. Here

ContourPlot$[-.2xy^3 + .2x^4 + .4x^3 y - x^2 y^2 + 1.5x + 3y, \{x, -10, 10\}, \{y, -1, 10\}]$

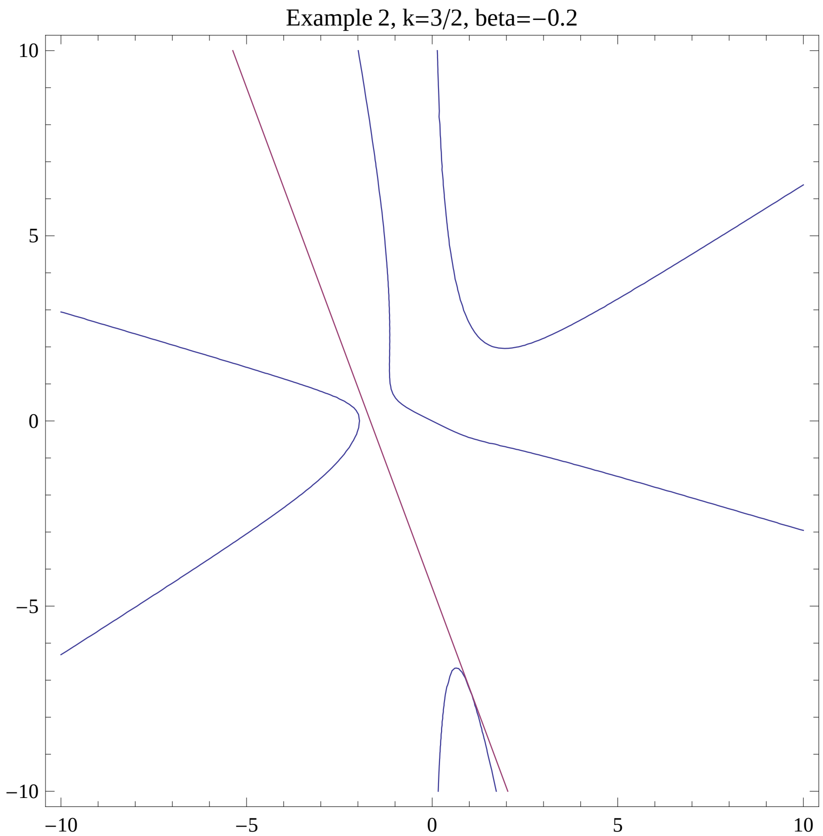

FIG.2, EX.2 , SYSTEM (1.6.6),($k = 3/2,\ \beta = -0.2$)

From the graph, a line such as $y + 7 = (-5.4/2)(x - 1)$ i.e. $y = -2.7x - 4.5$ seems reasonable. Solving (via MATHEMATICA), this line has two real intersections with the zero curve of Z, specifically $x^* = 0.794305$, with $y^* = -6.205695$ and $x^* = 1.48284$ with $y^* = -8.2071$. See fig.2. Thus for each $y^*/x^* < -1$ and there exists a homogeneous ASFC at $k = 3/2$.

The construction for the existence of a homogeneous ASFC for $k \in (3/2, 2)$ and $k \in (2, 3)$ is similar.

For $k > 3$ we take $k = 4$ as an illustration, the construction being similar for

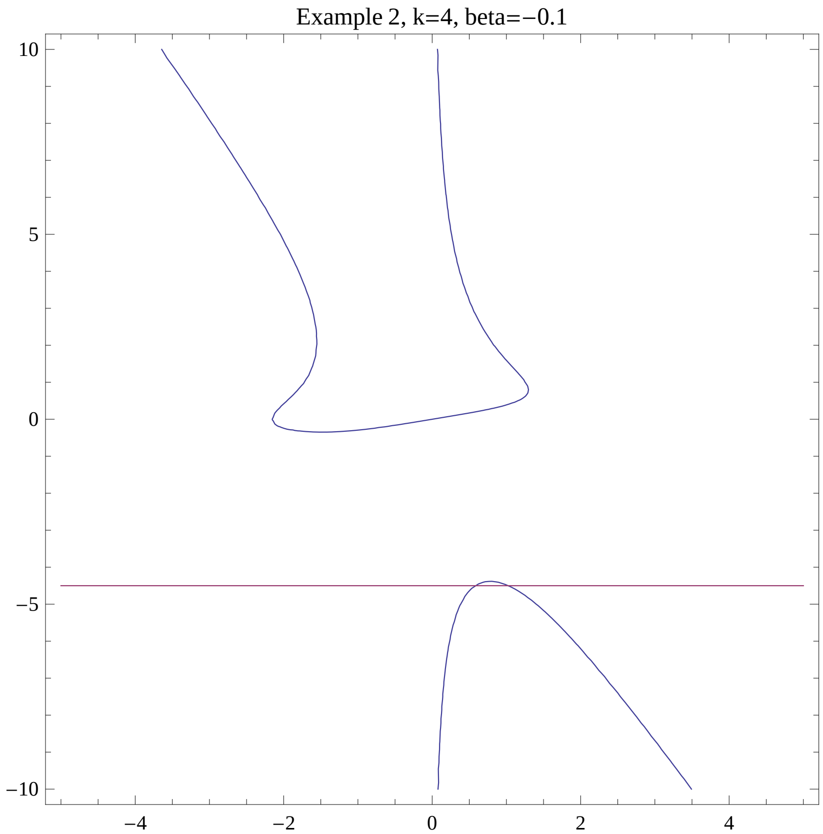

FIG. 3, EX. 2, SYSTEM (1.6.6), ($k = 4$, $\beta = -0.1$)

all $k > 3$.

With $k = 4$, choose $\beta = -0.1$ and:

ContourPlot $[-.4xy^3 - 0.1x^4 + 0.3x^3y - x^2y^2 - x + 3y, \{x, -10, 10\}, \{y, -10, 10\}]$.
A line, for example, $y = -4.5$ has two real crossing of the zero curve of Z at $x^* = 0.582467$ and $x^* = 1.02471$, both with $y^* = -4.5$, hence both crossings satisfy Theorem A-1 and system 2-1 has a homogeneous ASFC for k=4. Note that here $P_9(0, x_2) = \beta x_2^3$ so the horizontal line is permissible. See fig. 3.

For Example 3 the existence of an ASFC in $\mathcal{H}_1^r$ for $k \neq 3$ is shown in APPENDIX

A3-1 by direct use of Theorem A-1. ■

APPENDIX. The object of study in this section will be a reasonably general, three dimensional, homogeneous, affine control system for which the uncontrolled dynamics are unstable and the linearization is uncontrollable. We begin with a general setting for Sepulchre's methods, [Sep]. Specifically, we study the system

$$\dot{x}_1 = u, \quad \dot{x}_2 = k_2 x_2 + x_1^{2m+1}, \quad \dot{x}_3 = k_3 x_3 + P_{2l+1}(x_1, x_2) \tag{A.1}$$

where $0 \leq m < l$ are integers and $P_{2l+1}(x_1, x_2)$ is a homogeneous polynomial of degree $(2l + 1)$ with respect to the dilation $\delta_\varepsilon^r(x) = (\varepsilon x_1, \varepsilon^{2m+1} x_2, \varepsilon^{2l+1} x_3)$,i.e. $P_{2l+1}(\varepsilon x_1, \varepsilon^{2m+1} x_2) = \varepsilon^{2l+1} P_{2l+1}(x_1, x_2)$. Most cases of interest will have $k_2 > 0, k_3 > 0$, however $k_2 < 0$ is useful for example A-2. If u is continuous and homogeneous of degree one with respect to δ_ε^r we again write $u \in \mathcal{H}_1^r$. For such a control u, the vector field on the right side of (A.1) will be homogeneous of degree zero. We will assume STLC (which in some cases restricts the values of k) and seek an ASFC $u \in \mathcal{H}_1^r$ for (A.1). One should note that system (A.1) with $k_2 > 0, k_3 > 0$ does satisfy the Brockett necessary condition (iii). By choosing various polynomials P_{2l+1} many examples can be examined. The method used in the analysis of these examples is a generalization of homogeneous dimensional reduction introduced by Kawski, [K1], [K2], and an abstraction of the procedure used by Sepulchre [Sep].

The analysis of (A.1) begins by invoking the Coron-Praly theorem, [CP], on the elimination of the integrator. Let $w = x_1^{2m+1}$ or $x_1 = w^{1/(2m+1)}$ and consider the two dimensional system

$$\dot{x}_2 = k_2 x_2 + w, \quad \dot{x}_3 = k_3 x_3 + P_{2l+1}(w^{1/(2m+1)}, x_2). \tag{A.2}$$

Let $\gamma_\varepsilon^r(x_2, x_3) = (\varepsilon^{2m+1} x_2, \varepsilon^{2l+1} x_3)$. Then if w is homogeneous of order $2m + 1$ with respect γ_ε^r, we write $w \in \mathcal{H}_{2m+1}^\gamma$ and the vector field defined by the right side of (A.2) will be homogeneous of degree zero with respect to γ_ε^r as desired.

Coron-Praly theorem [CP]. *If the homogeneous system (A.2) admits an ASFC in* $\mathcal{H}^{\gamma}_{2m+1}$ *then the homogeneous system (A.1) admits an ASFC in* $\mathcal{H}^{r}_{1}$.

Remark A-1. The proof is not constructive, i.e. knowing the ASFC for (A-2) does not yield an ASFC for (A-1).

The Coron-Praly theorem reduces the existence problem from dimension three to dimension two. The reduction to dimension one is a homogeneity reduction, as given by Kawski [K1], [K2]. Let

$$\alpha = (2m+1)/(2l+1), \quad \theta = x_2/x_3^{\alpha}, \quad v = w/x_3^{\alpha}. \tag{A.3}$$

Notice that $\theta \in (-\infty, \infty)$ parameterizes homogeneous curves, i.e. $x_2 = \theta x_3^{\alpha}$ is a homogeneous curve in the plane. The transformation (A.3) may be viewed as homogeneous projection from the plane to the projective line $\mathbb{P}^1$. Since system (A.2) is homogeneous, the vector field induced on $\mathbb{P}^1$ by the transformation (A.3) is well defined and easily computed. Indeed

$$\dot{\theta} = k_2\theta + v - \alpha k_3\theta - \alpha\theta[(1/x_3)P_{2l+1}(w^{1/(2m+1)}, x_2)].$$

For ease of exposition we note that for $w \in \mathcal{H}^{\gamma}_{2m+1}, \quad w^{1/(2m+1)}$ has weight one while x_2 has weight $(2m+1)$ relative to γ^{r}_{ε}. Thus a possible term in P_{2l+1} would be $x_2^{2l/(2m+1)} w^{1/(2m+1)}$. In this case, $(1/x_3)x_2^{2l/(2m+1)} w^{1/(2m+1)} = v^{1/(2m+1)}\theta^{2l/(2m+1)}$.

In general

$$(1/x_3)P_{2l+1}(w^{1/(2m+1)}, x_2) = P_{2l+1}(v^{1/(2m+1)}, \theta). \tag{A.4}$$

Thus the induced equation on $\mathbb{P}^1$ is

$$\dot{\theta} = k_2\theta + v - \alpha k_3\theta - \alpha\theta P_{2l+1}(v^{1/(2m+1)}, \theta). \tag{A.5}$$

If $v^*(\theta)$ is a linear function of θ, say $v^*(\theta) = c_1\theta + c_2$, the control function w which corresponds to v^* via (A.3) is $w(x_2, x_3) = x_3^\alpha(c_1\theta + c_2) = c_1x_2 + c_2x_3^\alpha$ and this is a control in $\mathcal{H}^\gamma_{2m+1}$. If $(\overline{\theta}, v^*(\overline{\theta}))$ is a zero of the right side of (A.5) with v replaced by $v^*(\theta)$ then $\overline{\theta}$ corresponds to an invariant homogeneous curve of (A.2). Should values of $\overline{\theta}$ exist such that $(\overline{\theta}, v^*(\overline{\theta}))$ is a zero of the right side of (A.5), these will comprise the ω -limit set of (A.5) with $v = v^*(\theta)$. Kawski's theorem [K2], shows that if the origin of the restriction of system (A.2) to invariant, homogeneous curves corresponding to such values $\overline{\theta}$ in the ω-limit set of (A.5) is asymptotically stable, then system (A-2) is asymptotically stable. (For an odd system such as (A.1), on can replace S^1 by $\mathbb{P}^1$ in Kawski's theorem.)

Observe that the transformation (A.3) is only defined for $x_3 \neq 0$ hence the study of (A.5) may exclude part of the ω-limit set of system (A.2). Specifically, we also have to check if the point at infinity of $\mathbb{P}^1$ is an equilibrium point for (A-5) or equivalently that the ray through $x_3 = 0$ on S^1 in the (x_2, x_3) space is an invariant curve of (A.2). This will be the case if $w(x_2, x_3) = c_1x_2 + c_2x_3^\alpha$ in (A.2) and if $P_{2l+1}((c_1x_2 + c_2x_3^\alpha)^{1/(2m+1)}, x_2)$ vanishes at $x_3 = 0$, i.e. if $P_{2l+1}((c_1x_2)^{1/(2m+1)}, x_2) = 0$. Furthermore, in this case (A-2) shows $\dot{x}_2 = (k_2 + c_1)x_2$ on this ray. Hence there is stability if $c_1 < -k_2$, instability if $c_1 \geq -k_2$. (Certainly if $c_1 = 0$ and $P_{2l+1}(0, x_2) = 0$ then an unstable solution of (A.2) exists on the ray through the point $x_3 = 0$ on S^1 in the (x_2, x_3) space but this is not the most general case.) Let

$$Z = \{(\theta, v) : k_2\theta + v - \alpha k_3\theta - \alpha\theta P_{2l+1}(v^{1/(2m+1)}, \theta) = 0\}. \tag{A.6}$$

We seek a line $v^*(\theta) = c_1\theta + c_2$ which intersects the zero set of Z in points $(\overline{\theta}, v^*(\overline{\theta}))$ such that the flow of (A.2) on the invariant, homogeneous, curves corresponding to these values $\overline{\theta}$ is asymptotically stable, i.e. "contractive". In order that the flow of (A.2) be contractive on an invariant, homogeneous, curve it is necessary and sufficient that $x_3\dot{x}_3 < 0$ or $x_2\dot{x}_2 < 0$. Computing, using (A.4),

$$x_3\dot{x}_3 = x_3^2[k_3 + P_{2l+1}(v^{1/(2m+1)}, \theta)], \tag{A.7}$$

$$x_2\dot{x}_2 = x_2^2(k_2 + v/\theta). \tag{A.8}$$

Thus we have contraction at a point $(\theta, v) \in Z$ if

$$P_{2l+1}(v^{1/(2m+1)}, \theta) < -k_3 \quad \text{or} \quad v/\theta < -k_2. \tag{A.9}$$

We summarize this construction as

THEOREM A-1, Generalized Sepulchre construction, [Sep]. *Consider the system*

$$\dot{x}_1 = u, \quad \dot{x}_2 = k_2x_2 + x_1^{2m+1}, \quad \dot{x}_3 = k_3x_3 + P_{2l+1}(x_1, x_2) \tag{A.1}$$

where $0 \le m < l$ are integers and P_{2l+1} is a homogeneous polynomial of degree $(2l+1)$ with respect to the dilation $\delta_\varepsilon^r(x) = (\varepsilon x_1, \varepsilon^{2m+1}x_2, \varepsilon^{2l+1}x_3)$. Let $\alpha = (2m+1)/(2l+1)$. A sufficient condition that system (A.1) admit an ASFC in $\mathcal{H}_1^r$ is that there exist a line $v^*(\theta) = c_1\theta + c_2$ which intersects the zero set

$$Z = \{(\theta, v) : k_2\theta + v - \alpha k_3\theta - \alpha\theta P_{2l+1}(v^{1/(2m+1)}, \theta) = 0\} \tag{A.10}$$

only in points $(\overline{\theta}, \overline{v})$ satisfying (A.9) and if $P_{2l+1}((c_1x_2)^{1/2m+1}, x_2) \equiv 0$ then $c_1 < -k_2$.

Remark A-2. If a line $v^*(\theta) = c_1\theta + c_2$ exists which satisfies the above theorem, $w^*(x_2, x_3) = c_1x_2 + c_2x_3^\alpha$ will be an ASFC for (A.2) in $\mathcal{H}_{2m+1}^\gamma$. The construction of an ASFC in $\mathcal{H}_1^r$ for (A.1) from w^* is, to our knowledge, a difficult and open problem.

Remark A-3. The choice of $v^*(\theta)$ as a line is convenient but by no means necessary. Indeed, $\lim_{\theta\to\infty} v^*(\theta)/\theta = \lim_{\theta\to -\infty} v^*(\theta)/\theta$ suffices. Thus one could consider a function such as $v^*(\theta) = (c_1\theta^{1/3} + c_2)^3$. An interesting (and useful for considerations of necessity in the above theorem) is the conjecture that if system (A.2) admits an ASFC in $\mathcal{H}_{2m+1}^\gamma$ then it admits an ASFC of the form $w(x_2, x_3) = c_1x_2 + c_2x_3^\alpha$.

Remark A-4. In the case $(\overline{\theta}, \overline{v}) \in Z, \quad \overline{\theta} \neq 0$, one can see from (A.6) that

$$k_2 + \overline{v}/\overline{\theta} = \alpha[k_3 + P_{2l+1}(\overline{v}^{1/(2m+1)}, \overline{\theta})].$$

Hence since $\alpha > 0$ it follows that either the inequalities in (A.9) are both satisfied or both fail. Thus with $\overline{\theta} \neq 0$ it is sufficient to only check one.

Remark A-5. If $k_3 = k_2/\alpha$ and $P_{2l+1}(x_1, x_2)$ is not only a function of x_2 i.e. P_{2l+1} does depend on x_1, then $v = 0$ is part of the zero set in (A.5) and any line with $c_1 \neq 0$)must cross $v = 0$. But then $P_{2l+1}(0, \overline{\theta}) = 0$ hence the homogeneous, invariant, curve $x_2 = \overline{\theta} x_3^{\alpha}$ of (A.2) is not contracting, and there will not be a homogeneous ASFC for (A.2). Note that in the four examples of section 1, $\alpha = 1/3$ and $k_3 = 3k_2$ is precisely the condition which yields rank $(A - cR) \leq 1$, i.e. Kawski's theorem, to be invoked when $c = 1/3$. However, in general, negative results for (A.2) do not imply negative results for (A.1), i.e. the converse of the Coron-Praly theorem is not necessarily true, see [SEP, thm. 5.7].

Examples of the use of Theorem A-1.

Example A-0, (easy example). Consider the system

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = kx_3 + x_1^3, \quad k > 0. \tag{A.11}$$

This is example 2.1 in [H4] where a tedious computation using nonlinear regulator results showed there exists a continuous ASFC. Above, letting $X(x) = x_1\partial/\partial x_2 + (kx_3 - x_1^3)\partial/\partial x_3, \quad Y = \partial/\partial x_1$ one finds the Lie brackets $Y(0), [X, Y](0)$ and $[Y, [Y, [Y, X]]](0)$ are linearly independent showing the system is odd, STLC, [Su]. To use theorem A-1, we note that the system is homogeneous with respect to the dilation $\delta_\varepsilon^r x = (\varepsilon x_1, \varepsilon x_2, \varepsilon^3 x_3)$, i.e. $m = 0, \quad l = 1, \quad \alpha = 1/3, \quad P_3(x_1, x_2) = x_1^3, \quad k_2 = 0, \quad k_3 = k > 0$. Next graph the zero set $Z = \{(\theta, v) : v - (1/3)k\theta - (1/3)\theta v = 0\}$ or $\theta =$

$3v/(k - v^3)$. It thus follows that one can always construct a line $v^*(\theta) = c_1\theta + c_2$ with $c_1, c_2 > 0$ such that this line is tangent to the branch, in the fourth quadrant, of the zero set of Z and has only this tangency point, call it $(\overline{\theta}, \overline{v})$, in common with Z. Since $\overline{v} < -k$, $\overline{\theta} > 0$ in the fourth quadrant, $\overline{v}/\overline{\theta} < 0 = k_2$. Also, $P_3(\overline{v}, \overline{\theta}) = \overline{v}^3 < -k = -k_3$. Theorem A-1 applies to show the system (A.10) has an ASFC in $\mathcal{H}_1^r$.

If we modify equation (A.11) slightly we may lose the existence of an ASFC. Specifically, consider

$$\dot{x}_1 = u, \quad \dot{x}_2 = \epsilon x_2 + x_1, \quad \dot{x}_3 = 3\epsilon x_3 + x_1^3, \quad \epsilon > 0. \tag{A.12}$$

The dilation exponents are $r = (0, \epsilon, 3\epsilon)$. The eigenvalues of the linear part of the system are $(0, \epsilon, 3\epsilon)$ and there is homogeneous resonance. Let R be the matrix R=diag(1,1,3). The system obtained by subtracting ϵRx from (A.12) fails the Brocket condition (iii) hence has no continuous ASFC and hence by the Coron-Rosier lemma, system (A.12) does not have a continuous ASFC. This shows that a first order perturbation can destroy the existence of a continuous ASFC. Many of the listed results for the four examples in section 1 were obtained by use of theorem (A-1). The major effort, in most cases, is the graphing of the zero set Z.

APPENDIX A2-1, Analysis of system (1.2.1) for $k = 3$. For system (1.2.1), in theorem A-1, $m = 1$, $l = 4$, $k_2 = 1$, $k_3 = k$, $\alpha = 1/3$, $P_9(v^{1/3}, \theta) = v^2\theta$ and $(\theta, v) \in Z$ requires

$$\theta^2 v^2 - 3v + \theta(k - 3) = 0. \tag{A2-i}$$

Case $\mathbf{k} = 3$. For $k = 3$ the graph of $v(\theta^2 v - 3) = 0$ consists of the line $v = 0$ and the two branches of $v = 3/\theta^2$. (See Fig A2-1).

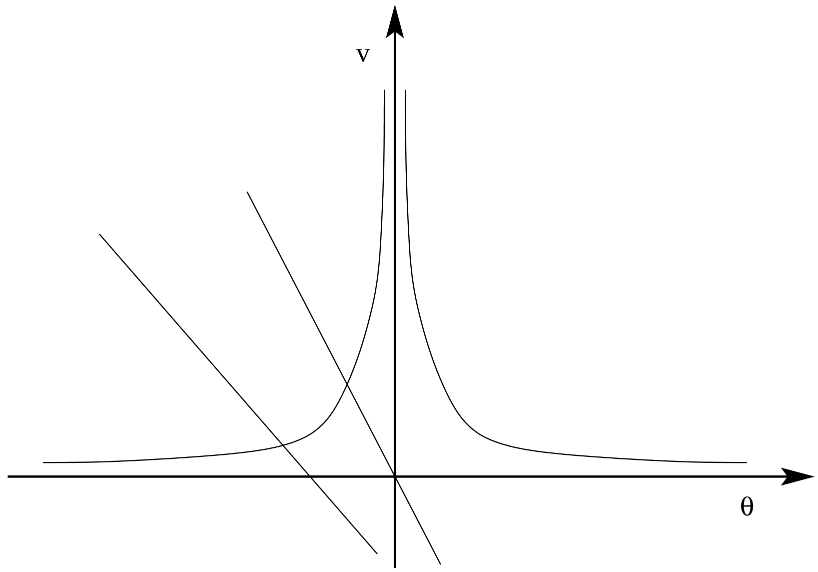

FIG A2-1 ($k = 3$)

Any line $v = c_1\theta + c_2, \quad c_1 \neq 0, c_2 \neq 0$, crosses the line $v = 0$ at a point which does not satisfy either of the conditions (A.9). A line of negative slope through the origin, e.g. say $v = c_1\theta, \quad c_1 < -1$, may cross the branch of the graph in the second quadrant at a permissible point; its crossing at the origin will not satisfy the first of conditions (A.9) while the second condition is the indeterminate form 0/0. This condition arises from the contraction condition $x_2\dot{x}_2 < 0$ which (when not simplified as in (A.9)) is

$$x_2\dot{x}_2 = x_2^2 + x_2 w = x_2^2 + x_2 v x_3^{1/3}. \tag{A.13}$$

Thus when $\theta = v = 0$, $x_2\dot{x}_2 \geq 0$, contraction does not occur, i.e. the second of conditions (A.9) is not satisfied. Thus, as expected, for $k = 3$ we cannot show the existence of an ASFC, which agrees with the knowledge from Kawski's theorem which shows there is no homogeneous ASFC.

Case k > 3. For $k > 3$ or $k < 3$ the zero set (A2-i) is most easily graphed via the two quadratic solutions $((\theta, v) \neq 0)$

$$v = \frac{3 \pm \sqrt{9 - 4\theta^3(k-3)}}{2\theta^2}, \quad \theta = \frac{(3-k) \pm \sqrt{(k-3)^2 + 12v^3}}{2v^2}.$$

(See figures A2-2,A2-3.)

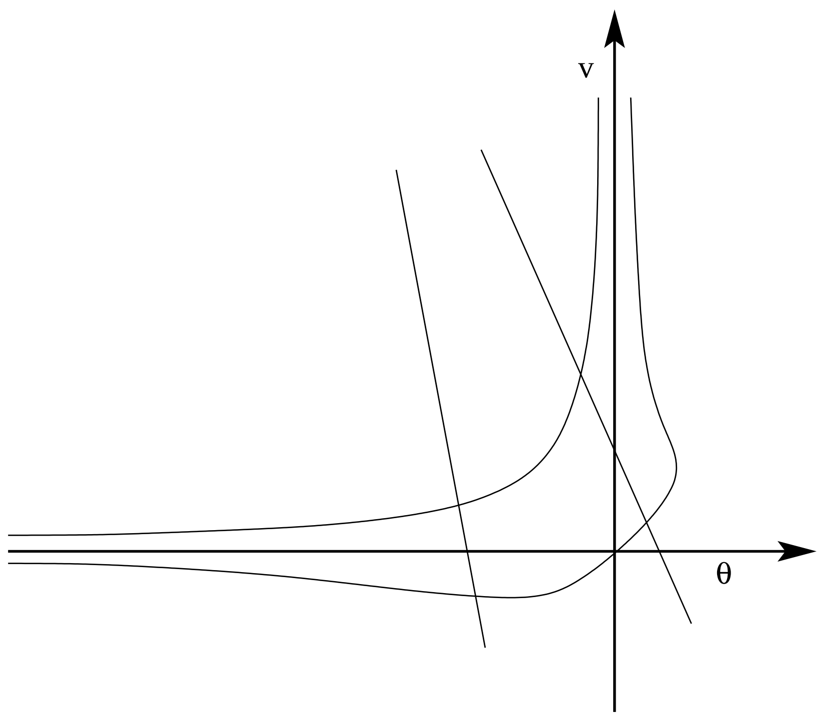

Fig A2-2 $(k > 3)$

Any line crossing the branch in the second quadrant must also cross the branch in the first or third quadrant, or at the origin. In the first and third quadrants, $v/\theta > 0$ and at the origin, (A.13) shows failure of $x_2\dot{x}_2 < 0$. A line tangent to the graph at the minimum point $(\frac{12^{2/3}}{2(3-k)^{1/3}}, -(\frac{(k-3)^2}{12})^{1/3}) = (\overline{\theta}, \overline{v})$ having zero slope need not be cosidered. Also, if a line crosses the branch of the graph twice, in the third quadrant, one of the crossings must fail the first of conditions (A.9). Since $\overline{v}/\overline{\theta}$ is positive in the first and third quadrants, the second of conditions (A.9) cannot be satisfied. Thus we cannot conclude the existence of a homogeneous ASFC for (1.2.1) when $k > 3$ using the direct approach.

Case k < 3. We refer to figure A2-3.

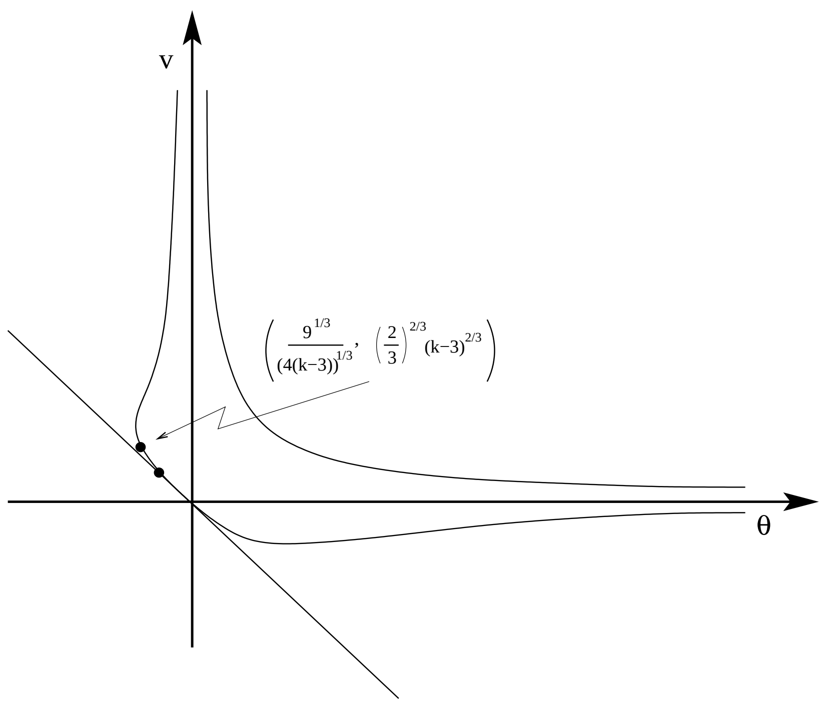

Fig A2-3 ($k < 3$)

Since $v^2\theta \geq 0$ in quadrants one and four, the first of conditions (A.9) will not apply. For the second of conditions (A.9), the best possibility will be a line tangent to the graph at a point on the graph just below the point $(\frac{9}{4(k-3)})^{1/3}, 3/2(\frac{4(k-3)}{9})^{2/3})$. At this point $v/\theta = 2/3(k-3)$ hence $(v/\theta) < -1$ if $k < 3/2$. One concludes that for $k < 3/2$ system (1.2.1) does admit a continuous, homogeneous, ASFC.

APPENDIX A3-1, Analysis of system (1.3.1). For system (1.3.1), in Theorem A-

1, $m = 1, \quad l = 4, \quad k_2 = 1, \quad k_3 = k, \quad \alpha = 1/3, \quad P_9(v^{1/3}, \theta) = v\theta^2$. Then $(\theta, v) \in Z$ requires

$$\theta + v - (k/3)\theta - \theta^3 v/3 = 0, \quad \text{or} \quad v = \theta(k-3)/(3-\theta^3).$$

For $k = 3$ the graph consists of the lines $v = 0$ and $\theta = 3^{1/3}$ and it easily follows (as expected) that a line satisfying either of conditions (A.9) does not exist. From Kawski's theorem, system (1.3.1) does not have a continuous ASFC at $k = 3$.

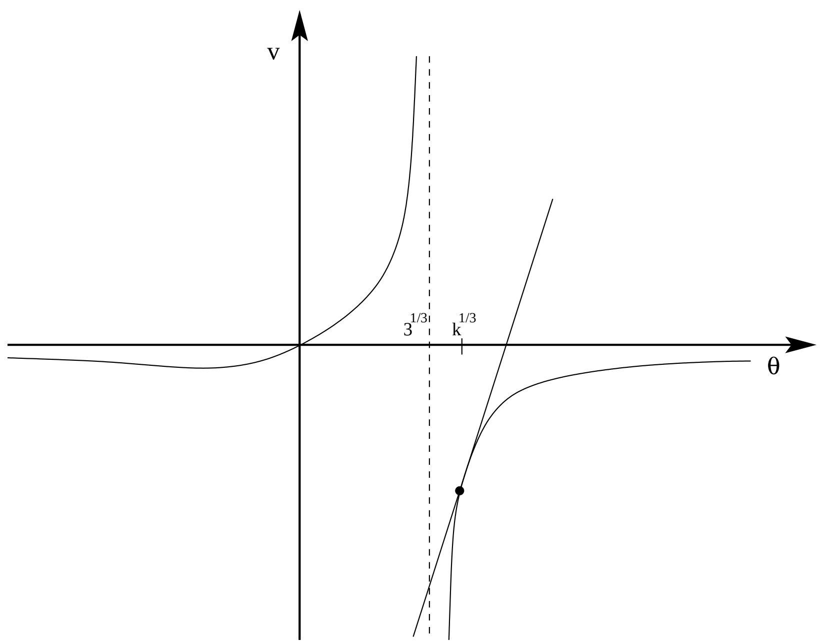

Fig A3-1 $(k > 3)$

From figures A3-1, A3-2, it readily follows that lines satisfying the second of conditions (A.9) exist and hence system (1.3.1) has a homogeneous ASFC for $k > 3, \quad k < 3$

Example A-2. This illustrates that a homogeneous system may have a matrix, A, of its linear part with resonance, be able to be transformed to Ancona normal form, but

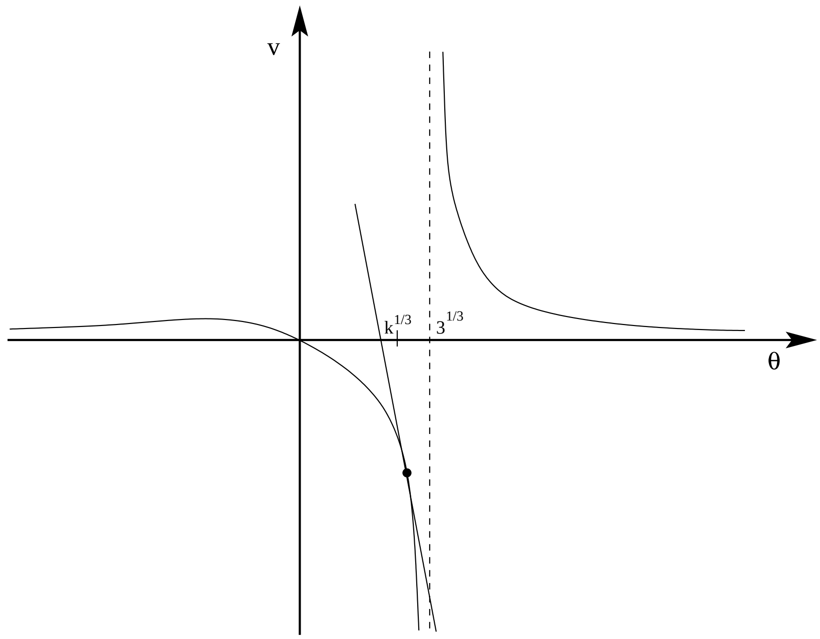

FIG A3-2 ($k < 3$)

if A has even one negative eigenvalue the system may have an ASFC in $\mathcal{H}_1^r$.

$$\dot{x}_1 = x_1 + w, \quad \dot{x}_2 = -x_2 + x_1^3, \quad \dot{x}_3 = x_3 + x_1^9. \tag{A.14}$$

Note that one could absorb the term x_1 into the control w in the first equation, which would change the eigenstructure, and resonance values, but not the existence of an ASFC. In order to remain in dimension three it is useful to retain the first equation as it is.

The system is homogeneous with respect to the dilation $\delta_\varepsilon^r(x) = (\varepsilon x_1, \varepsilon^3 x_2, \varepsilon^9 x_3)$ and STLC at zero. The homogeneous variable change $y_1 = x_1, \quad y_2 = -x_2 + x_1^3, \quad y_3 = x_3 + x_1^9$ transforms the system to Ancona normal form $\dot{y}_1 = y_1 + w, \quad \dot{y}_2 = -y_2 + 3y_1^2 w, \quad \dot{y}_3 = y_3 + 9y_1^8 w$. The eigenvalues of the linear part are $\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 1$ while $r_1 = 1, \quad r_2 = 3, \quad r_3 = 9$ and $3\lambda_1 + 2\lambda_2 = \lambda_3, \quad 3r_1 + 2r_2 = r_3$

showing resonance. To show an ASFC in $\mathcal{H}_1^r$ exists we use theorem (A-1) with, now, $x_1 + w$ replaced by u. Then $k_2 = -1$, $k_3 = 1$, $\alpha = 1/3$, $m = 1$, $l = 4$, $P_9(x_1, x_2) = x_1^9$ in (A-1). The zero set of Z is the graph, in the (θ, v) plane, of $\theta = 3v/(4+v^3)$ and one can readily construct a line $v = c_1\theta + c_2$ which is tangent to the branch of the graph lying in the fourth quadrant at a point $(\overline{\theta}, \overline{v})$ with $\overline{v}/\overline{\theta} < -k_2 = 1$ and having no other points in common with Z. Theorem A-1 gives the existence of an ASFC in $\mathcal{H}_1^r$ for system (A.14).

References

[AL] C. O. Aguilar & A.D. Lewis; Small Time Local Controllability for a Class of Homogeneous Systems, SIAM J. Control & Opt.,50, no 3, (2012) MR 2968064.

[A1] F. Ancona; Decomposition of Homogeneous Vector Fields of Degree One and Representation of the Flow, *Annales non lineare, Annales de l'Institut Henry Poincare*, vol 13, N2 (1996) 135–169.

[A2] F. Ancona; Normal Forms for Vector Fields with Respect to an Arbitrary Dilation, *Nodea Nonlinear Differential Equations & Applications*, vol 3, N3, (1996),306–322.

[BS] R. Bianchini & G. Stefani, Graded Approximations and Controllability along a trajectory, *SIAM J. Control and Opt.*, **28**, 903–924 (1990).

[B1] R. W. Brockett; Asymptotic Stability and Feedback Stabilization, *Differential Geometric Control Theory*, (Brockett, Millman, Sussmann, eds.), Birkhauser, Boston, **27** (1983), 181–191.

[C1] J. M. Coron; A Necessary Condition for Feedback Stabilization, Systems & Control Lett., **14** (1990), 227–232.

[C2] J. M. Coron; Stabilization of Controllable Systems, Sub- Riemannian Geometry, (Eds. A. Bellaiche and J. J. Risler), Progress in Mathematics, 144, 365-388, Birkhauser, Basel (1996) (MR 97h:93068).

[G] L. Grüne; Homogeneous State Feedback Stabilization of Homogeneous Systems, SIAM J. Control & Opt., 38, (2000), no 4, 1288- 1308, (MR 2001d: 93069.

[CP] J. M. Coron & L. Praly; Adding an Integrator for the Stabilization Problem, *Systems and Control Letters*, **17**, (1991), pp 89–104.

[H1] H. Hermes; Homogeneous Coordinates and Continuous Asymptotically Stabilizing Feedback Controls, *Diff. Eqs., Stability and Control*, (S. Elaydi, ed.), Lecture Notes in Pure & Appl. Math #127, Marcel Dekker, Inc. (1991), 249–260.

[H2] H. Hermes; Nilpotent and High-Order Approximations of Vector Field Systems, SIAM Review **33** (1991), 238—264.

[H3] H. Hermes; Control Systems which Generate Decomposable Lie Algebras, J. Diff. Eqs. **44** (1982), 166–187.

[H4] H. Hermes; Asymptotically Stabilizing Feedback Controls and the Nonlinear Regulator Problem, SIAM J. Control & Opt. **29** (1991), 185–196.

[H5] H. Hermes; Large Time Local Controllability via Homogeneous Approximations, SIAM J. Control & Opt., no 4, 1291-1299, (1996), (MR97f:93005)

[HK] H. Hermes and M. Kawski; Local Controllability of a Single-Input Affine System, Lecture Notes in Pure & Appl. Math #109, (V. Lakshmikantham, ed.), Marcel Dekker, Inc. (1987), 235–248.

[K1] M. Kawski; Stabilization of Nonlinear Systems in the Plane, Systems & Control Lett. **12** (1989), 169–175.

[K2] M. Kawski; Homogeneous Stabilizing Feedback Laws, Control Theory and Advanced Technology, Mita Press, Tokyo, vol. 6 (1990), 497–516.

[K3] M. Kawski; Homogeneous Feedback Laws in Dimension Three, Proceedings IEEE Conference on Decision & Control, Tampa, FL., **II** (1989), 1370–1375.

[K4] M. Kawski; High-Order Small Time Local Controllability, Nonlinear Controllability and Optimal Control, Textbooks Pure Appl. Math., 133, Dekker, N.Y., H. Sussmann, Ed., (1990), 431-467, (MR 91j:93010)

[R] R. Rosier; Homogeneous Lyapunov function for continuous vector field, Systems & Control Lett. **19** (1992), 467–473.

[SEP] R. Sepulchre; Contributions to Nonlinear Control Systems Analysis by Means of the Direct Method of Liapunov, *Ph.D Dissertation, Universite Catholique de Louvain.* Av. G. Lemaitre, 4-B 1348 Luvain-La-Neuve, Belgium, (1994)

[St] G. Stefani; Local properties of nonlinear control systems, in *Geometric Theory of Nonlinear Control Systems*, (Edited by B. Jakubczyk, W. Respondek & K. Tchon), pp 219–226, Tech. Univ., Wroclaw, (1984).

[Su] H. J. Sussmann; A General Theorem on Local Controllability, SIAM J. Control & Opt. **25** (1987), 158–194.