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Abstract

In this paper we consider a class of quasilinear elliptic systems of PDEs which arise in the mean field games theory of J-M Lasry and P-L. Lions. We provide a wide range of sufficient conditions for existence of solutions to these systems: on one hand the Hamiltonians (first-order terms) need to be at most quadratic in the gradients, on the other they can even grow arbitrarily provided that they do not "oscillate extremely much in the space variable" (a conditions expressed rigorously by means of certain inequalities that involve the space derivatives of the Hamiltonians). We concentrate on periodic conditions, but the same techniques allow to handle Dirichlet, Neumann...boundary conditions, or even the evolutive counterpart of these equations.

1 Introduction

The systems of equations we deal with in this paper are of the following type

$$\begin{cases} \mathcal{L}^i v_i + H^i(x, Dv_i) + \lambda_i = V^i[m] \quad \text{in } Q, \\ \\ \mathcal{L}^{i*} m_i - div\,(g^i(x, Dv_i) m_i) = 0 \quad \text{in } Q, \\ \\ \int_{Q^d} m_i(x)dx = 1, \quad m_i > 0, \quad \int_{Q^d} v_i(x)dx = 0, \quad i = 1, \ldots, N, \end{cases} \tag{1}$$

where the unknowns are functions $v = (v_1, \ldots, v_N)$, probability measures $m = (m_1, \ldots, m_N)$ defined on the d-dimensional torus Q (which turn out to be continuous functions too) and constants $\lambda = (\lambda_1, \ldots, \lambda_N)$. Throughout the paper, the $\mathcal{L}^i = -a^i \cdot D^2$ are always second-order uniformly elliptic operators with Lipschitz coefficients in Q. (We use the notations

$\operatorname{tr} b$ for the trace of a square matrix b, and $a \cdot b = \operatorname{tr} ab^t$, $|b| := (b \cdot b)^{1/2}$ for the Frobenius matrix norm). Whereas $\mathcal{L}^{i*} = -\sum_{h,k} D^2_{hk}(a^i_{hk} v)$ is the formal adjoint of $\mathcal{L}^i$, which is to be interpreted in the sense of distributions:

$$< \mathcal{L}^{i*} v, \phi >= \int_Q v\mathcal{L}\phi dx \qquad \forall \phi \in C^\infty(Q).$$

The Hamiltonians $H^i = H^i(x,p)$ ($x \in Q$, $p \in I\!R^d$) satisfy two kinds of conditions:

1. Either they are locally Lipschitz in both variables, superlinear in p uniformly in x, i.e.,
$$\inf_{x\in Q} |H^i(x,p)|/|p| \to +\infty \quad \text{as} \quad |p| \to \infty, \tag{2}$$
and $\exists \theta^i \in (0,1)$, $C > 0$
$$\operatorname{tr}(a^i) D_x H^i \cdot p + \theta^i (H^i)^2 \geq -C|p|^2 \quad \text{for} \quad |p| \text{ large, and for a.e. } x \in Q. \tag{3}$$

2. Or they are locally α-Hölder continuous ($0 < \alpha < 1$) and grow at most quadratically
$$|H^i(x,p)| \leq C_1|p|^2 + C_2 \quad \forall x \in Q,\ p \in I\!R^d,\ i = 1, \ldots, N \tag{4}$$
for some C_1, $C_2 > 0$, the so-called *natural growth* condition.

Let $P(Q)$ be the set of probability measures on Q (which is a convex subset of $C(Q)^*$, the topological dual of $C(Q)$, and compact for the weak*-topology by Prokhorov's theorem). We assume that the operators

$$V^i : P(Q)^N \to C(Q), \quad m \to V^i[m],$$

send $C(Q)^N \cap P(Q)^N$ (the continuous functions of $P(Q)^N$) into a bounded set of Lipschitz functions on Q if the first set of hypotheses for the Hamiltonians H^i hold, and into a bounded set of α-Hölder continuous functions on Q if the second set of hypotheses on H^i hold. Suppose also that the V^i, when restricted to $C(Q)^N \cap P(Q)^N$ are continuous for the uniform convergence topology.

Finally, we assume also that

$$g^i : Q \times I\!R^d \to I\!R^d \quad \text{are measurable and locally bounded.}$$

Pay attention, the hypotheses on $\mathcal{L}^i$, V^i and g^i are assumed to hold throughout the paper and they will not be recalled anymore; unless explicitly stated otherwise, they are always assumed to be in force.

Under the above conditions, we show the existence of a solution $\lambda_i \in I\!R$, $v_i \in C^{2,\alpha}(Q)$, $m_i \in W^{1,p}(Q)$, for all $1 \leq p < \infty$, $i = 1, \ldots, N$, to the system (1) (α is any number in $(0,1)$ for the first set of assumptions on H^i).

If the operators $\mathcal{L}^i$ are assumed merely degenerate elliptic, under appropriate conditions on H^i, we can only assert that the v_i are Lipschitz on Q and they have to be interpreted as vanishing viscosity solutions of the first equations in (1).

If the Hamiltonians are merely measurable and continuous in p, under the second set of assumptions (H^i with quadratic growth) we can conclude that $v_i \in W^{2,p}(Q)$, $1 \le p < \infty$ and that solve the first equations in (1) a.e. in Q.

As to the methods of proof, we owe our results to J. Serrin [31], [32], P L. Lions [27] and P-L. Lions, P. E. Souganidis [28], for the first set of assumptions, and to A. Bensoussan, J. Frehse[8] (see also [10]) for the second set of assumptions. Of course, results from the classical books of D. Gilbarg, N. Trudinger [20], and O. A. Ladyzhenskaya and N. N. Uraltseva [24] are used extensively. While most of the results of this paper are an adaptation of what is known for scalar quasilinear equations to systems of the form (1), we wish to point out that there are also some results that seem to be new even for the scalar case, or that might have gone unnoticed. However, we have avoided making use of existence results for scalar equations, and have only used local a priori estimates which simplify things considerably because in the periodic setting "local is global". Thus, the paper is quite self-contained.

Most of these existence results extend (up to some slight modifications of course) also to Dirichlet, Neumann... boundary conditions (think of Q as a (bounded) domain of the space now and drop (some of) the conditions in the third line of (1)). The ideas contained in [27] should be very useful in this respect. The same can be said also for the parabolic counterparts of these equations.

These systems arise in the mean field games theory (MFG for short) of J-M. Lasry, P-L. Lions [25, 26] in (at least) two ways. First, for $N = 1$ we recover the stationary MFG system considered in [25, 26], and for N arbitrary, the stationary MFG system corresponding to N groups or populations of large players mentioned in [26]. (Each population consists of a large number of identical players, but the characteristics of the players vary from one population to the other. The aim of this part of MFGs is to model segregation or repartition of population phenomena.)

Second, systems of the form (1) comprise also certain systems of PDEs associated with a class of N-player ergodic stochastic differential games coupled only through the costs, which were introduced in [25, 26] in order to derive the MFG system (for a single population) from Nash equilibria as the number of the individuals/players $N \to \infty$.

Moreover, J-M. Lasry and P-L. Lions [25, 26] gave also a first existence result for a system of the form (1) (with $\mathcal{L}^i = -\nu^i \Delta$, $\nu^i > 0$, $g^i = D_p H^i$) assuming (2) and (3) with $C = 0$. They indicate that their existence proof relies on a priori estimates for Dv_i that can be obtained by the Bernstein method (see also [1]). For our first set of assumptions on H^i, i.e., (2) and (3), we follow this suggestion and we give the other necessary technical details. It is also worth to note that these a priori estimates are independent of N, and this allows to carry out the abovementioned limit procedure, for $N \to \infty$ in the derivation of the MFG equations.

The paper is organized as follows. Section 2 contains the statements and proofs of the existence results for (1) and related systems arising from discounted infinite horizon problems. As an application of these results, in Section 3 we show the existence of Nash equilibria for a class of non-cooperative ergodic games where the players influence each other only through the costs. We consider two types of costs: ergodic or long-time-average costs and discounted infinite horizon costs. Finally, the Appendix contains the proofs of some technical lemmas.

2 Main results

Theorem 2.1. *Assume $H^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are locally Lipschitz functions, $\mathbb{Z}^d$-periodic in x, and satisfy (2) and (3). Then there exist $\lambda_1, \dots, \lambda_N \in \mathbb{R}$, $v_1, \dots, v_N \in C^{2,\alpha}(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, for all $0 < \alpha < 1$, $1 \le p < \infty$, which solve the system (1).*

We need the following two lemmas for linear equations. We believe they are well-known, but for lack of a precise reference we give their proofs in the Appendix.

Lemma 2.1. *Let*

$$\mathcal{L} = -a_{hk}(x)D_{hk} + b_h(x)D_h$$

be a second-order uniformly elliptic linear differential operator in the d-dimensional torus Q with coefficients a_{hk}, b_h, $\in C^\alpha(Q)$, $h, k = 1, \dots, d$, $0 < \alpha < 1$.

Then, for any $f \in C^\alpha(Q)$, the problem

$$\begin{cases} \mathcal{L}v + \lambda &= f \\ \int_Q v(x)dx &= 0 \end{cases} \tag{5}$$

has a unique solution $(v, \lambda) \in C^{2,\alpha}(Q) \times \mathbb{R}$. Moreover,

$$|\lambda| \le \|f\|_\infty, \tag{6}$$

$$\|v\|_{C^{1,\alpha}(Q)} \le C\|f\|_\infty, \tag{7}$$

$$\|v\|_{C^{2,\alpha}(Q)} \le C\|f\|_{C^\alpha(Q)} \tag{8}$$

for some constant $C > 0$ which depends only on (the coefficients of) $\mathcal{L}$.

Lemma 2.2. *Let*

$$\mathcal{L} = -a_{hk}(x)D_{hk}, \qquad g : Q \to \mathbb{R}^d$$

be a (symmetric) second-order uniformly elliptic linear differential operator in the d-dimensional torus Q with coefficients $a_{hk} \in C^{0,1}(Q)$, $h, k = 1, \dots, d$, and a bounded measurable vector field, respectively. Then, the problem

$$\begin{cases} \mathcal{L}^* m - div(g(x)m) &= 0 \\ \int_Q m(x)dx &= 1 \end{cases} \tag{9}$$

has a unique solution $m \in W^{1,p}(Q)$ for all $1 \le p < \infty$. Moreover, m is positive and

$$\|m\|_{W^{1,p}(Q)} \le C(\|g\|_\infty), \tag{10}$$

for some constant $C(\|g\|_\infty)$ which depends (continuously) on g only through $\|g\|_\infty$ (and also on p and the coefficients a_{hk} in a way which we are not interested to clarify here).

Proof of Theorem 2.1.

The proof is based on Schauder's fixed point theorem (see for instance [33, Theorem 4.1.1, p. 25] or [20, Corollary 11.2, p. 280]) and on a priori estimates for the gradients of v_i that are obtained by Bernstein's method, as suggested in [25, 26].

We first assume, instead of (2) and (3), that the Hamiltonians are bounded, that is,

$$\exists M > 0 \quad \text{such that} \quad |H^i(x,p)| \leq M \qquad \forall i,\ x,\ p.$$

Let

$$\mathcal{B} = \left\{ u = (u_1, \ldots, u_N) \in (C^{1,\alpha}(Q))^N \ : \ \int_Q u dx = 0 \right\},$$

which, as a closed linear subspace of the Banach space $C^{1,\alpha}(Q)^N$, is itself a Banach space. We define an operator

$$T : \mathcal{B} \to \mathcal{B},$$

according to the scheme

$$u \to m \to (v, \lambda) \to v,$$

as follows. Given $u = (u_1, \ldots, u_N) \in \mathcal{B}$, we plug it in place of v in the second N scalar linear equations of the system (1) and solve them for the unknowns $m = (m_1, \ldots, m_N)$, requiring that these unknowns satisfy conditions in the third line of (1). That these $\{m_i\}$ exist, are uniquely defined and have the required properties, is a consequence of Lemma 2.2. Then, with these $\{m_i\}$, we solve the N scalar linear equations obtained from the first equations in (1) after plugging into the H^i the u_i place of v_i, i.e.,

$$\mathcal{L}^i v_i + H^i(x, Du_i) + \lambda_i = V^i[m]$$

for the unknowns $v = (v_1, \ldots, v_N)$, $\lambda = (\lambda_1, \ldots, \lambda_N) \in I\!R^N$. And, if it is required in addition that the v_i have zero mean, then $v = (v_1, \ldots, v_N) \in C^{2,\alpha}(Q)$ and $\lambda = (\lambda_1, \ldots, \lambda_N) \in I\!R^N$ are uniquely defined. . This is a consequence of Lemma 2.1 of course. We set $Tu = v$. By (6), (8), (10), and a standard embedding theorem, T is continuous and compact. Moreover, the $C^{1,\alpha}$-estimate (7) and the boundedness of H^i, V^i gives

$$\|v\|_{C^{1,\alpha}(Q)} \leq C$$

for some $C > 0$ independent of v; thus $T\mathcal{B}$ is bounded. Therefore, by Schauder's fixed point theorem, T has a fixed point (in the convex hull of the closure of $T\mathcal{B}$).

Now we turn to consider Hamiltonians H^i that satisfy the assumptions of the theorem. We introduce the modified Hamiltonians H^i_R, where the parameter $R > 0$ is to be fixed in the sequel sufficiently large, defined as follows

$$H^i_R(x,p) = \begin{cases} H^i(x,p), & \text{if } |p| \leq R, \\ H^i\left(x, R\frac{p}{|p|}\right), & \text{if } |p| > R, \end{cases} \qquad x \in Q,\ p \in I\!R^d. \tag{11}$$

Let $R_1 > 0$ be such that (3) is verified for all $x \in Q$, $|p| \geq R_1$. Then

$$\inf_{x\in Q} \left((\mathrm{tr}\, a^i) D_x H^i_R \cdot p + \theta^i (H^i_R)^2\right) \geq -C|p|^2 \quad \text{for all} \quad |p| \geq R_1,\ R > R_1, \tag{12}$$

with the same θ^i, C ($i = 1, \ldots, N,$) as in (3).

Clearly the H^i_R are bounded and Lipschitz continuous. So let $\lambda^R_1, \ldots, \lambda^R_N \in \mathbb{R}$, $v^R_1, \ldots, v^R_N \in C^{2,\alpha}(Q)$, $m^R_1, \ldots, m^R_N \in W^{1,p}(Q)$ ($0 < \alpha < 1$, $1 \leq p < \infty$) be a solution of (1) with H^i_R in place of H^i.

The crucial step of the proof is an a priori estimate for $\|Dv^R_i\|_\infty$ uniform in R, obtained by Bernstein's method. We drop the indices i and R in the following estimates. Let $w = Dv$ and $\psi = (1/2)|w|^2$. We have these identities

$$D\psi = wD^2v \tag{13}$$

$$D^2\psi = \sum_h w_h D^2 w_h + (D^2v)^2 \tag{14}$$

$$\mathcal{L}\psi = -a \cdot D^2\psi = -\sum_h w_h(a \cdot D^2 w_h) - a \cdot (D^2v)^2. \tag{15}$$

Let us apply to the first equations in (1) the operator[1] $w \cdot D$ and use (15), (13) to obtain

$$\mathcal{L}\psi + a \cdot (D^2v)^2 - \delta a \cdot D^2 v + D_x H \cdot w + D_p H \cdot D\psi = G \cdot w, \tag{16}$$

where

$$\delta a := (\delta a_{hk})_{h,k=1,\ldots,d} \quad \text{with} \quad \delta a_{hk} := \sum_l D_l a_{hk} w_l \tag{17}$$

and G is a function whose L^∞-norm does not exceed a universal constant which does not depend on v (recall the assumptions on operators V^i). We use the following inequalities which are a simple consequence of Cauchy-Schwarz inequality: for any a, b symmetric matrices with $a \geq 0$ and c, $e \in \mathcal{M}^{d\times m_i}$

$$(a \cdot b)^2 \leq (a \cdot b^2)\mathrm{tr}\, a \quad \text{and} \quad (\mathrm{tr}\, ce^t b)^2 \leq |e|^2 cc^t \cdot b^2. \tag{18}$$

Assume that

$$a = \frac{1}{2}\sigma\sigma^t, \tag{19}$$

where the matrix $\sigma(x)$ is Lipschitz; such a decomposition is always possible for $a(x)$ is Lipschitz and its smallest eigenvalue (which is positive) is bounded away from zero uniformly in $x \in Q$ (e.g., let σ be the square root of $2a$). By (19) and (17) we have

$$\delta a = (\delta\sigma)\sigma^t.$$

[1] Actually, this requires u to be of class C^3 and a_{hk}, H, F to be of class C^1. However, under our assumptions (a variant of) this equation holds a.e. (in $\mathbb{R}^d$). This is so because $u \in W^{3,p}$, $p > n$ (by classical elliptic regularity theory for linear equations), hence u is three times differentiable in the usual sense a.e. and also the coefficients, being Lipschitz continuous, are differentiable a.e. (see [17, Theorem 1, p. 235] for instance). The application of the chain rule is actually problematic: one needs tor use a sophisticated result in [4].

Using this identity, the second of inequalities (18) and (19), we obtain

$$\begin{aligned}\delta a\cdot D^2v &\le |\delta\sigma|\left(\sigma\sigma^t\cdot(D^2v)^2\right)^{1/2}=\sqrt{2}|\delta\sigma|\left(a\cdot(D^2v)^2\right)^{1/2}\\ &\le \varepsilon a\cdot(D^2v)^2+\frac{1}{2\varepsilon}|\delta\sigma|^2,\end{aligned}\tag{20}$$

for any $0<\varepsilon<1$.On the other hand, using the first of inequalities (18) and the first equations in (1), we have

$$\begin{aligned}(a\cdot(D^2v)^2)\operatorname{tr}a &\ge (a\cdot D^2v)^2=(\mathcal{L}v)^2\\ &\ge(\lambda+H-V[m])^2\\ &\ge \omega H^2-c_\omega\end{aligned}\tag{21}$$

for $0<\omega<1$ and some constant c_ω independent of R, where $\overline{F}$ is the right hand side of (**??**). The last inequality is obtained by the boundedness of operators V and constants λ. In fact, looking at the minima and maxima of v in the first equations of (1) we obtain

$$|\lambda|\le\sup_{x\in Q}(|\overline{F}(x)|+V[m](x)).$$

Multypling (16) by $\operatorname{tr}a$, and using (20), (21), we get

$$(\operatorname{tr}a)(\mathcal{L}\psi+D_pHD\psi+D_xH\cdot w)+(1-\varepsilon)\omega H^2\le(\operatorname{tr}a)\left(G\cdot w+\frac{1}{2\varepsilon}|\delta\sigma|^2\right)+c_\omega$$

If we choose ε and ω such that $(1-\varepsilon)\omega>\theta$, where θ is the constant appearing in (12), using (12), we get

$$(\operatorname{tr}a)(\mathcal{L}\psi+D_pHD\psi)+((1-\varepsilon)\omega-\theta)H^2\le(\operatorname{tr}a)\left(G\cdot w+\frac{1}{2\varepsilon}|\delta\sigma|^2\right)+C|w|^2+c_\omega \tag{22}$$

for $|w|>R_1$. At a maximum point of ψ_i^R, (now we reintroduce i and R in order to avoid any possible confusion), say x_R^i, taking into account that $|\delta\sigma^i|$ is at most linear in w_i^R, we have[2]

$$((1-\varepsilon^i)\omega^i-\theta^i)(H_R^i)^2(x_R^i,w_i^R(x_R^i))\le C(|w_i^R(x_R^i)|^2+1)\tag{23}$$

for some $C>0$ (independent of R) and $|w_i^R(x_R^i)|\ge R_1$. But by (2), the left-hand side above is superlinear in $|w_i^R(x_R^i)|$ and thus $|w_i^R(x_R^i)|$ must be bounded.

Thus, we have shown that

$$\|Du_i^R\|_\infty\le R_2$$

for some $R_2>0$ independent of $R>R_1$. So if take any $R>\max\{R_1,R_2\}$ in (11), we discover that $\lambda_1^R,\dots,\lambda_N^R\in I\!R$, $v_1^R,\dots,v_N^R\in C^{2,\alpha}(Q)$, $m_1^R,\dots,m_N^R\in W^{1,p}(Q)$ $(0<\alpha<1$, $1\le p<\infty)$ is also a solution of the original system of PDEs (1). □

[2]In the general case

$$\operatorname{ess}\limsup_{x\to x_R^i}\mathcal{L}\psi_i^R\ge 0.$$

as shown in [13].

Remark 2.1. i) In dimension $d = 1$ condition (3) is not needed at all. Note that $\frac{d^2 v_i}{dx^2}$ is bounded from below and - only that H^i be bounded from below is needed, superlinearity is unnecessarily too much - has zero mean in $(0, 1)$, hence it is bounded in L^1-norm. Therefore $\frac{dv_i}{dx}$ is bounded.

ii) Condition (2) might be quite strong sometimes. It may happen that instead of (2) we only know that

$$\liminf_{|p|\to\infty} \frac{\inf_{x\in Q} |H^i(x,p)|}{|p|} = \nu^i > 0. \tag{24}$$

A typical condition which guarantees this for Hamiltonians H^i, as in the nest section, defined by (37) is (using the notation of that section)

$$\exists \nu^i > 0 \text{ s.t. } \overline{co}\{f^i(x,\alpha) \ : \ \alpha \in A_0^i\} \supset B(0,\nu^i) \qquad \forall x \in Q,$$

where $A_0^i \subset A^i$ is some compact set. If we still want to guarantee the validity of Theorem 2.1, we have to substitute (3) with the stronger condition: $\exists \theta^i$, $\eta^i \in (0, 1)$ such that

$$\liminf_{|p|\to\infty} \frac{1}{|p|^2} \inf_{x\in Q} \left(\theta^i (\operatorname{tr} a) D_x H^i \cdot p + (1-\theta^i)\theta^i \eta^i (H^i)^2 - |p|^2 \frac{(\operatorname{tr} a^i)|D\sigma^i|^2}{2} \right) \geq 0, \tag{25}$$

where $a^i = (1/2)\sigma\sigma^t$, $D\sigma^i = (D\sigma^i_{hk})$ is a matrix (whose entries are vectors) and $|D\sigma^i|^2 = \sum_{h,k} |D\sigma^i_{hk}|^2$. To see this take $\varepsilon^i = \theta^i$ in (20), multiply (22) by θ^i and choose ω^i in (21) so that $\omega^i > \eta^i$. Let $s > 0$, $R_s > 0$ be such that the quantity under the "lim inf" sign in the left-hand side of (25) is $> -s$ for all $|p| \geq R_s$. Noticing that $|\delta\sigma^i| \leq |D\sigma^i||w_i^R|$, we deduce

$$(\omega^i - \eta^i)(1-\theta^i)\theta^i (H_R^i)^2 (x_R^i, w_i^R(x_R^i)) - s|w_i^R(x_R^i)|^2 \leq C(|w_i^R(x_R^i)| + 1)$$

for some $C > 0$ (independent of R and s) and $|w_i^R(x_R^i)| \geq R_s$. Now if we choose $s > 0$ so small that $s < \nu^i(\omega^i - \eta^i)(1-\theta^i)\theta^i$, by (24) we find that $|w_i^R(x_R^i)|$ is bounded uniformly in R.

iii) If inequality (25) with $\eta^i = 1$ holds with sign "$>$", then (24) is superfluous.

These existence results for (1) may be interpreted as follows. The Hamiltonians H^i can grow arbitrarily provided that they "do not oscillate too much in x" which rigorously means that they should satisfy certain technical conditions of the abovementioned kind. On the other hand, if the Hamiltonians have at most quadratic growth (the so called natural growth condition), that is, (4) hold, we do not need any additional condition of the aforementioned type in order to ensure existence. Indeed, we have the following

Remark 2.2. An alternative existence result con be stated for Hamiltonians with (at most) quadratic growth. Precisely, assume that the Hamiltonians H^i, $i = 1, \ldots, N$, are α-Hölder continuous, $0 < \alpha < 1$, and satisfy (4). Then the conclusion of Theorem 2.1 holds with this particular value of α.

The proof is based upon the "discounted case", see Theorem 2.2 below. Let $v_1^\rho, \ldots, v_N^\rho \in C^{2,\alpha}(Q)$, $m_1^\rho, \ldots, m_N^\rho \in W^{1,p}(Q)$ $(1 \leq p < \infty,\ i = 1, \ldots, N)$ be a solution of (27) below with $\rho^i = \rho > 0$. Let $< v_i^\rho >= \int_Q v_i^\rho dx$ be the mean of v_i^ρ. The crucial observation is that

$$\|v_i^\rho - < v_i^\rho > \|_\infty \leq C$$

for some $C > 0$ independent of ρ, which can be shown by using the techniques of [8]. Then we leave it as an exercise to the reader to finish the proof by showing that there exists a sequence $\rho_n \to 0$ such that

$$(v_i^{\rho_n} - < v_i^{\rho_n} >, \, \rho_n v_i^{\rho_n}, \, m_i^{\rho_n}) \to (v_i, \, \lambda_i, \, m_i) \qquad \text{in } C^2(Q) \times C(Q) \times C(Q),$$

where $(v_i, \, \lambda_i, \, m_i)$, $i = 1, \ldots, N$, is a solution of (1).

Remark 2.3. Let us now drop the uniform ellipticity assumption on $\mathcal{L}^i$ ($i = 1, \ldots, N$) and assume that $\mathcal{L}^i$ are merely degenerate elliptic (even $\mathcal{L}^i \equiv 0$ are allowed). Suppose that, for some $C > 0$,

$$|D_x H^i| \leq C(|H^i| + 1) \quad \forall x \in Q, \; p \in I\!R^d \tag{26}$$

and, in addition, assume (25) holds with the sign "$>$" (even $\eta^i = 1$ is allowed, compare with [28]). Under these conditions we can still conclude the existence of $\lambda_1, \ldots, \lambda_N \in I\!R$, $v_1, \ldots, v_N \in C^{0,1}(Q)$, $m_1, \ldots, m_N \in W^{1,p}(Q)$ (for all $1 \leq p < \infty$) which solve (1), by which we mean that the v_i solve the first N equations in (1) in the sense of vanishing viscosity solutions.

To see this let $\mathcal{L}^i_\varepsilon = \mathcal{L}^i - \varepsilon\Delta$, $\varepsilon > 0$, which are of course uniformly elliptic, and let $\lambda^\varepsilon_1, \ldots, \lambda^\varepsilon_N \in I\!R$, $v^\varepsilon_1, \ldots, v^\varepsilon_N \in C^2(Q)$, $m^\varepsilon_1, \ldots, m^\varepsilon_N \in W^{1,p}(Q)$ (for all $1 \leq p < \infty$) solve (1) with $\mathcal{L}^i_\varepsilon$ instead of $\mathcal{L}^i$. The point is that

$$\|Dv_i^\varepsilon\|_\infty \leq C \quad i = 1, \ldots, N,$$

for some $C > 0$ independent of ε sufficiently small. Then, using also (10) and some compactness argument, one concludes that $\exists \varepsilon_n \to 0$, $\lambda_1, \ldots, \lambda_N \in I\!R$ $v_1, \ldots, v_N \in C^{0,1}(Q)$, $m_1, \ldots, m_N \in W^{1,p}(Q)$ ($1 \leq p < \infty$) such that

$$(\lambda_i^{\varepsilon_n}, \, v_i^{\varepsilon_n}, \, m_i^{\varepsilon_n}) \to (\lambda_i, \, v_i, \, m_i) \quad \text{in } I\!R \times C(Q) \times C(Q)$$

(actually, the convergence of the functions can be somewhat better, but we do not elaborate further on this) where $(\lambda_i, \, v_i, \, m_i)$, $i = 1, \ldots, N$ solve (1) in the claimed sense.

Finally, observe also that together (26) and (2) imply (25).

Remark 2.4. With the same reasoning as in the proof of Theorem 2.1, with some obvious modifications, we have also proved the solvability of the problem

$$\begin{cases} \mathcal{L}v + \lambda + H(x, Dv) = 0 \\ \int_Q v(x)dx = 0 \end{cases}$$

for the unknowns $v \in C^2(Q)$, $\lambda \in I\!R$, where the Hamiltonian H is locally Lipschitz continuous and satisfies (2), (3), or, alternatively, any condition mentioned in the remarks above and $\mathcal{L}$, as usual, is a second-order uniformly elliptic linear differential operator with Lipschitz continuous coefficients in the d-dimensional torus Q. This result should be known, and in fact Dirichlet and/or Neumann boundary value problems for quasilinear elliptic PDEs with first-order terms (here called Hamiltonians) satisfying similar conditions have been studied in [32], [27]. However, to find its analog for the periodic setting in the literature is not that easy, hence we prefer to bypass it in the proof of Theorem 2.1 (thus doing, we actually reprove it).

If $\mathcal{L}$ is degenerate elliptic, then, in the presence of (26) and if (25) holds with "$>$" instead of "$\geq$" (even $\eta = 1$ is allowed), (of course, we mean that these conditions are referred to H, a,... so the index i is to be ignored), we can only conclude the existence of $v \in C^{0,1}(Q)$ as a vanishing viscosity solution of the equation above. This result is contained in [28].

Theorem 2.2. *Assume that the Hamiltonians H^i are locally α-Hölder continuous, $0 < \alpha < 1$, and satisfy (4). Let $\rho^1, \dots, \rho^N$ be positive constants. Then there exist $v_1, \dots, v_N \in C^{2,\alpha}(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$ (for all $1 \leq p < \infty$, $i = 1, \dots, N$) which solve*

$$\begin{cases} \mathcal{L}^i v_i + H^i(x, Dv_i) + \rho^i v_i = V^i[m] \quad \text{in } Q, \\ \\ \mathcal{L}^{i*} m_i - div\,(g^i(x, Dv_i) m_i) = 0 \quad \text{in } Q, \\ \\ \int_{Q^d} m_i(x)dx = 1, \quad m_i > 0, \quad i = 1, \dots, N, \end{cases} \tag{27}$$

For the proof we need the following lemma which is proved in the Appendix.

Lemma 2.3. *Let*

$$\mathcal{L} = -a_{hk}(x) D_{hk} + b_h(x) D_h + c(x)$$

be a second-order uniformly elliptic linear differential operator in the d-dimensional torus Q with coefficients a_{hk}, b_h, $c \in C^\alpha(Q)$, $h, k = 1, \dots, d$, $0 < \alpha < 1$. Assume also that $c > 0$. Then, for any $f \in C^\alpha(Q)$, the equation

$$\mathcal{L} v = f \tag{28}$$

has one and only one solution $v \in C^{2,\alpha}(Q)$. Moreover, for some constant $C > 0$ which depends only on (the coefficients of) $\mathcal{L}$,

$$\|v\|_{C^{2,\alpha}(Q)} \leq C \|f\|_{C^\alpha(Q)}. \tag{29}$$

Proof of Theorem 2.2. We define an operator

$$T : C^{1,\alpha}(Q)^N \to C^{1,\alpha}(Q)^N,$$

$$u \to m \to v,$$

in the following way. Given $u = (u_1, \dots, u_N)$, we solve the second N equations in (27) with u_i plugged into g^i in place of v_i and with the corresponding conditions, and find $m = (m_1, \dots, m_N)$. With these m_i and the u_i plugged into the Hamiltonians H^i, $i = 1, \dots, N$, in place of v_i we solve the first N linear equations of (27), that is,

$$\mathcal{L}^i v_i + \rho^i v_i + H^i(x, Du_i) = V^i[m], \qquad i = 1, \dots, N \tag{30}$$

and find $v = (v_1, \dots, v_N) \in C^{2,\alpha}(Q)^N$. This is possible in virtue of Lemma 2.3. We set $Tu = v$.

It is standard to verify that $T : C^{1,\alpha}(Q) \to C^{1,\alpha}(Q)$, $u \to Tu$ is continuous and compact.

By Schaefer's version of Leray-Schauder theorem (see [33, Theorem 4.3.2, p 29] or [20, Theorem 11.3, p. 280]), we need only prove that the set of the fixed points of the operators sT, $0 \le s \le 1$, that is,

$$\{u \in C^{1,\alpha}(Q)^N \ : \ sTu = u \text{ for some } 0 \le s \le 1\} \tag{31}$$

is bounded in $C^{1,\alpha}(Q)^N$.

In fact, first note that if $u = sTu$, for some $o \le s \le 1$, then, by looking at the extrema of u_i which satisfies equation (30) with $u_i = v_i$ and H^i, F^i multiplied by s, we have

$$\|u_i\|_{C(Q)} = \frac{s}{\rho^i} \max_{x\in Q}(|H^i(x,0)| + |F^i(x)|) \le C$$

for some $C > 0$ independent of u and s. Then, this together with a classical a priori interior estimate for the gradients of solutions of elliptic quasilinear equations, that is[3] [24, Theorem 3.1, p. 266] - and it here where growth assumption (4) is used - imply

$$\|u_i\|_{C^1(Q)} \le C$$

for some $C > 0$ independent of u and s. This estimate combined with [20, Theorem 8.32, p. 210] yields

$$\|u_i\|_{C^{1,\alpha}(Q)} \le C$$

for some $C > 0$ again independent of u and s.

Thus T has at least one fixed point. □

Remark 2.5. An alternative existence result can be stated, that is, the conclusions of the preceding theorem hold (with any $0 < \alpha < 1$), if we assume that the Hamiltonians H^i are locally Lipschitz continuous and, instead of (4), we assume (2), (3). To see this requires only a slight modification of the proof of Theorem **??**.

Moreover, the considerations made in Remark 2.1 continue to hold also for Theorem 2.2. Actually, if we assume (24), then (25) can be improved a bit by requiring $\exists\theta^i$, $\eta^i \in (0,1)$ such that

$$\liminf_{|p|\to\infty} \frac{1}{|p|^2} \inf_{x\in Q} \left(\theta^i(\operatorname{tr} a) D_x H^i \cdot p + (1-\theta^i)\theta^i\eta^i(H^i)^2 - |p|^2 \frac{(\operatorname{tr} a^i)|D\sigma^i|^2}{2} \right) \ge -\rho^i, \tag{32}$$

If the inequality above with $\eta^i = 1$, holds with the sign ">", then (24) is not needed at all.

Of course the sense of these results is that we can allow for Hamiltonians with arbitrary growth provided that some of the previous technical conditions are satisfied.

[3] This theorem assumes that the leading coefficients of the quasilinear equation are of class C^1. We then first prove our existence theorem with this additional assumption and then remove it by approximation; we can do so because the resulting estimates depend only on the L^∞-norm and not on the modulus of continuity of the derivatives of the coefficients. The fact that the said theorem deals with equations in divergence form is not a problem either because we can put our equations in divergence form by perturbing the Hamiltonians with a term at most linear in the gradient.

3 Applications to ergodic games

As an application of the previous results, in this section we show the existence of Nash equilibria (under suitable assumptions of course) for a class of stochastic differential N-player games with ergodic costs, which are such that the state of each player evolves independently from the states of the other players, and the only coupling (i.e., reciprocal influence of players on each-other) comes through the costs. These games were introduced by J-M. Lasry and P-L. Lions [25, 26] in order to derive their stationary mean field game equations, by letting the number of the players $N \to \infty$.

Now we describe these games in detail. Consider a control system driven by the stochastic differential equations

$$dX_t^i = f^i(X_t^i, \alpha_t^i)dt + \sigma^i(X_t^i)dW_t^i, \quad X_0^i = x^i \in I\!R^d, \quad i = 1, \dots, N \tag{33}$$

where: $\{W_t^i\}$ are N independent Brownian motions in $I\!R^d$, $d \geq 1$, A^i is a metric space,

$$f^i : I\!R^d \times A^i \to I\!R^d \qquad \sigma^i : I\!R^d \to I\!R^{d \times d}$$

are continuous, $\mathbb{Z}^d$-periodic and Lipschitz continuous in x uniformly in α, the matrix $\sigma^i(x)$ is nonsingular for any value of x, α_t^i is an *admissible control* of the i-th player, that is, a stochastic process taking values in A^i and adapted to W_t^i.

In view of the assumed periodicity in x^i of all data we will often consider functions as defined on $Q = [0,1]^d$ instead of $I\!R^d$. The i-th player seeks to minimize the long-time-average or ergodic cost

$$J^i(X_0, \alpha^1, \dots, \alpha^N) = \liminf_{T \to +\infty} \frac{1}{T} E\left[\int_0^T L^i(X_t^i, \alpha_t^i) + F^i(X_t^1, \dots, X_t^N)dt\right]. \tag{34}$$

On the cost of the i-th player (34) we assume

$$L^i : Q \times A^i \to I\!R \tag{35}$$

are measurable and locally bounded whereas

$$F^i(x^1, \dots, x^N) : Q^N \to I\!R \quad \text{Lipschitz continuous.} \tag{36}$$

Define

$$\mathcal{H}^i(x^i, p, \alpha) := -p \cdot f^i(x^i, \alpha) - L^i(x^i, \alpha), \quad H^i(x^i, p) := \sup_{\alpha \in A^i} \mathcal{H}^i(x^i, p, \alpha), \quad p \in I\!R^d. \tag{37}$$

Of course we assume that the supremum on the right side is finite for any choice of $x \in Q$, $p \in I\!R^d$. This is certainly so if A^i is compact. If not, a sufficient condition is

$$\lim_{\alpha \to \infty} \mathcal{H}^i(x^i, p, \alpha) = -\infty \quad x \in Q,\ p \in I\!R^d,$$

which in turn is implied by the following[4]

$$|f^i(x,\alpha)| = o(L^i(x,\alpha)) \quad \text{as} \qquad \alpha \to \infty, \quad x \in Q,$$
$$\liminf_{\alpha\to\infty} |f^i(x,\alpha)| > 0.$$

Note that under both of the above conditions, the supremum in (37) is indeed a maximum, that it, it is achieved at some point $\overline{\alpha}^i(x,p)$ and this is of fundamental importance for the construction of Nash feedbacks, see the following assumptions (38), (39).

In order to construct a Nash equilibrium in feedback form for the N-person stochastic differential game described above we follow a standard procedure.

We assume there exist functions $\overline{\alpha}^i : I\!R^d \times I\!R^d \to A^i$ such that

$$\overline{\alpha}^i \text{ is locally Lipschitz and } \mathbb{Z}^d - \text{periodic in } x, \tag{38}$$

$$\overline{\alpha}^i(x,p) \text{ is a maximum point for } \alpha^i \to -f^i(x,\alpha^i)\cdot p - L^i(x,\alpha^i),\ \forall x,p. \tag{39}$$

Define

$$g^i(x,p) = -f^i(x,\overline{\alpha}^i(x,p)), \qquad i = 1,\dots N. \tag{40}$$

Finally, let

$$a^i = \sigma^i(\sigma^i)^t/2 \quad \text{and} \quad \mathcal{L}^i = -a^i \cdot D^2. \tag{41}$$

The system of Bellman equations of ergodic type introduced and studied by Bensoussan and Frehse [9, 10] becomes in our case

$$\sum_{j=1}^N \mathcal{L}^j v_i + H^i(x^i, D_{x^i}v_i) + \sum_{j\neq i} g^j(x^j, D_{x^j}v_i)\cdot D_{x^j}v_i + \lambda_i = F^i(X) \quad \text{in } I\!R^{dN}, \tag{42}$$

where the unknowns are the constants λ_i and the functions $v_i(X)$, $X = (x^1,\dots,x^N) \in I\!R^{dN}$, $i = 1,\dots,N$. This is a system of nonlinear elliptic equations in diagonal form (only $D_{x_j}v_j$ appears in the i-th equation).

We want to derive from (42) another system of elliptic equations , hopefully simpler (this part of the argument is heuristic) following an idea introduced by Lasry and Lions in the seminal paper [25]. Assume that the data and solutions are not only $\mathbb{Z}^d$-periodic in the x^i variables but also smooth. An optimal diffusion process solves

$$dX_t^i = f^i(X_t^i, \overline{\alpha}^i(X_t^i, D_{x^i}v_i(X_t)))dt + \sigma^i(X_t^i)dW_t^i, \quad X_0^i = x^i \in I\!R^d, \quad i = 1,\dots,N. \tag{43}$$

In [26] Lasry and Lions conjecture that, as the number of players N, increases each v_i behaves asymptotically as a function of only x^i (they expect that $D_{x_j}v_i$ should be of order $1/N$ for $j \neq i$). Let us assume that each v_i depends only on x^i outright (we repeat, this part of the argument is only heuristics). Then each component X_t^j of the optimal diffusion above

[4] By the first in (3) we mean that for any $\varepsilon > 0$ there is a compact set $A^i_\varepsilon \subset A^i$ such that $|f^i(x,\alpha)| \le \varepsilon L^i(x,\alpha)\ \forall \alpha \notin A^i_\varepsilon$.

evolves independently from the other components and is ergodic: its invariant measure solves the Fokker-Planck equation

$$\mathcal{L}^{j*} m_j - div_{x^j}\left(g^j(x^j, D_{x_j} v_j) m_j\right) = 0, \tag{44}$$

where $\mathcal{L}^{j*}$ is the formal adjoint of $\mathcal{L}^j$. Suppose these elliptic PDEs have positive solutions $m_j = m_j(x^j)$ (depending only on x_j) such that

$$\int_Q m_j(x)dx = 1. \tag{45}$$

Multiply the i-th equation in (42) by $\prod_{j\neq i} m_j(x^j)$ and integrate over Q^{N-1} with respect to $dx^j, j \neq i$. Observe that, integrating by parts,

$$\int_{Q^{N-1}} \sum_{j\neq i} \left(\mathcal{L}^j v_i + g^j(x^j, D_{x^j} v_j) \cdot D_{x^j} v_i\right) \prod_{j\neq i} m_j(x^j) dx^j = 0,$$

hence we arrive at

$$\mathcal{L}^i v_i + H^i(x, Dv_i) + \lambda_i = V^i[m] \quad \text{in } I\!R^d. \tag{46}$$

with

$$V^i[m](x) = \int_{Q^{N-1}} F^i(x^1, \ldots, x^{i-1}, x, x^{i+1}, \ldots, x^N) \prod_{j\neq i} m_j(x^j) dx^j. \tag{47}$$

One checks easily that the above operators V^i satisfy the hypotheses of the Introduction.

Observe also that $H^i(x,p)$ is convex in p. Let us assume further that $H^i(x,p)$ satisfy the assumptions mentioned in the Introduction, i.e., either they are locally α-Hölder continuous, $0 < \alpha < 1$, in both variables and grow at most quadratically in p as in (4), or they are locally Lipschitz in both variables and (2), (3) hold.

By Theorem 2.1 there exist $\lambda_1, \ldots, \lambda_N \in I\!R$, $v_1, \ldots, v_N \in C^2(Q)$, $m_1, \ldots, m_N \in W^{1,p}(Q)$, $1 \leq p < \infty$ which solve (46), (44), (45).

Thus, with the above assumptions on f^i, σ^i, F^i, H^i, we show the existence of Nash equilibria for the above game be means of the following

Theorem 3.1. *Let $\lambda_1, \ldots, \lambda_N \in I\!R$, $v_1, \ldots, v_N \in C^2(Q)$, $m_1, \ldots, m_N$ be a solution of the system* (46), (44), (45). *Then*

$$\overline{\alpha}^i(x) = \overline{\alpha}^i(x, Dv_i(x)), \qquad i = 1, \ldots, N \tag{48}$$

define a feedback which is a Nash equilibrium for all initial positions $X \in Q^N$ of the control system (33). *In addition, for each $X \in Q^N$,*

$$\lambda_i = J^i(X, \overline{\alpha}^1, \ldots, \overline{\alpha}^N) = \liminf_{T\to+\infty} \frac{1}{T} E\left[\int_0^T L^i(\overline{X}_t^i, \overline{\alpha}^i(\overline{X}_t^i)) + F^i(\overline{X}_t^1, \ldots, \overline{X}_t^N) dt\right], \tag{49}$$

where $\overline{X}_t^i$ is the optimal diffusion corresponding to the feedback $\overline{\alpha}^i$ obtained by solving (43).

Proof. It is a standard verification argument by Ito's formula and the ergodicity of $\overline{X}_t^i$. For the sake of completeness we provide the details. Consider the feedback law (48) and let us check (49). By Ito's formula,

$$dv_i(\overline{X}_t^i) = \Big(-g(\overline{X}_t^i), Dv_i(\overline{X}_t^i)) \cdot Dv_i(\overline{X}_t^i) - \mathcal{L}^i(\overline{X}_t^i)\Big)\, dt + (\dots)dW_t^i.$$

Hence,

$$\begin{aligned}
&\frac{1}{T}\Big(E[v_i(\overline{X}_T^i)] - v_i(\overline{X}_0^i)\Big)\\
&\qquad = \frac{1}{T}E\left[\int_0^T \Big(-g(\overline{X}_t^i), Dv_i(\overline{X}_t^i)) \cdot Dv_i(\overline{X}_t^i) - \mathcal{L}^i(\overline{X}_t^i)\Big)\, dt\right]\\
&= \lambda_i - \frac{1}{T}E\left[\int_0^T L^i(\overline{X}_t^i, \overline{\alpha}^i(\overline{X}_t^i)) + \int_{Q^{N-1}} F^i(x^1,\dots,x^{i-1},\overline{X}_t^i,x^{i+1},\dots,x^N)\prod_{j\neq i} dm_j(x^j)dt\right].
\end{aligned}$$

Letting $T \to \infty$, we obtain (49) by applying [12, Theorem 3.2., p. 373] (note also that the joint process $\overline{X}_t^i$ is ergodic with invariant measure $\prod_{i=1}^N m_i(x^i)$).

Next let us check that the feedback law (48) defines a Nash point. For any admissible control α_t^i we have

$$\begin{aligned}
&\frac{1}{T}\Big(E[v_i(\overline{X}_T^i)] - v_i(\overline{X}_0^i)\Big)\\
&\qquad = \frac{1}{T}E\left[\int_0^T \Big(f(\overline{X}_t^i), \alpha_t^i) \cdot Dv_i(\overline{X}_t^i) - \mathcal{L}^i(\overline{X}_t^i)\Big)\, dt\right]\\
&\geq \lambda_i - \frac{1}{T}E\left[\int_0^T L^i(\overline{X}_t^i, \alpha_t^i) + \int_{Q^{N-1}} F^i(x^1,\dots,x^{i-1},\overline{X}_t^i,x^{i+1},\dots,x^N)\prod_{j\neq i} dm_j(x^j)dt\right].
\end{aligned}$$

Again, by ergodicity we get

$$J^i(X, \overline{\alpha}^1, \dots, \overline{\alpha}^{i-1}, \alpha^i, \overline{\alpha}^{i+1}, \dots, \overline{\alpha}^N) \geq \lambda_i = J^i(X, \overline{\alpha}^1, \dots, \overline{\alpha}^N)$$

for all $X \in Q^N$. $\square$

Example 3.1. For Hamiltonians of the form

$$H^i(x,p) := \sup_{\alpha \in A^i}\{-p \cdot f^i(x,\alpha) - L^i(x,\alpha)\}$$

assume $A^i = I\!R^d$ and the system affine in the control, i.e.,

$$f^i(x,\alpha) = f_o^i(x) + \sum_{k=1}^d \alpha_k f_k^i(x) = f_o^i(x) + \Phi^i(x)\alpha,$$

where Φ^i is a square matrix whose columns are the vector fields $f^i_k, k = 1, \dots, d$, and all vector fields are Lipschitz. Assume L^i is Lipschitz in x, uniformly as α varies in any bounded subset, and

$$\lim_{\alpha\to\infty} \inf_{x\in Q} L^i(x,\alpha)/|\alpha| = +\infty.$$

Then the sup in the definition of H^i is attained and

$$H^i(x,p) = L^{i^*}\left(x, -\Phi^i(x)^t p\right) - p \cdot f^i_o(x), \quad L^{i^*}(x,q) := \max_{\alpha\in\mathbb{R}^d}\{q\cdot\alpha - L^i(x,\alpha)\},$$

i.e., $L^{i^*}(x,\cdot)$ is the convex conjugate of $L^i(x,\cdot)$. Therefore H^i is locally Lipschitz. Moreover, since L^{i^*} is superlinear in q, H^i is superlinear in p if the matrix $\Phi^i(x)$ is nonsingular. Now the other conditions of the existence theorems for the elliptic system can be checked on the last explicit expression of H^i. For instance, if $L^i(x,\alpha) = c(x)|\alpha|^\gamma/\gamma$ for some Lipschitz $c > 0$ and $\gamma > 1$, then $L^{i^*}(x,q) = c^{1/(1-\gamma)}|q|^{\gamma/(\gamma-1)}(\gamma-1)/\gamma$, so one can compute $D_x H^i$ and see (using again that $\Phi^i(x)$ is nonsingular) that (3) is satisfied for any choice of $\theta^i > 0$ (and therefore also (3) holds true).

Another example is $L^i(x,\alpha) = \alpha^t B(x)\alpha$ with a positive definite matrix $B(x)$ Lipschitz in x. Then $L^{i^*}(x,q) = q^t B(x)^{-1} q/4$, so H^i grows at most quadratically in p, condition (4), even at points where $\Phi^i(x)$ is singular.

Finally, a sufficient condition for (38) and (39) is that $D_\alpha L^i(x,\cdot)$ be invertible (a fact related to the strict convexity of L^i in α). In fact, the sup in the definition of H^i is attained at a unique value $\overline{\alpha}^i(x,p) = (D_\alpha L^i)^{-1}\left(x, -\Phi^i(x)^t p\right)$, which is a locally Lipschitz function of x and p. This condition is easily checked in the two examples above. More precisely, for $L^i = c|\alpha|^\gamma$ we get

$$\overline{\alpha}^i(x,p) = c(x)^{1/(1-\gamma)}|q|^{(2-\gamma)/(\gamma-1)}q, \qquad q = -\Phi^i(x)^t p,$$

whereas in the quadratic case, $L^i = \alpha^t B\alpha$, the Nash equilibrium feedback is linear in p:

$$\overline{\alpha}^i(x,p) = -\frac{1}{2}B(x)^{-1}\Phi^i(x)^t p.$$

Consider again the controlled stochastic dynamics (33) with the same assumptions on the regularity and periodicity of the drifts f^i and diffusions σ^i, $i = 1, \dots, N$. Fix a set of controls $(\alpha^i)_{i=1}^N$. The corresponding solution processes X^i_t, $i = 1, \dots, N$, are ergodic with invariant measures $m_i(x) = m_i(x;\alpha_i)$, $x \in Q$. By this we mean that each X^i_t satisfies the conclusions of [12, Theorem 3.2., p. 373]. The same is true for the joint process $X_t = (X^i_t, \dots, X^N_t)$ with invariant measure $\prod_{i=1}^N m_i(x^i)$.

Each player i is interested in minimizing its own cost functional

$$J^i_{\rho^i}(\alpha^1, \dots, \alpha^N) = E\left[\int_0^\infty e^{-\rho^i t}\left(L^i(X^i_t, \alpha^i_t) + V^i[m](X^i_t)\right) dt\right]. \tag{50}$$

Here L^i, V^i, $i = 1, \dots, N$, are the same as in (35), (47), respectively, and $\rho^i > 0$ is some fixed constant called the *discount rate* of the ith player. In order to find Nash points we are led to the following Hamilton-Jacobi-Bellman system of PDEs.

$$\mathcal{L}^i v_i + \rho^i v_i + H^i(x, Dv_i) = V^i[m] \tag{51}$$

$$\mathcal{L}^{i*}m_i - div\left(g^i(x^i, Dv_i)m_i\right) = 0 \quad \text{in } Q, \tag{52}$$

$$\int_Q m_i(x)dx = 1, \quad m_i > 0, \quad i = 1, \dots N, \tag{53}$$

where $\mathcal{L}^i$, H^i, g^i are the same as in (41), (37), (40), respectively.

By Theorem 2.2 there exists a solution $v_1, \dots, v_N \in C^2(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, $1 \leq p < \infty$ to (51), (52) (53). Thus we may assert the existence of Nash equilibria for this game by the following

Theorem 3.2. *Let $\rho^1, \dots, \rho^N > 0$ be given discount rates. For any solution $v_1^{\rho^1}, \dots, v_N^{\rho^N} \in C^2(Q)$, $m_1, \dots, m_N \in W^{1,p}(Q)$, $1 \leq p < \infty$, of the system* (51), (52), (51)*, the feedback law $\overline{\alpha}^i$, $i = 1, \dots, N$, defined by* (48)*, provides a Nash point for all $X \in Q^N$. Moreover, for each $X = (x^i)_{i=1}^N \in Q^N$,*

$$v_i^{\rho^i}(x^i) = J^i_{\rho^i}(X, \overline{\alpha}^1, \dots, \overline{\alpha}^N) = E\left[\int_0^\infty e^{-\rho^i t}\left(L^i(\overline{X}_t^i, \overline{\alpha}^i(\overline{X}_t^i)) + F^i(\overline{X}_t^1, \dots, \overline{X}_t^N)\right)dt\right], \tag{54}$$

where $\overline{X}_t^i$ is the solution of (43)*.*

We conclude with some bibliographical remarks. The connection between Nash feedback equilibria and systems of elliptic or parabolic equations was first observed by Friedman [18] and a systematic study of such systems began with [9]. More references can be found in the book of Bensoussan and Frehse [10] and in their recent paper [11], where they also weaken the standard quadratic growth condition on the Hamiltonians (although their results do not apply to our system). The difficulties arising from constraints on the controls and the appearence of discontinuous feedbacks were treated by Mannucci [29] for parabolic equations. The existence of Nash equilibria for some stochastic N-person differential games was also proved by probabilistic methods, see e.g. [14, 19] and the references therein. Mean-Field Games were also studied independently by Huang, Caines, and Malhamé using different methods, see, e.g., [22, 23]. For a general recent presentation of Mean-Field Games and their applications we refer to the lecture notes by Gueant, Lasry, and Lions [21] and Cardaliaguet [16], and for numerical methods to Achdou, Camilli, Capuzzo Dolcetta [2] and the references therein.

4 Appendix: proofs of some technical results

Proof of Lemma 2.1. For any solution $(v, \lambda) \in C^{2,\alpha}(Q) \times I\!R$ of (5), we immediately deduce (6) by looking at the extrema of v.

Uniqueness. Thus $f \equiv 0$ implies $\lambda = 0$. On the other hand $v \equiv 0$ otherwise the strong maximum principle (e.g., [20, Theorem 3.5, p. 35]) would be contradicted.

Existence. Consider (5) with $\mathcal{L}_0 = -\Delta$ instead of $\mathcal{L}$. It is clearly solvable for any $f \in C^\infty$ with $\lambda = \int_Q f dx$ and v that can be determined by Fourier series. Moreover, by

Bessel's identity one deduces[5] $\|D^\beta v\|_p \le \|f\|_p$ for all $2 \le p < \infty$, $|\beta| \le 2$. By letting $p \to \infty$, this estimate holds also for $p = \infty$ (by duality it holds also for $1 \le p < 2$, but we do not need this here). Thus (5) with $\mathcal{L}_0$ instead of $\mathcal{L}$ is solvable for any $f \in C(Q)$ and the solution $v \in C^2(Q)$. Actually, by Schauder interior estimates (see [20, Theorem 4.8, p. 62]) $f \in C^\alpha(Q)$ implies $v \in C^{2,\alpha}(Q)$.

In order to apply a continuity method, see [20, Theorem 5.2, p. 75], and deduce the solvability of (5), we introduce the operators

$$\begin{aligned} T_i :& C^{2,\alpha}(Q)/I\!R \times I\!R \to C^\alpha(Q), \qquad & i = 0, 1, \\ &(u,\lambda) \to T_i(u,\lambda) = \mathcal{L}_i u + \lambda \qquad & \mathcal{L}_1, \equiv \mathcal{L}. \end{aligned}$$

Remark that

$$[v]_{2,\alpha,Q} = \sup_{x,\,y\in Q,\,|\beta|=2} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x-y|^\alpha}$$

is a norm in $C^{2,\alpha}(Q)/I\!R$ equivalent to the natural one.

Define also $T_s = (1-s)T_0 + sT_1$, $0 \le s \le 1$. These operators are clearly linear and bounded. T_0 is also an isomorphism (of Banach spaces). We need only prove that they are bounded from below in order to finish. This requires a careful look at Schauder interior estimates, see [20, Theorem 6.2, p. 90]. Let $\Omega \supset Q$ be an open set, a ball for instance. We need some notation from [20]. For any function g (provided that it is differentiable as many times as needed for the following to make sense), if we write $d_x = \mathrm{dist}\,(x, \partial\Omega)$, $d_{x,y} = \min\{d_x,\, d_y\}$, we set

$$\begin{aligned} [g]^*_{2,\alpha,\Omega} &= \sup_{x,y\in\Omega,\,|\beta|=2} d_{x,y}^{2+\alpha} \frac{|D^\beta g(x) - D^\beta g(y)|}{|x-y|^\alpha}, \\ [g]^{(j)}_{l,\alpha,\Omega} &= \sup_{x,y\in\Omega,\,|\beta|=2} d_{x,y}^{j+l+\alpha} \frac{|D^\beta g(x) - D^\beta g(y)|}{|x-y|^\alpha} \end{aligned}$$

$$[g]^{(j)}_{l,\Omega} = \sup_{x\in\Omega,\,|\beta|=l} d_x^{l+j} |D^\beta g(x)|, \qquad |g|^{(j)}_{i,\alpha,\Omega} = \sum_{l=1}^{i} [g]^{(j)}_{l,\Omega} + [g]^{(j)}_{i,\alpha,\Omega}$$

for any nonnegative integers i, j, l. Let $T_s(v,\lambda) = f$. Then, by [20, Theorem 6.2, p. 90],

$$[v]^*_{2,\alpha,\Omega} \le C(\|v\|_{C(\Omega)} + |f-\lambda|^{(2)}_{0,\alpha,\Omega}), \tag{55}$$

where C depends only on a constant of ellipticity of $\mathcal{L}_s = (1-s)\mathcal{L}_0 + s\mathcal{L}_1$ which clearly can be taken to be independent of s, and at most linearly on

$$\max_{h,\,k}\{|1 - s + sa_{hk}|^{(0)}_{0,\alpha,\Omega},\ |sb_h|^{(1)}_{0,\alpha,\Omega}\} \le D^{1+\alpha} \max_{h,\,k}\{\|a_{hk}\|_{C(Q)},\ \|b_h\|_{C^\alpha(Q)}\},$$

[5] $\beta = (\beta_1, \ldots, \beta_d)$ is a multiindex, $|\beta| = \beta_1 + \cdots + \beta_d$ its length and $D^\beta v$ stands for $\frac{\partial^{|\beta|} v}{\partial^{\beta_1} x_1 \cdots \partial^{\beta_d} x_d}$.

where D is the diameter of Ω Therefore, taking also into account (6), by (55) we obtain

$$[v]_{2,\alpha,Q} \le C\frac{D^{1+\alpha}}{(D-\sqrt{d})^{2+\alpha}}\left(\|v\|_{C(Q)} + D^{2+\alpha}\|f\|_{C^{\alpha}(Q)}\right)$$

for some C independent of D. Now it is sufficient to take an Ω with a suitably large diameter D in order to obtain (8). Finally, the $C^{1,\alpha}$-estimate (7) follows from [20, Theorem 8.32, p. 210]. This concludes the proof of this lemma.

An alternative proof of the existence relies on the "discounted case". That is, for $\rho > 0$ we consider the solution v^ρ of $\mathcal{L}v + \rho v = f$, see Lemma 2.3. Then we prove that $(v^\rho - \int_Q v^\rho dx,\ \rho v^\rho) \to (v,\ \lambda)$ in $C^2(Q) \times C(Q)$ as $\rho \to 0$, where $(v,\ \lambda)$ is a solution of (5). The reader can either fill in the details himself/herself as an exercise or see [5] and/or [8], where more general nonlinear equations are treated. □

Proof of Lemma 2.2. That $m \in W^{1,2}(Q)$ exists, is unique and positive is proved e.g., in [12, Theorem 3.4, 378] or [6, Theorem 4.2, p. 133, Theorem 4.3, p. 136]. Actually, in [6] only the case $\mathcal{L} = -\Delta$ is treated but the techniques used there adapt easily to our more general operator.

The fact that $m \in W^{1,p}(Q)$ should also be well-known, but since we have difficulties in providing a reference, we sketch a proof here based on ideas of [30].

Split the operator $\mathcal{L} = \mathcal{L}_0 + R$ into a (formally) selfadjoint part $\mathcal{L}_0 = -D_k(a_{hk}(x)D_h)$ and a reminder $R = D_k a_{ak} D_h$. We use the deep fact that

$$\mathcal{L}_0 : W^{2,p}(Q)/I\!R \to L^p(Q)/I\!R, \qquad 1 < p < \infty$$

is an isomorphism of Banach spaces. Then, by duality and interpolation,

$$\mathcal{L}_0 : W^{1,p}(Q)/I\!R \to W^{-1,p}(Q)/I\!R, \qquad 1 < p < \infty$$

is also an isomorphism of Banach spaces. So we need only show that $\mathcal{L}_0 m \in W^{-1,p}(Q)$ in order to conclude.

Note that, by Sobolev's embedding lemma, $m \in L^p(Q)$, where $p > 2$ is given by $1/p = 1/2 - 1/d$ if $d \ge 3$, or, otherwise, for any $1 \le p < \infty$. By the equation (9), we have $\mathcal{L}_0 m = -R^* m + div(gm)$. Hence, for any $\varphi \in C^\infty(Q)$,

$$\begin{aligned}|<\mathcal{L}_0 m, \varphi>| &= \left|\int_Q mD_k a_{hk} D_h \varphi dx - \int_Q mg \cdot D\varphi dx\right| \\ &\le (\|g\|_\infty + C)\|m\|_p \|D\varphi\|_{p'}\end{aligned}$$

for some $C > 0$ (independent of g), $p' = p/(p-1)$. Therefore, $\mathcal{L}_0 m \in W^{-1,p}(Q)$ and $m \in W^{1,p}(Q)$. We are done if $d \le 2$. Otherwise, again Sobolev's lemma implies that $m \in L^p(Q)$, where p now is given by $1/p = 1/2 - 2/d$ if $d \ge 5$, or, otherwise, for any $1 \le p < \infty$. In the same manner we conclude that $m \in W^{1,p}(Q)$. Thus, by a bootstrap argument, we deduce that $m \in W^{1,p}(Q)$ for all $1 \le p < \infty$.

Moreover, by the estimates above and the fact that $\mathcal{L}_0 : W^{1,p}(Q)/I\!R \to W^{-1,p}(Q)/I\!R$ is an isomorphism, we deduce that

$$\|m\|_{W^{1,p}(Q)} \le C_1(\|g\|_\infty + C_2)\|m\|_p.$$

for some $C_1,\ C_2 > 0$ independent of g. Taking also into account [6, Theorem 4.3, p. 136] which states that $\delta_1 < m < \delta_2$ for some constants $\delta_1,\ \delta_2 > 0$ that depend only on $\|g\|_\infty$ (and in our case also on the coefficients a_{hk}, in a way which we do not specify because we will not need it in the sequel) we obtain (10). □

Proof of Lemma 2.3. Uniqueness is easy. Indeed, consider the free term $f \equiv 0$ in (28). A corresponding solution v needs to have a nonpositive maximum and a nonnegative minimum. This is possible only if $v \equiv 0$.

For the existence, we use a continuity method, e.g., [20, Theorem 5.2, p. 75], and Schauder a priori estimates to reduce to the equation corresponding to a simpler operator, say

$$\mathcal{L}_0 = -\Delta + 1.$$

That $\mathcal{L}_0 v = f$ has a solution $v \in C^\infty(Q)$ for each $f \in C^\infty(Q)$ can be shown, e.g., by Fourier series. Moreover, by the equation, $\|v\|_{C(Q)} \le \|f\|_{C(Q)}$. Then, by a Schauder estimate, see [20, Theorem 4.8, p. 62], $\|v\|_{C^{2,\alpha}(Q)} \le C\|f\|_{C^\alpha(Q)}$. Now for an arbitrary $f \in C^\alpha(Q)$ consider a sequence $\{f_n\} \subset C^\infty(Q)$ such that $f_n \to f$ in $C^\alpha(Q)$. The sequence of the corresponding solutions $\{v_n\}$ is Cauchy in $C^{2,\alpha}(Q)$ and its limit v verifies $\mathcal{L}_0 v = f$.

Next, introduce the standard family of operators

$$\mathcal{L}_s : C^{2,\alpha}(Q) \to C^\alpha(Q), \qquad 0 \le s \le 1,$$

$$u \to \mathcal{L}_s u = (1-s)\mathcal{L}_0 u + s\mathcal{L}u.$$

Given $u \in C^{2,\alpha}(Q)$, by looking at its extrema, we have

$$\|u\|_{C(Q)} \le \max\{1, \|1/c\|_{C(Q)}\}\|\mathcal{L}_s u\|_{C(Q)}.$$

Combining this with the interior Schauder estimates, see [20, Theorem 6.2, p. 90], we obtain

$$\|u\|_{C^{2,\alpha}(Q)} \le C\|\mathcal{L}_s u\|_{C^\alpha(Q)}$$

for all $u \in C^{2,\alpha}(Q)$, $0 \le s \le 1$ and some $C > 0$ independent of u and s. Since $\mathcal{L}_0$ is onto, by the method of continuity [20, Theorem 5.2, p. 75], $\mathcal{L}_1$ is also onto, which is what we wanted to prove. With these considerations we also proved (29). □

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