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Abstract. We analyse the complexity of algebraic algorithms for solving systems of linear equations with *noise*. Such systems arise naturally in the theory of error-correcting codes as well as in computational learning theory. More recently, linear systems with noise have found application in cryptography. The *Learning with Errors* (LWE) problem has proven to be a rich and versatile source of innovative cryptosystems, such as fully homomorphic encryption schemes. Despite the popularity of the LWE problem, the complexity of algorithms for solving it is not very well understood, particularly when variants of the original problem are considered. Here, we focus on and generalise a particular method for solving these systems, due to Arora & Ge, which reduces the problem to non-linear but noise-free system solving. Firstly, we provide a refined complexity analysis for the original Arora-Ge algorithm for LWE. Secondly, we study the complexity of applying algorithms for computing Gröbner basis, a fundamental tool in computational commutative algebra, to solving Arora-Ge-style systems of non-linear equations. We show positive and negative results. On the one hand, we show that the use of Gröbner bases yields an exponential speed-up over the basic Arora-Ge approach. On the other hand, we give a negative answer to the natural question whether the use of such techniques can yield a subexponential algorithm for the LWE problem. Under a mild algebraic assumption, we show that it is highly unlikely that such an improvement exists.
We also consider a variant of LWE known as BinaryError-LWE introduced by Micciancio and Peikert recently. By combining Gröbner basis algorithms with the Arora-Ge modelling, we show under a natural algebraic assumption that BinaryError-LWE can be solved in subexponential time as soon as the number of samples is quasi-linear, e.g. $m = \mathcal{O}(n \log\log n)$. We also derive precise complexity bounds for BinaryError-LWE with $m = \mathcal{O}(n)$, showing that this new approach yields better results than best currently-known generic (exact) CVP solver as soon as $m/n \geq 6.6$. More generally, our results provide a good picture of the hardness degradation of BinaryError-LWE for a number of samples ranging from $m = n\left(1 + \Omega\big(1/\log(n)\big)\right)$ (a case for which BinaryError-LWE is as hard as solving some lattice problem in the worst case) to $m = \mathcal{O}\left(n^2\right)$ (a case for which it can be solved in polynomial-time). This addresses an open question from Micciancio and Peikert. Whilst our results do not contradict the hardness results obtained by Micciancio and Peikert, they should rule out BinaryError-LWE for many cryptographic applications. The results in this work depend crucially on the assumption the algebraic systems considered systems are not easier and not harder to solve than a random system of equations. We have verified experimentally such hypothesis. We also have been able to prove formally the assumptions is several restricted situations. We emphasize that these issues are highly non-trivial since proving our assumptions in full generality would allow to prove a famous conjecture in commutative algebra known as Fröberg's Conjecture.

1 Introduction

Whilst linear systems of polynomial equations can be solved very efficiently and with low data requirements, the situation changes dramatically once we introduce some kind of *error* or *noise* to these equations. In this work we consider the hardness of solving such systems of linear equations with *noise*, a problem that can be informally defined as follows:

Definition 1 (Linear System with Noise). *Let q be a prime. Given a matrix $G \in \mathbb{Z}_q^{n\times m}$ and a vector $\mathbf{c} \in \mathbb{Z}_q^m$. The* Linear System with Noise *problem is the task of finding a vector $\mathbf{s} \in \mathbb{Z}_q^n$ such that:*

$$\mathbf{c} = \mathbf{s} \times G + \mathbf{e}, \text{where } \mathbf{e} \in \mathbb{Z}_q^m \text{ is a “small” error-vector.}$$

A classical way to define smallness is to consider the Hamming *weight* of $\mathbf{e} \in \mathbb{Z}_q^m$, i.e. the number of non-zero components of $\mathbf{e}$. Equipped with this norm, the problem defined above is a classical NP-hard problem from coding theory known as BOUNDED DISTANCE DECODING [10]. The hardness of this problem has been used by McEliece to design the first public-key code-based encryption scheme [34]. The McEliece cryptosystem, which was proposed in the late seventies, still belongs to the small group of public-key schemes that remain unbroken. The alphabet $\mathbb{Z}_q$ in the McEliece cryptosystem is usually small (typically, $q = 2$) and independent of the security parameter n.

More recently, Regev [38, 39] considered an alternative way of introducing errors. He suggested a larger modulus q (typically, taken to be polynomial in n), and sample the components of the error-vector according to a discrete Gaussian distribution χ on $\mathbb{Z}$ (considered modulo q) with mean 0 and standard deviation $\sigma = \alpha\, q/\sqrt{2\pi}$, for some $\alpha \in (0, 1)$. In this context, LINEAR SYSTEM WITH NOISE is better known as the LEARNING WITH ERRORS (LWE) problem. The probability distribution used for error sampling implies that each of its components will have a small norm (w.r.t. the size of the field) with high probability.

Since its introduction, LWE has proven to be a rich and versatile source of many innovative cryptosystems, such as the oblivious transfer protocol by Peikert et al. [37], a cryptosystem by Akavia et al. [1] that is secure even if almost the entire secret key is leaked, homomorphic encryption [2, 14, 31] and many others. Below we reproduce the definition of LWE from [38, 39].

Definition 2 (LWE). *Let $n \geq 1$ be an integer, q be an odd integer, χ be a probability distribution on $\mathbb{Z}_q$ and $\mathbf{s} \in \mathbb{Z}_q^n$ be a secret vector. We denote by $L_{\mathbf{s},\chi}^{(n)}$ the probability distribution on $\mathbb{Z}_q^{n\times m} \times \mathbb{Z}_q^m$ obtained by choosing $G \in \mathbb{Z}_q^{n\times m}$ uniformly at random, sampling $\mathbf{e}$ according to χ^m, and returning $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n\times m} \times \mathbb{Z}_q^m$.* LWE *is the problem of finding $\mathbf{s} \in \mathbb{Z}_q^n$ from $(G, \mathbf{s} \times G + \mathbf{e})$ sampled according to $L_{\mathbf{s},\chi}^{(n)}$.*

In what follows, $\chi_{\alpha,q}$ will denote a discrete Gaussian distribution over $\mathbb{Z}$ with standard deviation $\alpha\, q/\sqrt{2\pi}$ considered modulo q. A typical setting for the standard deviation (std) is $\sigma = n^{\epsilon}$, with $0 \leq \epsilon \leq 1$. It has been shown that as soon as $\epsilon > 1/2$, (worst-case) $\mathrm{GAPSVP} - \tilde{\mathcal{O}}(n/\alpha)$ classically reduces to (average-case) LWE [13, 36, 38, 39]. Thus, any algorithm solving LWE (when $\epsilon > 1/2$) can be used for $\mathrm{GAPSVP} - \tilde{\mathcal{O}}(n/\alpha)$. We note that it is widely believed that only exponential (classical or quantum) algorithms exist for solving $\mathrm{GAPSVP} - \tilde{\mathcal{O}}(n/\alpha)$.

1.1 The Arora-Ge Algorithm for Solving LWE

In [3], Arora & Ge show that solving LWE can be reduced to solving a system of (error-free) high-degree non-linear equations. The total complexity (time and space) of this approach is $2^{\tilde{\mathcal{O}}(n^{2\epsilon})}$. Hence, it is sub-exponential when $\epsilon < 1/2$, but remains exponential when $\epsilon \geq 1/2$. As a corollary, this result also shows that Regev's reduction in [38,39] is tight. The strategy in [3] is to construct a non-linear but noise-free system of equations; from each noisy linear sample a non-linear equation of degree $2T + 1$ is formed, encoding the information that any noise value e_i is in the interval $\{-T, \ldots, T\}$. If $T < q/2$ these equations still carry information and restrict the space of the secret. Hence, collecting many such equations and solving the resulting system allows to recover the secret. In [3] linearisation is used to solve this noise-free high-degree system of equations. Hence, $\mathcal{O}\left(n^{2T+1}\right)$ equations are required. However, since $\chi_{\alpha,q}$ is a discrete Gaussian, requesting more samples also increases the probability that the noise of at least one sample falls outside of the chosen interval $\{-T, \ldots, T\}$ implying that, as the number of samples grows, so does T, which then requires a further increase in the number of samples. More samples make the problem easier but at the same time more samples make the problem harder (cf. Section 3).

These opposing characteristics of the Arora-Ge algorithm naturally propose the application of algorithms for solving non-linear systems of equations when the number of samples is restricted. Indeed, linearisation is a special case of computing a Gröbner basis, a notion going back to the seminal work of Buchberger [15, 16, 19]. Gröbner basis algorithms are fundamental tools in computational commutative algebra and allow to solve systems of equations in degree d when less than $\mathcal{O}\left(n^d\right)$ equations are available – at the cost of increased computational complexity. Applying the theory of Gröbner bases promises the following advantages over merely applying linearisation. We may hope to reduce the complexity of solving the system even if we are given access to as many samples as we would like. Since Gröbner basis algorithms need less than $\mathcal{O}\left(n^d\right)$ equations to solve systems and since the number of equations increases the required degree in the Arora & Ge modelling, we may hope to improve the complexity of solving LWE by reduction to non-linear polynomial system solving. After deriving a precise complexity of the basic Arora-Ge algorithm (Section 4), we show that the use of Gröbner bases (cf. Section 2) does indeed yield an exponential speed-up for solving LWE but is unlikely to yield a subexponential algorithm (Section 5).

1.2 LWE with Binary Errors

We then apply these ideas to a recent variant of LWE proposed by Micianccio and Peikert in [35]: LWE instances where the components in the error vectors are sampled uniformly in a fixed interval. This generalises an earlier result of Döttling and Müller-Quade [22] who first introduced a variant of LWE with uniform errors whilst keeping a strong security reduction to lattice problems which we reproduce informally below:

Theorem 1 (UniformError-LWE **[22]).** *Let n, m, q be in $k^{\mathcal{O}(1)}$ and $T \geq n^{\frac{1}{2}+\epsilon}$ for any constant $\epsilon > 0$. Then, solving* LWE *with parameters n, m, q and independent uniformly distributed errors in $\{-T, \ldots, T\}$ is at least as hard as solving worst-case problems on $(n/2)$-dimensional lattices to within a factor $\frac{mq}{T} \cdot n^{\frac{1}{2}+\epsilon}$.*

In [35], the authors further expand this result. They consider uniform errors in a much smaller interval, i.e. T is a constant. For binary errors [35, Theorem 1.2], they obtain the following result.

Theorem 2 (BinaryError-LWE **[35]).** *Let $n, m = n\left(1 + \Omega\big(1/\log(n)\big)\right)$ be integers, and $q \geq n^{\mathcal{O}(1)}$ be a sufficiently large polynomially bounded (prime) modulus. Then, solving* LWE *with parameters n, m, q and independent uniformly random binary errors is at least as hard as approximating lattice problems in the worst-case on $\Theta\big(n/\log(n)\big)$ -dimensional lattices within a factor $\tilde{\mathcal{O}}(\sqrt{n} \cdot q)$.*

From now on, we shall denote by BinaryError-LWE the problem of finding $\mathbf{s} \in \mathbb{Z}_q^n$ from $(G, \mathbf{s} \times G + \mathbf{e})$ sampled according to $L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}$, i.e. LWE with uniform binary errors. Note, though, that the result of Theorem 2 can be generalised to any uniform error in a bounded interval $\{-T, \ldots, T\}$ [35, Theorem 4.6].

We note that this problem instance immediately matches the assumption made in the Arora & Ge algorithm where the noise is considered to be bounded by some T. Hence, it is clear that the complexity of solving with uniform error is at most the cost of computing a Gröbner basis for m equations of degree $2T + 1$ in n variables. In the binary case, the approach of [3] yields a polynomial-time algorithm as soon as the number of samples $m = \mathcal{O}\left(n^2\right)$. This is already acknowledged in [22, 35], which only show that BinaryError-LWE remains hard if the number of samples is severely limited. As emphasized in [35], though, it is a natural open question to investigate the hardness of such LWE variants when the number of sample is (strictly) smaller than the upper bound provided by the Arora & Ge approach. Put differently, there is a gap between $m = n\left(1 + \Omega\big(1/\log(n)\big)\right)$ where the hardness reduces to standard LWE and $O(n^2)$ where problem is known to be easy due to the Arora-Ge algorithm. Indeed, applications in lattice-based cryptography typically require the provision of $m = \mathcal{O}\left(n\right)$ or $m = \tilde{\mathcal{O}}(n)$ samples, i.e. from within that gap.

In Section 6, we show – under a mild assumption, cf. Section 7 – that BinaryError-LWE can be solved in subexponential time as soon as the number of samples is quasi-linear, typically, $m = \mathcal{O}\left(n \log\log n\right)$. Precisely, for such number of samples, there is an algorithm solving BinaryError-LWE with complexity:

$$\mathcal{O}\left(m^2\, 2^{\frac{\omega\, n\, \log\log\log n}{8 \log\log n}}\right),$$

with $\omega, 2 \leq \omega < 3$ being the linear algebra constant. Currently, the best known value for ω is 2.3728639 [18, 30]. We also derive precise complexity bounds for BinaryError-LWE with $m = \mathcal{O}\left(n\right)$. Let

$$H_2(x) = -x \log_2(x) - (1 - x) \log_2(1 - x)$$

be the binary entropy, and assume that $m = C \cdot n$, with $C \geq 1$. Then we show that there is an algorithm solving BinaryError-LWE in

$$\mathcal{O}\left(n^2\, 2^{\omega\, n(1+\beta)\, H_2\left(\frac{\beta}{1+\beta}\right)}\right) \text{ (time) and } \mathcal{O}\left(n^2\, 2^{2n(1+\beta)\, H_2\left(\frac{\beta}{1+\beta}\right)}\right) \text{ (memory)}$$

with $\beta = \left(C - \frac{1}{2} - \sqrt{C(C-1)}\right)$.

More concretely, we can solve BinaryError-LWE in dimension n in time $n^2 \cdot 2^{0.344\,n}$ using a memory $n^2 \cdot 2^{0.289\,n}$ as soon as $m \geq 6.6\,n$. This is better than the best currently-known generic (exact) CVP solver [9] (time $2^{0.377\,n}$ using memory $2^{0.029\,n}$). As a consequence, our results provide a good picture of the hardness degradation of BinaryError-LWE for a number of samples ranging from $m = n\left(1 + \Omega\big(1/\log(n)\big)\right)$ to $m = \mathcal{O}\left(n^2\right)$ which addresses an open question by Micciancio and Peikert. Whilst our results does not contradict the hardness result of [35], they certainly rule out BinaryError-LWE for cryptographic applications requiring a quasi-linear number of samples.

1.3 Fröberg's Conjecture and Arora-Ge Equations

The results in this work depend crucially on two algebraic assumptions on the algebraic systems considered. The assumption is as follows for BinaryError-LWE:

Assumption 2. *Let* $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n\times m} \times \mathbb{Z}_q^m$ *be sampled according to* $L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}$, *and let* $P(x) = X(X-1)$. *We define:*

$$f_1 = P\big(c_1 - \sum_{j=1}^{n} s_j G_{j,1}\big) = 0,\ \ldots,\ f_m = P\big(c_m - \sum_{j=1}^{n} s_j G_{j,m}\big) = 0.$$

It holds that $\langle f_1, \ldots, f_m\rangle$ *is semi-regular (Definition 3, Section 2).*

For LWE, Assumption 1 is similar but we consider a polynomial $P(X) = X\prod_{i=1}^{C_{\mathrm{GB}}\cdot\sigma}(X+i)(X-i)$, where C_{GB} depends on the Gaussian distribution.

The motivation of these assumptions is that the complexity of computing the Gröbner basis of semi-regular sequences is well mastered. One can compute a Gröbner basis (Proposition 1, Section 2) of a semi-regular sequence $f_1, \ldots, f_m \in \mathbb{Z}_q[x_1, \ldots, x_n]$ in

$$\mathcal{O}\left(mD_{reg}\binom{n+D_{reg}}{D_{reg}}^{\omega}\right), \text{as } D_{reg} \to \infty,$$

where D_{reg} can be computed explicitly from the Hilbert polynomial given in (1). Hence, semi-regular sequences are a family of algebraic systems for which the complexity can be explicitly and easily computed [4–6, 8].

It is believed that semi-regular sequences capture rather precisely the behaviour of random system of equations. Our semi-regularity assumptions essentially states that our systems are not easier and not harder to solve than a random system of equations. We note that that in algebraic cryptanalysis efficient attacks are only reported in the case that the degree of regularity for the systems considered is much smaller than the degree of regularity of semi-regular systems (for instance, see the attack against HFE [27]). If Arora-Ge-style systems were easier than random systems this would imply that the analysis of Section 5 could be much improved and lead to progress towards a subexponential classical algorithm for solving GAPSVP.

Furthermore, to verify that Assumption 1 and 2 hold, we experimentally confirmed that they hold for reasonably large parameters in Section 7.1. For BinaryError-LWE, we have checked the assumption for n up to 53. In LWE, we can only check our assumption up to $n = 8$ due to the high degree of equations ($n = 8$ already requires 65GB of RAM).

We note that the fact that semi-regular sequences captures the behaviour of random sequences of polynomials is related to a famous conjecture in algebraic geometry known as *Fröberg's conjecture* [28] which states that semi-regular sequences form a dense subset among the set of all sequences. More precisely, the Fröberg conjecture states that a property – i.e. the rank of some linear map associated to Macaulay matrices (the matrices occurring in a Gröbner basis computation) is maximal – holds *generically*.

A property is said to be generic if it holds on a Zariski open subset Z_O when the characteristic of $\mathbb{K}$ is 0. In Zariski's topology, a close subset is defined defined as the vanishing set of algebraic equations. Hence, we can find a polynomial $h(\mathbf{a})$ in $\mathbb{Z}[\mathbf{a}]$ which does not depend on the field $\mathbb{K}$ such that $h(\mathbf{a}) \neq 0 \Rightarrow \mathbf{a} \in Z_O$. The main difficulty in Fröberg's conjecture is to prove that the polynomial h is not identically zero or that Z_O is not empty (see [29]).

To prove Fröberg's conjecture, it is then sufficient to find one explicit family of polynomials which can be proven semi-regular for any m and n. Proving Assumption 1 or 2 would provide such family and hence solve Fröberg's conjecture. Furthermore, any non-trivial partial results on our assumptions would lead to progress on the general Fröberg's conjecture.

Fröberg's conjecture is proved in very few cases: bi-variate sequences, and tri-variate sequences in characteristic 0 or over a sufficiently big finite field, sequences with the same number of polynomials (m) and same number of variables (n), $m = n + 1$ in characteristic 0, m polynomials of degree 2 with $n \leq 11$, and m polynomials of degree 3 with $n \leq 8$ [20, 28, 29].

We note that Fröberg and Hollman [29] already investigated the question of semi-regularity for powers of generic linear forms. In characteristic 0, [29, Lemma 2.1] proves that a sequence of $n + 1$ squares of generic linear forms in n variables is generically semi-regular. Assumption 1 and 2 state that powers of random affine forms behave as semi-regular sequences.

A technical difficulty for proving results towards Fröberg's conjecture is that over a finite field, the notion of Zariski open set is meaningless due to the field equations. However, the notion of genericity can be understood via the classical Schwartz-Zippel-DeMillo-Lipton [21, 40, 44].

Lemma 1 (Schwartz, Zippel, DeMillo, Lipton [21, 40, 44]). *Let $\mathbb{K}$ be a field and $P \in \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial. Select $r_1, \ldots, r_n$ uniformly at random from a finite subset $\mathcal{X}$ of $\mathbb{K}$. Then, the probability that $P(r_1, \ldots, r_n) = 0$ is less than $\deg(P)/|\mathcal{X}|$.*

The Fröberg conjecture, being related to rank defects of certain matrices, can be easily interpreted algebraically as the vanishing of some determinants. The difficult task is to prove that these determinants are non-zero. The determinant polynomials involved being of high degree, they can not be expanded symbolically. But, these determinant polynomials can be efficiently evaluated. So, a classical strategy to prove that such polynomials are non-zero is to find an explicit family for which the determinants can be proven to be non-zero. This allows to understand why a core difficulty of Fröberg's conjecture is to find only one explicit example.

In this paper, we report some progress towards proving Fröberg conjecture by investigating our assumptions. The first step to prove our assumptions is to show that the algebraic equations constructed are linearly independent. In Theorem 9 (Section 7.2), we prove that the equations $f_1, \ldots, f_m$ generated for BinaryError-LWE are linearly independent with high probability. Another necessary condition implies by Assumption 2 is that the sequence $f_1, \ldots, f_m$ is semigeneric (Definition 4), i.e. $\{x_i \cdot f_j\}_{1 \leq i \leq n}^{1 \leq j \leq n}$ spans a vector space of maximal dimension. For BinaryError-LWE, we prove that such algebraic independence at low degree holds with $m \leq n + \lfloor \frac{n-2}{2} \rfloor$ (Theorem 13, Section 7.2). This improves on a result of [29, Theorem 2.2] where Fröberg and Hollman proved that the squares of m generic linear forms are semigeneric as long as $m \leq n + 15$ and $n \leq 6$. In Section 7.2, we finally consider algebraic independence at higher degree. We have been able to prove that the assumption for BinaryError-LWE for $m = n + 1$ (Theorem 12, Section 7.2) for a field which is big enough.

2 Preliminaries

Algorithms for computing Gröbner bases are a fundamental tool in computational commutative algebra and one of the main tools for solving systems of non-linear polynomial equations over finite fields. Lazard [33] showed that computing the Gröbner basis for a system of homogeneous polynomials $f_1 \ldots, f_m$ is equivalent to perform Gaussian elimination on the *Macaulay matrices* $\mathcal{M}_{d,m}^{\text{acaulay}}$ for d, $\min\big(\deg(f_1), \ldots, \deg(f_m)\big) \leq d \leq D$ for some integer D. The Macaulay matrix $\mathcal{M}_{d,m}^{\text{acaulay}}(f_1 \ldots, f_m)$ is defined as the coefficient matrix of $(t_{i,j} \cdot f_i)$ where $1 \leq i \leq m$ and $t_{i,j}$ runs through all monomials of degree $d - \deg(f_i)$. It can be shown that Macaulay matrices up to degree d can be used to compute a partial Gröbner basis, called d-Gröbner basis. For d big enough, a d-Gröbner basis is a Gröbner basis and we have the following result:

Theorem 3 ([33]). *Let q be a prime and let $\mathbf{f} = (f_1, \ldots, f_m) \in (\mathbb{Z}_q[x_1, \ldots, x_n])^m$ be homogeneous polynomials and $\prec$ be a monomial ordering. There exists a positive integer D for which Gaussian elimination on all $\mathcal{M}_{d,m}^{\text{acaulay}}(f_1, \ldots, f_m)$ matrices for $d, 1 \leq d \leq D$ computes a Gröbner basis of $\langle f_1, \ldots, f_m \rangle$ w.r.t. to $\prec$.*

It follows that the complexity of computing a Gröbner basis is bounded by the complexity of performing Gaussian elimination on the Macaulay matrices up to some degree D. In general, computing the maximum degree occurring in a Gröbner computation is a difficult problem. However, this degree is precisely known for a specific family of polynomial systems [4–6, 8].

Definition 3 (Semi-regular Sequence [6, 8]). *Let $m \geq n$, and $f_1, \ldots, f_m \in \mathbb{Z}_q[x_1, \ldots, x_n]$ be homogeneous polynomials of degrees $d_1, \ldots, d_m$ respectively and $\mathcal{I}$ the ideal generated by these polynomials. The system is said to be a* semi-regular sequence *if the Hilbert polynomial [19] associated to $\mathcal{I}$ w.r.t. the grevlex order is:*

$$\mathrm{HP}(z) = \left[\frac{\prod_{i=1}^{m}(1 - z^{d_i})}{(1 - z)^n}\right]_+, \tag{1}$$

with $[S]_+$ being the polynomial obtained by truncating the series S before the index of its first non-positive coefficient. We shall call degree of regularity of a semi-regular sequence the quantity :

$$1 + \deg\big(\mathrm{HP}(z)\big).$$

This degree of regularity is the degree D involved in Theorem 3 for a semi-regular sequence.
Finally, let $f_1, \ldots, f_m \in \mathbb{Z}_q[x_1, \ldots, x_n]$ be a sequence of affine polynomials. We denote by $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}} \in \mathbb{Z}_q[x_1, \ldots, x_n]$ the corresponding homogeneous components of highest degree. We shall say that $f_1, \ldots, f_m$ is semi-regular if the sequence $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ is semi-regular.

In this paper, we will use the assumptions that the algebraic equations considered in our approaches behave as semi-regular sequences. Throughout this paper, we use intensively the following complexity results about semi-regular sequences.

Proposition 1 (adapted from [7]). *Let $\mathbf{f} = (f_1, \ldots, f_m) \in (\mathbb{Z}_q[x_1, \ldots, x_n])^m$ be affine polynomials with $m > nn$. If $f_1, \ldots, f_m$ is semi-regular, then the number of operations in $\mathbb{Z}_q$ required to compute a Gröbner basis for any admissible order is bounded by:*

$$\mathcal{O}\left(mD_{reg}\binom{n + D_{reg}}{D_{reg}}^{\omega}\right), \text{as } D_{reg} \to \infty, \tag{2}$$

where $2 \leq \omega < 3$ is the linear algebra constant and D_{reg} is the degree of regularity of $\langle f_1, \ldots, f_m \rangle$.

Note that (2) is actually the cost of computing a Gröbner basis with a grevlex ordering. To change the ordering, we have also to use FGLM [26] whose complexity is polynomial in the degree of the ideals. For semi-regular sequences, this is equal to the number of solutions counted with multiplicities. This part can be ignored for $m > n$ (see for instance [11]).

Note that the complexity bound (2) is rather pessimistic as we do not take into account the particular structure of the matrices involved. Typically, the Macaulay matrices considered have huge rank defects which correspond to useless computations. More recent algorithms such as F_4 and F_5 [23, 24]) are actually trying to take advantage as much as possible of the structure of Macaulay matrices. This leads to considerable speed-up for practical applications [25, 27] and in theory as well [7]. However, to simplify the asymptotical analysis we will consider Lazard's algorithm (Theorem 3) which performs row reductions on Macaulay matrices.

Finally, the following classical approximation of the binomial coefficient due to Stirling will be useful to prove some of our results below.

Lemma 2. *Let $H_2(x) = -x\log_2(x) - (1 - x)\log_2(1 - x)$ be the binary entropy. For n and k large enough, we have $\log_2\binom{n}{k} \approx n\,H_2\big(\frac{k}{n}\big)$.*

Similarly, we have the following lemma.

Lemma 3.

$$\log\binom{n + D}{D} \approx \begin{cases} D\log(n/D), & \text{if } D \in o(n), \\ n\log(D/n), & \text{if } n \in o(D). \end{cases}$$

This follows easily from Stirling's expansion of the binomial.

3 Arora & Ge

The Arora & Ge algorithm proceeds by generating a non-linear noise-free system of equations from LWE samples. In the construction of such a system, it makes use of the following well-known fact about the Gaussian distribution.

Lemma 4. *Let χ denote the Gaussian distribution with standard deviation σ. Furthermore, for $x > 0$, we denote $Q(x) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$. Then, for all $C > 0$, it holds that:*

$$\Pr[e \xleftarrow{\$} \chi : |e| > C \cdot \sigma] \approx 2 \times Q(C) \leq \frac{2}{C\sqrt{2\pi}} e^{-C^2/2} \in e^{\mathcal{O}\left(-C^2\right)}.$$

That is, for a $C > 0$, elements sampled from a Gaussian distribution take only values on the interval $[-C \cdot \sigma, \dots, C \cdot \sigma]$ of $\mathbb{Z}_q$ with probability at least $1 - e^{\mathcal{O}\left(-C^2\right)}$ if we represent elements in $\mathbb{Z}_q$ as integers in $[-\lfloor \frac{q}{2} \rfloor, \dots, \lfloor \frac{q}{2} \rfloor]$. Moreover, if $e \xleftarrow{\$} \chi$ then $P(e) = 0$ for

$$P(X) = X \prod_{i=1}^{C \cdot \sigma} (X + i)(X - i),$$

with probability at least $1 - e^{\mathcal{O}\left(-C^2\right)}$. Clearly P is of degree $2\, C \cdot \sigma + 1 \in \mathcal{O}\left(C \cdot \sigma\right)$.
It follows that if $(\mathbf{a}_i, b_i) = (\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$, and $e_i \xleftarrow{\$} \chi$, then

$$P\Big(-b + \sum_{j=1}^{n} (\mathbf{a}_i)_{(j)} x_j\Big) = 0, \tag{3}$$

with probability at least $1 - e^{\mathcal{O}\left(-C^2\right)}$. Each sample $(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i) = (\mathbf{a}_i, b_i) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$ allows to generate a non-linear equation of degree $2\, C \cdot \sigma + 1$ in the n components of the secret $\mathbf{s}$ which holds with probability $1 - e^{\mathcal{O}\left(-C^2\right)}$.
The Arora & Ge algorithm then proceeds by generating M_{AG} independent equations of the form (3), to be then solved by linearisation. However, a value for C – denoted by C_{AG} – occurring in Lemma 4 has to be chosen sufficiently large so that all errors e_i lie with high probability in the interval $[-C_{\mathrm{AG}} \cdot \sigma, \dots, C_{\mathrm{AG}} \cdot \sigma] \subseteq \mathbb{Z}_q$, i.e. such that the secret $\mathbf{s}$ is indeed a common solution of the M_{AG} equations. To this end, let $\mathcal{S}_{\mathrm{AG}}$ be the system of equations generated from M_{AG} equations as in (3) and bound the probability of failure by the union bound:

$$p_f = M_{\mathrm{AG}} \times \Pr[e \xleftarrow{\$} \chi_{\alpha,q} : |e| > C_{\mathrm{AG}} \cdot \sigma] \leq \frac{M_{\mathrm{AG}}}{e^{\mathcal{O}\left(C_{\mathrm{AG}}^2\right)}}.$$

Hence, p_f is the probability of failure of the Arora & Ge algorithm, i.e. it is an upper bound on the probability that the secret $\mathbf{s} \in \mathbb{Z}_q^n$ is not a solution to $\mathcal{S}_{\mathrm{AG}}$. Let also $D_{\mathrm{AG}} = 2\, C_{\mathrm{AG}}\, \sigma + 1$ be the degree of the equations occuring in $\mathcal{S}_{\mathrm{AG}}$. It is shown in [3] (cf. Section 4) that taking $C_{\mathrm{AG}} \in \tilde{\mathcal{O}}(\sigma)$ allows us to make the probability of failure negligible.
In summary, the Arora & Ge algorithm reduces solving LWE to linearisation of a system of M_{AG} equations of degree $D_{\mathrm{AG}} = 2\, C_{\mathrm{AG}}\, \sigma + 1 \in \tilde{\mathcal{O}}(\sigma^2)$. In particular, the following theorem holds:

Theorem 4 ([3]). *Let $D_{\mathrm{AG}} < q$. The system obtained by linearizing $M_{\mathrm{AG}} = \mathcal{O}\left(\binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}^{\omega} \sigma\, q \log q\right) = n^{\mathcal{O}(D_{\mathrm{AG}})} = 2^{\tilde{\mathcal{O}}(D_{\mathrm{AG}})}$ equations as in* (3) *has at most one solution with high probability.*

Note that $\mathcal{O}\left(\binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}\right)$ equations is sufficient to linearise the system. The extra factor $\sigma\, q \log q$ allows to prove that the linearised system has at most one solution with high probability [3]. The overall complexity of the Arora-Ge algorithm is the cost of performing Gaussian elimination on a matrix of size $M_{\mathrm{AG}} \times \binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}$, i.e.

$$\mathcal{C}_{\mathrm{AG}}^{\mathrm{plx}} = n^{\mathcal{O}(D_{\mathrm{AG}})} = 2^{\tilde{\mathcal{O}}(\sigma^2)} = 2^{\tilde{\mathcal{O}}(n^{2\epsilon})}.$$

Note also that, if we have the standard deviation $\sigma = n^{\epsilon}$, then the algorithm requires $2^{\tilde{\mathcal{O}}(n^{2\epsilon})}$ LWE samples for performing the linearisation step. It follows that the Arora & Ge algorithm is subexponential when $\epsilon < 1/2$.

4 Refined Analysis of Arora & Ge

However, the above analysis and the analysis in [3] leaves room for improvements as it hides not only constants *in the exponent* but also logarithm factors. In this section we address this issue so we can then compare potential improvements due to the application of Gröbner bases in Section 5.

As established in the previous section, the overall complexity of solving an LWE instance with the Arora & Ge algorithm is that of executing Gaussian elimination on a matrix of size $M_{\mathrm{AG}} \times \binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}$. Gaussian elimination on an $m \times n$ matrix of rank r has complexity $\mathcal{O}\left(mnr^{\omega-2}\right)$ [32]. The Arora & Ge algorithm hence has a complexity of

$$\mathcal{O}\left(M_{\mathrm{AG}} \cdot \binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}^{\omega-1}\right) = \mathcal{O}\left(M_{\mathrm{AG}} \cdot \binom{n+2\,C_{\mathrm{AG}}\,\sigma+1}{2\,C_{\mathrm{AG}}\,\sigma+1}^{\omega-1}\right).$$

Hence, we need to bound C_{AG}.

Lemma 5. *Let $n, q, \sigma = \alpha \cdot q$ be parameters of an $\mathrm{LWE}_{\chi_{\alpha,q}}$ instance where $q = \mathrm{poly}(n)$. Let $p'_f \in [0,1]$ be a constant upper bound on the probability of failure and*

$$C_{\mathrm{AG}} \leq 2\,\sigma \log n + a^{1/2} \approx 4\,\sigma \log n,$$

with $a = 4(\sigma \log n)^2 + 2\log(\sigma\, q \log q) - 2\log p'_f + 2\log n$. Finally, let also $D_{\mathrm{AG}} = 2\,C_{\mathrm{AG}}\,\sigma + 1$. Then, the system obtained by linearizing $\binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}\sigma\, q \log q$ equations of degree as in (3) *is correct, i.e. the secret is a zero of all the polynomials, with probability bigger than $1 - p'_f$.*

Proof. The probability of failure is upper bounded by by:

$$p_f = M_{\mathrm{AG}} \times \Pr[e \xleftarrow{\$} \chi_{\alpha,q} : |e| > C_{\mathrm{AG}} \cdot \sigma] \approx \frac{2\binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}\sigma\, q \log q}{\sqrt{2\pi}\, C_{\mathrm{AG}}\, e^{C_{\mathrm{AG}}^2/2}} < \frac{\binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}\sigma\, q \log q}{C_{\mathrm{AG}} \cdot e^{C_{\mathrm{AG}}^2/2}}.$$

We bound $\binom{n+D_{\mathrm{AG}}}{D_{\mathrm{AG}}}$ by $n^{D_{\mathrm{AG}}}$. While this approximation is rather loose, it allows to simplify our expression sufficiently to recover a closed form of the complexity. With this simplification, our goal is to find C_{AG} such that:

$$0 \leq \frac{n^{D_{\mathrm{AG}}} \cdot \sigma\, q \log q}{C_{\mathrm{AG}} \cdot e^{C_{\mathrm{AG}}^2/2}} = p'_f \leq 1.$$

That is:

$$\frac{e^{\log\left(\sigma\, q \log q\right)} e^{(2\,C_{\mathrm{AG}}\,\sigma+1)\log n}}{e^{\log p'_f} \cdot \left(e^{\log C_{\mathrm{AG}}} \cdot e^{C_{\mathrm{AG}}^2/2}\right)} = 1.$$

Namely, we want to solve

$$\begin{aligned} 0 &= \log(\sigma\, q \log q) + 2\,C_{\mathrm{AG}}\,\sigma \log n + \log n - \log C_{\mathrm{AG}} - \log p'_f - C_{\mathrm{AG}}^2/2 \\ &> \log(\sigma\, q \log q) + 2\,C_{\mathrm{AG}}\,\sigma \log n + \log n - \log p'_f - C_{\mathrm{AG}}^2/2 \end{aligned}$$

for C_{AG}. The last line has 2 roots:

$$[R_1 = 2\,\sigma \cdot \log(n) - a^{1/2}, R_2 = 2\,\sigma \cdot \log(n) + a^{1/2}],$$

with $a = 4(\sigma \log n)^2 + 2\log(\sigma\, q \log q) - 2\log p'_f + 2\log n$.

Note that $a^{1/2} > 2\,\sigma \log(n)$ and hence $R_1 < 0$. Thus, the smallest possible value for C_{AG} is R_2. Now, assume that $q \in \mathrm{poly(n)}$, i.e. $q \approx n^c$. Also, recall that p'_f is a constant. Thus, for n big enough:

$$\begin{aligned} a &= 4(\sigma \log n)^2 + 2\log(\sigma\, q \log q) - 2\log p'_f + 2\log n \\ &= 4(\sigma \log n)^2 + 2c\log(\sigma\, n\, c \log n) - 2\log p'_f + 2\log n \\ &\approx 4(\sigma \log n)^2. \end{aligned}$$

So, we have

$$C_{\mathrm{AG}} \leq 2\,\sigma \log n + a^{1/2} \approx 4\,\sigma \log n.$$

□

We hence arrive at the following theorem:

Theorem 5. *Let $n, q, \sigma = \alpha \cdot q$ be parameters of an $\mathrm{LWE}_{\chi_{\alpha,q}}$ instance. If $D_{\mathrm{AG}}(= 8\,\sigma^2 \log n + 1) \in o(n)$ then the Arora & Ge algorithm solves the computational* LWE *problem in time complexity*

$$\mathcal{O}\left(2^{\,\omega \cdot D_{\mathrm{AG}} \log \frac{n}{D_{\mathrm{AG}}}} \cdot \sigma\, q \log q\right) = \mathcal{O}\left(2^{\,8\,\omega\,\sigma^2 \log n (\log n - \log(8\,\sigma^2 \log n))} \cdot \mathrm{poly}(n)\right)$$

and memory complexity

$$\mathcal{O}\left(2^{\,2 \cdot D_{\mathrm{AG}} \log \frac{n}{D_{\mathrm{AG}}}} \cdot \sigma\, q \log q\right) = \mathcal{O}\left(2^{\,16\,\sigma^2 \log n (\log n - \log(8\,\sigma^2 \log n))} \cdot \mathrm{poly}(n)\right).$$

If $n \in o(\sigma^2 \log(n))$ then the Arora & Ge algorithm solves the computational LWE *problem in time complexity*

$$\mathcal{O}\left(2^{\,\omega \cdot n \log \frac{D_{\mathrm{AG}}}{n}} \cdot \sigma\, q \log q\right) = \mathcal{O}\left(2^{\,\omega\, n \log(8\,\sigma^2 \log n) - n \log n} \cdot \mathrm{poly}(n)\right)$$

and memory complexity

$$\mathcal{O}\left(2^{\,2n \log \frac{D_{\mathrm{AG}}}{n}} \cdot \sigma\, q \log q\right) = \mathcal{O}\left(2^{\,2n \log(8\,\sigma^2 \log n) - n \log n} \cdot \mathrm{poly}(n)\right).$$

Proof. The result follows immediately from plugging the Lemmata 3 and 5 into Theorem 4.

This establishes a baseline to compare any potential improvements due to the application of Gröbner basis algorithms with.

5 Solving LWE with Gröbner Bases

We are now ready to address the question if the complexity of the basic Arora & Ge algorithm can be improved by using Gröbner bases instead of linearisation. Let the notation be as in Section 4. The motivation for this section is that the constant C_{AG} (and hence the degree of the equations) depends on the number of equations M_{AG} considered. Hence, on the one hand, we may lower the number of equations to a value smaller than $\tilde{\mathcal{O}}(n^{2\,\epsilon})$ whilst keeping the probability of failure small enough. On the other hand, this means that the cost of solving the resulting system will grow compared to that of linearisation. The optimisation target is then to find a tradeoff allowing to improve upon linearisation.

We assume that $\sigma = n^{\epsilon}$, with $\epsilon, 0 \leq \epsilon \leq 1$. Let also $\theta, 0 \leq \theta \leq \epsilon \leq 1$. We consider a number of samples of the following form:

$$M_{\mathrm{GB}} = e^{\gamma_\theta}, \text{ with } \gamma_\theta = n^{2 \cdot (\epsilon - \theta)}.$$

Note that $\theta = 0$ corresponds up to polylog factors to the basic Arora-Ge approach. To explain the rational for the choosing this form for M_{GB} the number of samples we state below a simple lemma which relates the number of samples considered to the degree of the multivariate equations:

Lemma 6. *Let $(\mathbf{a}_1, b_1), \ldots, (\mathbf{a}_m, b_m)$ be elements of $\mathbb{F}_q^n \times \mathbb{F}_q$ sampled according to $\mathrm{LWE}_{\chi_{\alpha,q}}$. If $C = \sqrt{2\log(m)}$ then, the equations generated as in* (3) *vanish with probability at least:*

$$p_{\mathrm{g}} = 1 - \sqrt{\frac{1}{\pi \cdot \log(m)}}.$$

Proof. By Lemma 4, the probability of failure verifies

$$\leq \frac{2\,m}{C\sqrt{2\pi}}e^{-C^2/2} = \frac{2m}{\sqrt{4\pi\cdot\log(m)}}e^{-\left(\sqrt{2\log(m)}\right)^2/2} = \frac{m}{\sqrt{\pi\cdot\log(m)}}e^{-\log(m)} = \frac{1}{\sqrt{\pi\cdot\log(m)}}.$$

From this the probably of success $p_{\mathrm{g}} \geq 1 - p_{\mathrm{f}}$ follows. □

Remark 1. If $m \in \mathcal{O}(n)$ then it holds that $p_{\mathrm{g}} \in 1 - o(1)$.

We can then deduce the degree D_{GB} required for $M_{\mathrm{GB}} = e^{\gamma_\theta}$ equations. From Lemma 6, we have to fix $C_{\mathrm{GB}} = \sqrt{2\cdot\log(M_{\mathrm{GB}})} = \sqrt{2\cdot\gamma_\theta}$. Thus:

$$D_{\mathrm{GB}} = 2\sqrt{2\cdot\log(M_{\mathrm{GB}})}\cdot\sigma + 1 \in \mathcal{O}\left(\sqrt{\log(M_{\mathrm{GB}})}\cdot\sigma\right) = \mathcal{O}\left(\sqrt{\gamma_\theta}\cdot\sigma\right) = \mathcal{O}\left(n^{2\epsilon-\theta}\right) = \mathcal{O}\left(\gamma_\theta\cdot n^\theta\right).$$

But to ease the analysis below, we further simplify D_{GB} to:

$$D_{\mathrm{GB}} \approx \gamma_\theta\cdot n^\theta = \log(M_{\mathrm{GB}})\cdot n^\theta.$$

Furthermore, we restrict our attention to the case $\sigma = \sqrt{n}/\sqrt{2\pi}$. Now, in order to analyse the complexity of the Gröbner basis computation, we need to make the following assumption about the structure of the generated equations:

Assumption 1 *Let $(\mathbf{a}_1, b_1), \ldots, (\mathbf{a}_{M_{\mathrm{GB}}}, b_{M_{\mathrm{GB}}})$ be elements of $\mathbb{F}_q^n \times \mathbb{F}_q$ sampled according to $\mathrm{LWE}_{\chi_{\alpha,q}}$. Let $P(X) = X\prod_{i=1}^{C_{\mathrm{GB}}\cdot\sigma}(X+i)(X-i)$. We define:*

$$f_i = P\big(-b + \sum_{j=1}^{n}(\mathbf{a}_i)_{(j)}x_j\big) = 0, \forall i, 1 \leq i \leq M_{\mathrm{GB}}. \tag{4}$$

Then, $\langle f_1, \ldots, f_m\rangle$ is semi-regular.

We justify this assumption in Section 7.

From D_{GB} and M_{GB} we now need to establish the degree of regularity. Whilst there are classical results on the degree of regularity in the literature, these do not apply here. In particular, we need to consider systems of equations having a non-constant degree. For brevity and due to the fact a detailed analysis is beyond the scope of this paper, we only provide the general statement which allows to derive the result below in Appendix A (Proposition 3).

Lemma 7. *Let $A \geq 1$, and $f_1, \ldots, f_m \in \mathbb{Z}_q[x_1, \ldots, x_n]$ be semi-regular polynomials of degree $\frac{n}{A}$, and D_{reg} be the degree of regularity of these polynomials. If $m = e^{\frac{\pi\cdot n}{4\cdot A^2}}$, then*[3] *it holds that D_{reg} behaves asymptotically as*

$$C_A\cdot n, \textit{where } C_A \textit{ is a constant which depends on } A.$$

The constant C_A in the Lemma can be computed explicitly for any value of A as explained in Proposition 3. For $A = 1$, we get in particular that $D_{\mathrm{reg}} = 1.41\cdot n$, for n big enough. Putting all these results together we can now derive the complexity of solving LWE using a Gröbner basis algorithm.

Theorem 6. *Let $A \geq 1, \omega$, C_A be as defined in Lemma 7, $\omega, 2 \leq \omega < 3$, be the linear algebra constant, and $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$. Let $(\mathbf{a}_i, b_i)_{i\geq 1}$ be elements of $\mathbb{F}_q^n \times \mathbb{F}_q$ sampled according to $\mathrm{LWE}_{\chi_{\alpha,q}}$ with a standard deviation $\sigma = \frac{\sqrt{n}}{\sqrt{2\pi}}$ and $A \geq 1$. There is an algorithm recovering the secret in*

$$\mathcal{O}\left(2^{n\left(\omega\,(1+C_A)\,H_2\left(\frac{C_A}{1+C_A}\right)+\frac{\pi\cdot\log_2(e)}{4\cdot A^2}\right)}\right) \textit{(time) and} \quad \mathcal{O}\left(2^{n\left(2\,(1+C_A)\,H_2\left(\frac{C_A}{1+C_A}\right)+\frac{\pi\cdot\log_2(e)}{4\cdot A^2}\right)}\right) \textit{(memory)}, \tag{5}$$

The algorithm has success probability $\geq 1 - \frac{2}{\pi\sqrt{n}} = 1 - o(1)$.

[3] We will see that the constant $\pi/4$ in the exponent allows to adjust the success probability in Theorem 6.

Proof. Let $M_{\text{GB}} = e^{\frac{\pi \cdot n}{4 \cdot A^2}}$ and $D_{\text{GB}} = n/A$. We generate a system of M_{GB} non-linear equations of degree D_{GB} as (4). Under our regularity assumption 1, the complexity of computing a Gröbner basis for this system is:

$$\mathcal{O}\left(n\, e^{\frac{\pi \cdot n}{4 \cdot A^2}} \binom{n(1+C_A)}{C_A\, n}^{\omega}\right), \tag{6}$$

Combining this with Lemma 2 gives the complexity. By Lemma 4, the probability of failure verifies is $\leq \frac{2\,m}{C\sqrt{2\pi}} e^{-C^2/2}$. In our case, $C \approx \frac{\sqrt{2\pi n}}{2}$ which gives a failure probability $\leq \frac{2}{\pi\sqrt{n}}$. □

Note that the complexities in Theorem 6 are minimized by taking a constant $A = 1$. So, we get a complexity of $\mathcal{O}\left(2^{n\left(2.35\,\omega + 1.13\right)}\right)$ (time) and $\mathcal{O}\left(2^{5.85\,n}\right)$ (memory). As a consequence we have that using Gröbner bases yields an exponential speed-up (for $\sigma = \sqrt{n}/\sqrt{2\pi}$ and under Assumption 1) over the basic Arora-Ge approach (cf. Theorem 5). On the other hand, our results also give a negative answer to the natural question whether the combination of Gröner basis techniques with the Arora-Ge modelling can yield a subexponential algorithm for the LWE problem. From Lemma 7, one can notice that there is no choice of A (constant, $\log n, \dots$) which makes the number of samples sub-exponential whilst keeping the degree of regularity sub-linear.

6 Solving LWE with Bounded Errors

We now turn to studying the complexity of solving BinaryError-LWE using the modelling of Arora & Ge [3] and applying a Gröbner basis algorithm for solving the resulting system of equations. As discussed earlier, BinaryError-LWE is an LWE instance over $\mathbb{Z}_q$ but with errors restricted to the binary field, as in [35]. Generating noise-free non-linear equations is straightforward in this case: if $\mathbf{e} = (e_1, \dots, e_m) \in \{0,1\}^m$ and $P(X) = X(X-1)$, then we have $P(e_i) = 0$, for all $i, 1 \leq i \leq m$.

Now, let $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n \times m} \times \mathbb{Z}_q^m$ be sampled according to $L^{(n)}_{\mathbf{s}, \mathcal{U}(\mathbb{F}_2)}$. Then

$$e_i = c_i - \sum_{j=1}^{n} s_j G_{j,i}, \text{ for } 1 \leq i \leq m.$$

It follows that the secret $\mathbf{s} \in \mathbb{Z}_q^n$ is a solution of the following algebraic system:

$$f_1 = P\big(c_1 - \sum_{j=1}^{n} s_j G_{j,1}\big) = 0, \;\dots,\; f_m = P\big(c_n - \sum_{j=1}^{n} s_j G_{j,n}\big) = 0. \tag{7}$$

This is an algebraic system of m quadratic equations in $\mathbb{Z}_q[x_1, \dots, x_n]$. As already pointed out in [3,35], this system can be solved using linearization if $m = \mathcal{O}\left(n^2\right)$. However the case $m < \mathcal{O}\left(n^2\right)$ remained untreated. Here, we address this problem of evaluating the complexity of solving the algebraic system (7) with an arbitrary number m of equations.

As discussed in Section 2, answering this question in general is hard. But for one particular class of systems, namely *semi-regular systems of equations*, this question has in fact been settled. In particular, the following result [4–6, 8] allows us to classify the complexity of solving polynomial systems with respect to the number of equations.

Theorem 7. *(i) Let $m = C \cdot n$, with $C > 1$, and let $f_1, \dots, f_m \in \mathbb{Z}_q[x_1, \dots, x_n]$ be a semi-regular system of equations. The degree of regularity of $f_1, \dots, f_m$ behaves asymptotically as*

$$\begin{aligned} D_{\text{reg}} = \left(C - \frac{1}{2} - \sqrt{C(C-1)}\right) n - \frac{a_1}{2\big(C(C-1)\big)^{1/6}}\, n^{\frac{1}{3}} \\ - \left(2 - \frac{2C-1}{4\big(C(C-1)\big)^{1/2}}\right) + \mathcal{O}\left(\frac{1}{n^{\frac{1}{3}}}\right), \end{aligned}$$

where $a_1 \approx 2.3381$ is the largest zero of the classical Airy function.

(ii) *Let* $m = n \cdot \log^{1/\epsilon}(n)$*, for any constant* $\epsilon > 0$*, or* $m = n \log\log n$*. The degree of regularity of* $f_1, \ldots, f_m$ *behaves asymptotically as:*

$$D_{\text{reg}} = \frac{n^2}{8m}\left(1 + o(1)\right).$$

A proof of $i)$ can be found, for instance, in [8, Theorem 1]. A proof similar to the case of ii) can be found in [5]. However, there is slight difference between [5] (binary fields) and our case (generic prime fields). In Appendix A we briefly sketch a proof for $ii)$ of Theorem 7.

Hence, under the assumption that the system (7) behaves like a semi-regular system of equations, Theorem 7 allows one to compute an upper bound on the complexity for solving it with Gröbner basis algorithms. While no proof currently exists that would demonstrate that the system (7) does indeed behave like a semi-regular system, we make the following assumption based on the discussion in Section 7.

Assumption 2 *Let* $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n\times m} \times \mathbb{Z}_q^m$ *be sampled according to* $L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}$*, and let* $P(x) = X(X-1)$*. We define:*

$$f_1 = P\big(c_1 - \sum_{j=1}^{n} s_j G_{j,1}\big) = 0,\ \ldots, f_m = P\big(c_n - \sum_{j=1}^{n} s_j G_{j,m}\big) = 0. \tag{8}$$

It holds that $\langle f_1, \ldots, f_m \rangle$ *is semi-regular.*

Based on Assumption 2, we can now state the main result of this section. We classify the hardness of our approach with various number of samples. The first one corresponds to the number of equations required in the security proof [35, Theorem 1.2]. We then consider a slightly larger number of equations than what is required in the security proof, i.e. $m = 2n$ equations. In addition we give the results for a quasi-linear number of equations.

Theorem 8. *Let* $\omega, 2 \le \omega < 3$*, be the linear algebra constant, and* $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$*. Under Assumption 2, we have the following.*

(i) *If* $m = n\left(1 + \frac{1}{\log(n)}\right)$*, then there is an algorithm solving* BinaryError-LWE *with a time complexity:*

$$\mathcal{O}\left(n^2\, 2^{1.37\,\omega\, n}\right). \tag{9}$$

(ii) *If* $m = 2 \cdot n$*, then there is an algorithm solving* BinaryError-LWE *with a time complexity*

$$\mathcal{O}\left(n^2\, 2^{0.43\,\omega\, n}\right). \tag{10}$$

(iii) *More generally, if* $m = C \cdot n$*, with* $C > 1$*, there is an algorithm solving* BinaryError-LWE *in:*

$$\mathcal{O}\left(n^2\, 2^{\omega\, n(1+\beta)\, H_2\left(\frac{\beta}{1+\beta}\right)}\right) \text{ (time) and } \quad \mathcal{O}\left(n^2\, 2^{2n(1+\beta)\, H_2\left(\frac{\beta}{1+\beta}\right)}\right) \text{ (memory)}, \tag{11}$$

with $\beta = \left(C - \frac{1}{2} - \sqrt{C(C-1)}\right)$*.*

(iv) *If* $m = \mathcal{O}\left(n \log\log n\right)$*, then there is a subexponential algorithm solving* BinaryError-LWE *with complexity*

$$\mathcal{O}\left(m^2\, 2^{\frac{\omega\, n\, \log\log\log n}{8\log\log n}}\right) \text{ (time)}, \quad \mathcal{O}\left(m^2\, 2^{\frac{2n\, \log\log\log n}{8\log\log n}}\right) \text{ (memory)}. \tag{12}$$

(v) *Finally, if* $m = n \cdot \log^{1/\epsilon}(n)$*, for any* $\epsilon > 0$*, then there is a subexponential algorithm solving* BinaryError-LWE *whose complexity is:*

$$\mathcal{O}\left(m^2\, 2^{\frac{\omega\, n\, \log\left(\log^{1/\epsilon}(n)\right)}{8\log^{1/\epsilon}(n)}}\right) \text{ (time)}, \quad \mathcal{O}\left(m^2\, 2^{\frac{2n\, \log\left(\log^{1/\epsilon}(n)\right)}{8\log^{1/\epsilon}(n)}}\right) \text{ (memory)}. \tag{13}$$

Proof. As explained Section 2, the complexity of computing a Gröbner basis is:

$$\mathcal{O}\left(m\, D_{\mathrm{reg}} \binom{n + D_{\mathrm{reg}}}{D_{\mathrm{reg}}}^{\omega}\right) \text{(time)},\quad \mathcal{O}\left(m\, D_{\mathrm{reg}} \binom{n + D_{\mathrm{reg}}}{D_{\mathrm{reg}}}^{2}\right) \text{(memory)}. \tag{14}$$

Under our semi-regularity assumption, Theorem 7 gives: $D_{reg} = 0.5 \cdot n + o(n)$ for $m = n\left(1 + \frac{1}{\log(n)}\right)$, $D_{\mathrm{reg}} = 0.08 \cdot n + o(n)$ for $m = 2 \cdot n$ and more generally $D_{\mathrm{reg}} = \left(C - \frac{1}{2} - \sqrt{C(C-1)}\right) n + o(n)$ for $m = C \cdot n$, for any constant $C > 1$. In these cases, the binomial coefficient in (14) has the following form:

$$\binom{\alpha \cdot n}{\beta \cdot n}, \text{ for some } \alpha > \beta > 0.$$

We obtain (9) – (11) by taking $\beta = \left(C - \frac{1}{2} - \sqrt{C(C-1)}\right)$ by applying Lemma 2. For (12) and (13), we combine Lemma 3 and Theorem 7. □

It follows from Theorem 8 that we can solve BinaryError-LWE in dimension n in time $n^2\, 2^{0.344\, n}$ using memory $n^2\, 2^{0.289\, n}$ as soon as as soon as $m \geq 6.6\, n$. We note that this is better than the best currently-known generic (exact) CVP solver [9]. Theorem 8 also provides a good picture of the hardness degradation of BinaryError-LWE for the number of available samples ranging from $m = n\left(1 + \Omega\big(1/\log(n)\big)\right)$, a case for which BinaryError-LWE is as hard as solving some lattice problem in the worst case (as shown in [35]) to $m = \mathcal{O}\left(n^2\right)$, the case for which it can be solved in polynomial-time. In view of items *(iv)-(v)* of Theorem 8, we conclude that BinaryError-LWE should be ruled out for cryptographic applications that require a quasi-linear number of samples.

7 Justifications of our Assumptions

The results in this work depend crucially on two assumptions, namely that all systems of equations occurring in this work are semi-regular. While no proof currently exists that would demonstrate either Assumption 1 or 2, we argue below why we believe these assumptions do indeed hold. As already mentioned, we note that each semi-regularity assumption essentially states that our systems are not easier and not harder to solve than a random system of equations. If Arora-Ge-style systems were easier than random systems this would imply that the analysis of Section 5 could be much improved and lead to progress towards a subexponential classical algorithm for solving Bounded Distance Decoding. Furthermore, this subexponential classical algorithm would work despite ignoring the particular error distribution and would consist of applying a generic Gröbner basis algorithm. We consider this case to be unlikely. Furthermore, we note that Arora & Ge essentially showed in [3] that Assumption 1 holds for $m = 2^{\tilde{\mathcal{O}}(\sigma^2)}$ (Theorem 4). Following [3], we prove in Section 7.2 that Assumption 2 holds for $m = \mathcal{O}\left(n^2\right)$. We also prove in 7.2 several partial results regarding our assumptions. Before this, we report in Section 7.1 on experimental results confirming that our assumptions hold for reasonably large parameters.

7.1 Experimental Verification

We experimentally confirmed that our assumptions hold for reasonably large parameters. Namely, we verified Assumption 1 for systems up to $n = 8$ variables. In particular we computed for $n = 8$ and $m = 256$, $\alpha \cdot q/\sqrt{2\pi} = 1$ using MAGMA [12] (V2.20-4) and Sage [41]. The generated system of equations has degree 9 and the degree of semi-regularity is 13. The highest degree reached was indeed degree 13.

While $n \leq 8$ might seem rather small, we point out that it is the last n for which we can reasonably expect to run experiments on current hardware. Theorem 6 bounds the memory complexity by $\mathcal{O}\left(2^{5.85n}\right)$. We note that for $n = 8$, our computation required 65GB of memory and 68 hours to complete. Hence, we would require about $2^{5.85} \cdot 65$GB of memory to perform this computation for $n = 9$ which is beyond our reach.

Also, to verify Assumption 2, we have generated systems as in (8) with $m = \lfloor n \log_2(n) \rfloor$ equations and $n \in \{5, \ldots, 45\}$ variables. We take q as the next prime larger than n (or n^2 in some instances). We then computed a Gröbner basis of the equations using MAGMA. Below we report the maximal degree reached D_{real} in our experiments, and the theoretical degree of regularity D_{reg}, as given by Assumption 2. We note that the largest of these experiments took 7 days to complete.

	D_{reg}	D_{real}
$n \in \{5, \ldots, 25\}$	3	3
$n \in \{26, \ldots, 53\}$	4	4

7.2 Formal Proofs in Limited Cases

In this part we provide formal proofs of some statements implied by Assumption 2 as first steps towards proving the assumption itself.

Linear Independence Assumption 2 for BinaryError-LWE implies in particular that the equations (8) are linearly independent. Below, we prove that this indeed holds with high probability for any $1 \le m \le \binom{n+1}{2}$.

Theorem 9. *Let $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n \times m} \times \mathbb{Z}_q^m$ be sampled according to $L_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}^{(n)}$, and let $P(x) = X(X-1)$. Assume that $q > 2m$. Then, for all $1 \le m \le \binom{n+1}{2}$, the equations*

$$f_1 = P\big(c_1 - \sum_{j=1}^{n} x_j G_{j,1}\big), \ \ldots \ , f_m = P\big(c_m - \sum_{j=1}^{n} x_j G_{j,m}\big), \tag{15}$$

are linearly independent with probability $\ge 1 - \frac{2m}{q}$. More precisely, the homogeneous components $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ of degree 2 are linearly independent with probability $\ge 1 - \frac{2m}{q}$.

Proof. The coefficients of the f_is can be viewed as polynomials of degree ≤ 2 in the components of the matrix G. We denote by N be the number of monomials of degree 2, and by Mac_2 the $m \times N$ matrix whose rows are the coefficients of the f_i^{H}s. This the Macaulay matrix of the f_i^{H}s at degree 2. We assume that the monomials are sorted with respect to a graded reverse lexicographical order. Let Mat_2 be a $m \times m$ sub-matrix of Mac_2. We can view $\mathrm{Det}(\mathrm{Mat}_2)$ as a polynomial p of degree $2m$ whose variables are the components of G. According to Lemma 9 (Appendix A.2), the polynomial p is non-zero for all $1 \le m \le \binom{n+1}{2}$. The Schwartz-Zippel-DeMillo-Lipton Lemma (Lemma 1) yields that $p(G) \neq 0$ with probability $\ge 1 - \frac{2m}{q}$. $\square$

One can remark that the notion of semi-genericity only depends on the homogeneous components of highest degree. Thus, the polynomial P in Theorem 9 could be replaced by X^2 and the proof will remain the same (also, the constants c_is are irrelevant in the proof). This illustrates that it is equivalent to consider the semi-regularity of the systems as in Assumption 2 or the semi-regularity of the square of linear forms as done by Fröberg and Hollman [29].

A consequence of Theorem 9 is:

Corollary 1. *Let $q > 2m$. There is a polynomial-time algorithm solving* BinaryError-LWE *with probability $1 - \frac{2m}{q}$ as soon as $n < m \le \binom{n+1}{2}$.*

In [35], it was mentioned without a proof that BinaryErrors-LWE can be solved in polynomial-time as soon as the number of samples is $\mathcal{O}\left(n^2\right)$. Precisely, a direct adaptation of [3, Theorem 3.1] to binary noise gives:

Theorem 10. *Let $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n \times m} \times \mathbb{Z}_q^m$ be sampled according to $L_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}^{(n)}$, and let $P(x) = X(X-1)$. The system obtained by linearizing the f_i's, as defined in* (15)*, has unique solution with probability $\ge 1 - 2^m \cdot q^{N_2 - m}$, with $N_2 = \binom{n+2}{2}$. The bound is then non-trivial (< 1) if $m \ge cN_2$, for some constant $c > 0$.*

Proof. We know that $\mathbf{s} \in \mathbb{Z}_q^n$ is a solution of $f_1, \ldots, f_m$. Given $\mathbf{s}' \neq \mathbf{s}$, the idea is to bound the probability that $\mathbf{s}'$ vanishes simultaneously the linearized system corresponding to $f_1, \ldots, f_m$.

By definition, $\mathbf{c} = (\mathbf{s} \times G) + \mathbf{e}$. So, we can write:

$$f_i = P\big(e_i - \sum_{j=1}^{n} (\mathbf{s}_j - x_j) G_{j,i}\big).$$

Now by setting $x_j^* = (\mathbf{s}_j - x_j)$, we have:

$$f_i^*(x_1^*, \ldots, x_n^*) = f_i(\mathbf{s}_1 - x_1^*, \ldots, \mathbf{s}_n - x_n^*) = P\big(e_i - \sum_{j=1}^{n} x_j^* G_{j,i}\big).$$

Thus, $\exists \mathbf{s}' \neq \mathbf{s}$ such that $f_i(\mathbf{s}') = 0 \iff \exists$ a non-zero $\mathbf{s}^* \in \mathbb{Z}_q^n$ such that $f_i^*(\mathbf{s}^*) = 0$. We can view $f_i^*(\mathbf{s}^*)$ has a multivariate polynomial of degree 2 in the components of G. Thus, assuming that $f_i^*(\mathbf{s}^*)$ is non-identically zero, it holds that:

$$\mathrm{Pr}_{(G,\mathbf{c}) \leftarrow_\$ L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}} \big(f_i^*(\mathbf{s}^*) = 0 \mid \mathbf{e}_i = b\big) \leq 2/q, \text{ with } b \in \{0, 1\}.$$

The fact that the $f_i^*(\mathbf{s}^*)$ is a non-zero polynomial – viewed as a polynomial whose variables in the components of G follows easily from [3, Lemma 3.4].

The same result holds if you replace f_i^* by its linearization $L_{f_i^*}$. Thus, for any $\mathbf{S}' = \mathbf{s}' \otimes \mathbf{s}' \in \mathbb{Z}_q^{N_2}$ with $\mathbf{s}' \neq \mathbf{s} \in \mathbb{Z}_q^n$, it holds that:

$$\mathrm{Pr}_{(G,\mathbf{c}) \leftarrow_\$ L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}} \big(L_{f_1}(\mathbf{S}') = 0, \ldots, L_{f_m}(\mathbf{S}') = 0\big) = \prod_{i=1}^{m} \mathrm{Pr}_{(G,\mathbf{c}) \leftarrow_\$ L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}} \big(L_{f_i}(\mathbf{S}') = 0\big).$$

We then have $\mathrm{Pr}_{(G,\mathbf{c}) \leftarrow_\$ L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}} \big(L_{f_i}(\mathbf{S}') = 0\big) =$

$$\frac{1}{2}\mathrm{Pr}_{(G,\mathbf{c}) \leftarrow_\$ L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}} \big(L_{f_i}(\mathbf{S}') = 0 \mid \mathbf{e}_i = 0\big) + \frac{1}{2}\mathrm{Pr}_{(G,\mathbf{c}) \leftarrow_\$ L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}} \big(L_{f_i}(\mathbf{S}') = 0 \mid \mathbf{e}_i = 1\big) \leq 2/q.$$

Finally, we consider the event $E_{\mathbf{S}'} =$ "$L_{f_1}(\mathbf{S}') = 0, \ldots, L_{f_m}(\mathbf{S}') = 0$". The probability that the linearized system has more than one solution is the probability of the event $\cup_{\mathbf{S}'} E_{\mathbf{S}'}$ which is:

$$\mathrm{Pr}_{(G,\mathbf{c}) \leftarrow_\$ L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}} (\text{linearized system has not unique solution}) \leq 2^m \cdot q^{N_2 - m}.$$

□

Linearization (Theorem 10) requires a number of samples which is a constant times the number of monomials of degree ≤ 2 in n variables $\geq c\binom{n+2}{2}$ whilst a Gröbner basis approach (Corollary 1) requires a number of samples equal to the number of monomials of degree exactly 2 in n variables, i.e. $\binom{n+1}{2}$. So, Gröbner bases already require a smaller number of samples than linearization to solve BinaryError-LWE in polynomial-time (we compute a Gröbner basis by performing a row reduction on the Macaulay matrix at the degree of the equations).

Algebraic Independence at Low Degree. Semigenericity. We consider algebraic independence at the degree of the equations plus 1. More precisely, following the terminology as in [29], we consider the notion of semigenericity.

Definition 4. *Let $f_1, \ldots, f_m \in \mathbb{Z}_q[x_1, \ldots, x_n]$ be homogeneous equations of degree d. We shall say that a sequence of polynomials $f_1, \ldots, f_m$ is semigeneric if $\{x_i \cdot f_j\}_{1 \leq i \leq n}^{1 \leq j \leq n}$ spans a vector space of maximal dimension, i.e.* $\min\big(n \cdot m, \binom{d-1+n}{d}\big)$*. For affine polynomials $f_1, \ldots, f_m \in \mathbb{Z}_q[x_1, \ldots, x_n]$, we shall say that the sequence $f_1, \ldots, f_m$ is semigeneric if $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ is semigeneric.*

For BinaryError-LWE, this corresponds to investigate Assumption 2 for a Macaulay matrices at degree 3. In [29, Theorem 2.2], the authors prove that the square of m generic linear forms are semigeneric as long as $m \leq n+15$ and $n \leq 6$. Here, we prove that system (15) is semigeneric for $m \leq n + \lfloor \frac{n-2}{2} \rfloor \approx (\frac{3}{2})n$; improving then towards [29].

Theorem 11. *We assume $q = \Omega(n^2)$. Let $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n \times m} \times \mathbb{Z}_q^m$ be sampled according to $L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}$. For any $m, 1 \leq m \leq n + \lfloor \frac{n-2}{2} \rfloor$, the sequence* (15) *is semigeneric with probability $\geq 1 - \frac{2mn}{q}$.*

Proof. The strategy is similar to the proof of Theorem 9. Now, let N be the number of monomials of degree 3 and let Mat_3 be a sub-matrix of size $m \cdot n \times m \cdot n$ of the Macaulay matrix $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ at degree 3. We can view $\mathrm{Det}(\mathrm{Mat}_3)$ as a polynomial p of degree $2mn$ whose variables are the components of G. According to Lemma 11, p is non-zero. Hence, Lemma 1 yields that $p(G) \neq 0$ with probability $\geq 1 - \frac{2mn}{q}$. □

We have a found a particular example which could allow to extend Theorem 13 up to $m \approx n^2/\log n$ (Remark 2). However, although we verified experimentally that the example is semi-generic, we have not been able to prove it formally. Still, a first difficulty in this type of proof is to actually to describe a particular example. The second step is of course to make the proof.

Full Proof of Assumption 2 for $m = n + 1$. We conclude this part by proving that Assumption 2 holds for $m = n+1$ equations. The proof requires however to have field size big enough.

Theorem 12. *We assume $q = \mathcal{O}(\exp(n))$. Let $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n \times m} \times \mathbb{Z}_q^m$ be sampled according to $L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}$, and let $P(x) = X(X-1)$. With probability $\geq 1 - o(1)$, it holds that the sequence $f_1 = P\big(c_1 - \sum_{j=1}^n x_j G_{j,1}\big),\ \ldots\ , f_{n+1} = P\big(c_{n+1} - \sum_{j=1}^n x_j G_{j,n+1}\big)$ is semi-regular.*

Proof. It is well known that the degree of regularity of a semi-generic sequence of $m = n+1$ equations is $\lceil \frac{n+1}{2} \rceil$. So, we need to prove that the Macaulay matrices associated to $f_1^{\mathrm{H}}, \ldots, f_{n+1}^{\mathrm{H}}$ of degree 2 to $\frac{n+1}{2}$ are of maximal possible rank. That is, the only linear dependencies occurring in the Macaulay matrices are the one induces by the trivial syzygies, i.e. $f_i^{\mathrm{H}} f_j^{\mathrm{H}} = f_j^{\mathrm{H}} f_i^{\mathrm{H}}$. Until now, we investigated degrees 2 and 3 for which there is no trivial syzygies. Let $[t^d]\mathrm{HP}(z)$ be the dth coefficient of the Hilbert polynomials (1). This coefficient gives the rank defects, and then the expected rank, of the Macaulay matrix of $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ at degree $d \geq 2$. As in the previous proofs, we can write easily that Macaulay matrix of $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ at each degree d has the expected rank if a minor is non-zero. The degree of this minor is $\mathcal{O}\left(n^{d-1}\right)$. We then conclude the proof by providing an explicit example of a sufficiently generic system of $m = n+1$ equations. This is the purpose of the next Lemma.

Lemma 8. *Let $P(x) = X(X-1)$. We consider a matrix $G^* \in \mathbb{Z}_q^{n \times n+1}$ such that all coefficients are zero except:*

- *$G^*[i,i] = 1$, for all $i, 1 \leq j \leq n$.*
- *$G^*[i, n+1] = 1$, for all $i, 1 \leq i \leq n$.*

Let $\mathbf{c} = \mathbf{s} \times G^* + \mathbf{e}$, $\mathbf{s} \in \mathbb{Z}_q^n$ be chosen uniformly at random, $\mathbf{e} \in \{0,1\}^{n+1}$ be sampled uniformly. Then, the sequence $P\big(c_1 - \sum_{k=1}^n x_k G^*_{k,1}\big),\ \ldots\ , P\big(c_{n+1} - \sum_{k=1}^n x_k G^*_{k,n+1}\big) \in \mathbb{Z}_q[x_1, \ldots, x_n]$ is semi-regular.

The proof of this result is exactly the proof of [29, Lemma 2.1]. □

Example 1 *For $n = 5$, the matrix G^* in Lemma 8 is as follows:*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

To conclude this part, we mention that we can adapt all the results of Section 7.2 on BinaryError-LWE to UniformError-LWE (for instance, we have Theorem 14).

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A Appendix

A.1 Degree of Regularity

The proof of Theorem 7-ii) is derived from the following more general result.

Proposition 2. *Let $\epsilon > 0$, and $F(n) \in \{\log^{1/\epsilon}(n), \log\log n\}$. Assuming $m = F(n)\, n$, then the degree of regularity of a system of quadratic semi-regular equations $f_1, \ldots, f_m \in \mathbb{Z}_q[x_1, \ldots, x_n]$ behaves asymptotically as:*

$$D_{\mathrm{reg}} = \left(F(n) - 1/2 - \sqrt{F^2(n) - F(n)}\right) n = \frac{1}{8}\frac{n}{F(n)} + \mathcal{O}\left(\frac{n}{F^2(n)}\right).$$

Proof. We assume that we have m quadratic equations in n variables. In this case, we have to consider the Hilbert series:

$$\mathrm{H}_{m,n}(z) = \frac{\left(1-z^2\right)^m}{\left(1-z\right)^n} = \sum_{d=0}^{\infty} h_d\, z^d.$$

The degree of regularity is the index D_{reg} such that $h_{D_{\mathrm{reg}}} < 0$. We try to find $D_{\mathrm{reg}} = \ell(n)\, n = \ell\, n$ such that $h_{\ell\, n} < 0$. To do so, we consider :

$$h_{\ell\, n} = \oint \frac{\mathrm{H}_{m,n}(z)}{z^{\ell\, n}} dz = \oint e^{f(z)\, n} dz,$$

where the contour is a circle centered in 0 whose radius is smaller than 1.

In our context:

$$f(z) = \frac{\log(\mathrm{H}_{m,n}(z))}{n} = \frac{m \log\left(1-z^2\right) - n \log\left(1-z\right) - n\, \ell\, \log\left(z\right)}{n}.$$

Laplace's Method gives then:

$$h_{\ell\, n} \approx \sum_{\{a \mid f'(a)=0\}} e^{f(a) n}.$$

More details about this preliminary part can be found in the literature, for instance [4–6, 8]. As n increases, the integral concentrates in the neighbourhood of one or several saddle points, i.e. the solutions of $f' = 0$.

When the equation $f'(z) = 0$ has two solutions, we have $h_{\ell\, n} \approx e^{f(z^-) n} + e^{f(z^+) n} \to \infty$. Hence, since when $d = D_{\mathrm{reg}} = \ell\, n$ we must have $h_d = 0$ this implies that the equation $f'(z) = 0$ has a multiple root.

In our case, we have:

$$f'(z) = \frac{1}{1-z} - \frac{2\, m\, z}{n(1-z^2)} - \frac{\ell}{z}.$$

Now, we set $m = n\, F(n)$. We have multiple root if the discriminant of f' is 0. As a consequence, $\ell = \ell(n)$ is the smallest real root of :

$$\left(4\, n^3\, F(n)^2 + 4\, n^3 - 8\, n^3\, F(n)\right)\ell^2 + \left(-8\, n^3\, F(n)^3 - 16\, n^3\, F(n) + 20\, n^3\, F(n)^2 + 4\, n^3\right)\ell - 2\, n^3\, F(n) + n^3\, F(n)^2 + n^3.$$

This yields :

$$\ell(n) = \left(F\left(n\right) - 1/2 - \sqrt{F^2(n) - F(n)}\right)\, n = \frac{1}{8} \frac{n}{F(n)} + \mathcal{O}\left(\frac{n}{F^2(n)}\right).$$

$\square$

In Section 5, we use the following result.

Proposition 3. *Let α and β be constants > 0, and $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ be semi-regular polynomials of degree αn. We define the function $F(X, \ell) =$*

$$\log(1+\ell) - \ell \log(\ell) + \ell \log(1+\ell) - \log(1+\ell-X) + \log(\ell-X)\ell - \log(\ell-X)\alpha - \log(1+\ell-X)\ell + \log(1+\ell-X)\alpha - \log(\beta).$$

Assuming $m = \beta^n$, then it holds that:

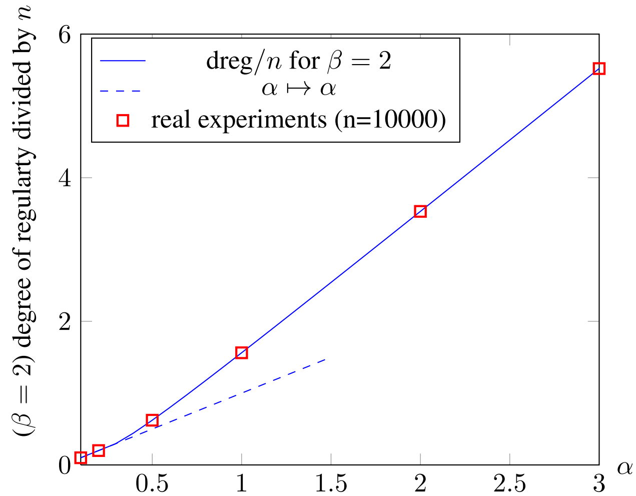
$$\frac{D_{\mathrm{reg}}}{n} \approx \begin{cases} \alpha \text{ if } \alpha < \alpha_0 \\ \mathrm{PositiveRealRoot}\big(F(\alpha, \ell)\big), \text{ if } \alpha_0 \leq \alpha < 6, \end{cases}$$

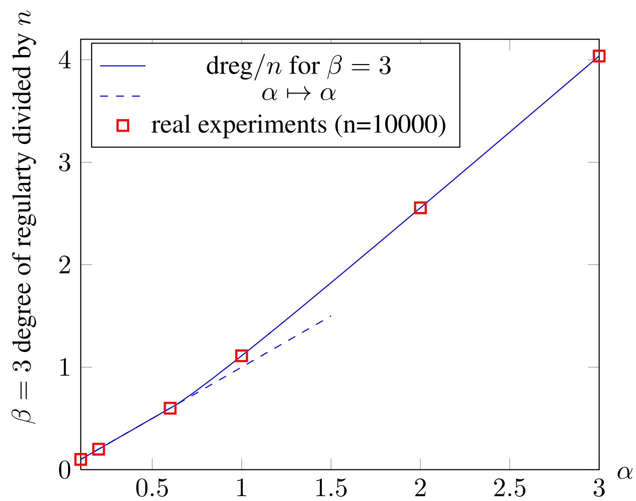
and α_0 is the real value such that $F(\alpha_0, \alpha_0) = 0$.

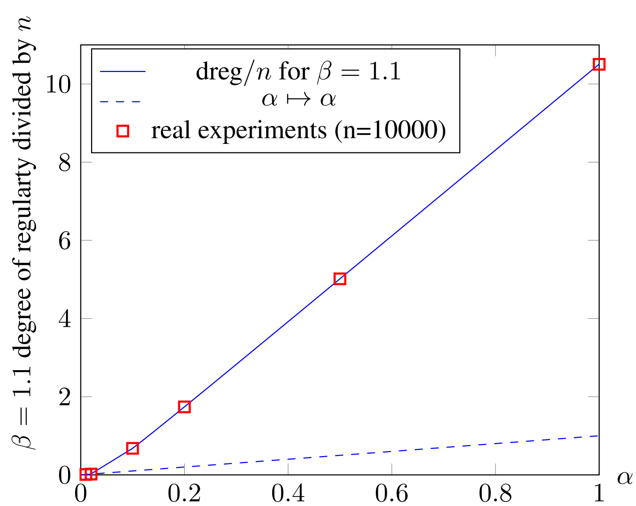
We give below the value of α_0 for various β.

β	α_0
$e^{\pi/4}$	0.3595671731
2	0.293815373
3	0.641794121
1.1	0.019208159

For $\beta = e^{\pi/4}$, the degree of regularity is then for instance $\approx 1.41\, n$. Below, we compare the theoretical degree of regularity obtained from Proposition 3 for various β and α with the degree of regularity obtained by computing the generic Hilbert series (Definition 3).







The proof of this result is beyond the scope of this paper. The first step is similar to the the previous Proposition 2, i.e. we use Laplace's method to approximate the coefficient $h_{\ell n}$ in the Hilbert series. The next step requires different tools that the ones used classically to cancel the coefficient asymptotically. The proof will be presented in an extended version of this paper.

To simplify the analysis, it is possible to upper bound uniformly the degree of regularity; that is:

Proposition 4. *Let α and β be constants > 0, and $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ be semi-regular polynomials of degree αn. If $m = \beta^n$:*

$$D_{\mathrm{reg}} \leq n \cdot \frac{\beta}{\beta - 1}$$

For $\beta = e^{\frac{\pi}{4}}$, we have $\frac{\beta}{\beta-1} = 1.83$.

A.2 Formal Proof in Limited Cases

The example below is used in Theorem 9 to show that the determinant considered is non-zero.

Lemma 9. *For all $i, 1 \leq i \leq n$, construct a $n \times \big(n-(i-1)\big)$ matrix G_i as follows. All the coefficients of G_i are zero except:*

- $G_i[i,j] = 1$, *for all* $j, 1 \leq j \leq \big(n-(i-1)\big)$.
- $G_i[j+(i-1), j] = 1$, *for all* $j, 1 \leq j \leq \big(n-(i-1)\big)$.

Now, let $G^* = G_1 \| G_2 \| \cdots \| G_n$ be a block matrix, $\mathbf{s} \in \mathbb{Z}_q^n$ chosen uniformly at random, and $\mathbf{e} \in \{0,1\}^m$ sampled uniformly. We set $\mathbf{c} = \mathbf{s} \times G^* + \mathbf{e}$ and $P(x) = X(X-1)$ and define:

$$f_1 = P\big(c_1 - \sum_{j=1}^{n} x_j G^*_{j,1}\big),\ \ldots\ , f_m = P\big(c_m - \sum_{j=1}^{n} x_j G^*_{j,m}\big).$$

Then, the homogeneous components $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ of degree 2 are linearly independent.

Proof. Let $f_{i,j}$ be the the jth equation derived from the matrix G_i (the equation corresponding to the jth column of the ith matrix G_i) . We start with the simple case $m = n$ where $G^* = G_1$. The monomial of highest degree in $f_{1,1} = P(c_1 - x_1)$ is simply x_1^2. More generally, for all $i, 1 \leq j \leq n$, the monomials of degree 2 in $f_{1,j} = P(c_j - x_1 - x_j)$ are $x_1^2, x_1 x_j$ and x_j^2. Remark then that the system

$$F_1 := [\tilde{f}_{1,1} = f_{1,1}, \tilde{f}_{1,2} = -f_{1,1} + f_{1,2}, \ldots, \tilde{f}_{1,n} = -f_{1,1} + f_{1,n}]$$

has a triangular shape: the leading monomial of $\tilde{f}_{1,j}$ is $x_1 \cdot x_j$ (all the terms of degree 2 divisible by x_1) and hence distinct.

More generally, let $G^* = G_1 \| G_2 \| G_3 \| \cdots \| G_n$. We consider, for all $i, 1 \leq i \leq n$:

$$F_i := [\tilde{f}_{i,1} = f_{i,1}, \tilde{f}_{i,2} = -f_{i,1} + f_{i,2}, \ldots, \tilde{f}_{i,n-i+1} = -f_{i,1} + f_{i,n-i+1}]. \tag{16}$$

All these equations are in triangular form, and leading monomials of F_i are the monomials $x_i x_j$, with $j \geq i$. Consequently the set of equations $\bigcup_{1 \leq i \leq n} F_i$ are linearly independent. Finally, the numbers of rows of G^* is $n + (n-1) + (n-2) + \cdots (n-(n-1)) = n^2 - n(n-1)/2 = n(n+1)/2 = \binom{n+1}{2}$. $\square$

Example 2 *For $n = 4$, and $m = n(n+1)/2 = 10$ the matrix G^* is as follows:*

$$\begin{bmatrix} 1\,1\,1\,1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,1\,1\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0\,1\,0\,1\,1\,0 \\ 0\,0\,0\,1\,0\,0\,1\,0\,1\,1 \end{bmatrix}.$$

The equations $f_{i,j}$ corresponding are then

$$[x_1^2+15\cdot x_1+5, x_1^2+2\cdot x_1\cdot x_2+x_2^2+4, x_1^2+2\cdot x_1\cdot x_3+x_3^2+10\cdot x_1+10\cdot x_3+12, x_1^2+2\cdot x_1\cdot x_4+x_4^2+9\cdot x_1+9\cdot x_4+3,$$
$$x_2^2+5\cdot x_2+6, x_2^2+2\cdot x_2\cdot x_3+x_3^2+4, x_2^2+2\cdot x_2\cdot x_4+x_4^2+16\cdot x_2+16\cdot x_4,$$
$$x_3^2+13\cdot x_3+8, x_3^2+2\cdot x_3\cdot x_4+x_4^2+7\cdot x_3+7\cdot x_4+12,$$
$$x_4^2+14\cdot x_4+2].$$

By performing the reductions as in (16)*, we get:*

$$[\mathbf{x_1^2}+15\cdot x_1+5, \mathbf{2\cdot x_1\cdot x_2}+x_2^2+2\cdot x_1+16, \mathbf{2\cdot x_1\cdot x_3}+x_3^2+12\cdot x_1+10\cdot x_3+7, \mathbf{2\cdot x_1\cdot x_4}+x_4^2+11\cdot x_1+9\cdot x_4+15,$$
$$\mathbf{x_2^2+5\cdot x_2}+6, \mathbf{2\cdot x_2\cdot x_3}+x_3^2+12\cdot x_2+15, \mathbf{2\cdot x_2\cdot x_4}+x_4^2+11\cdot x_2+16\cdot x_4+11,$$
$$\mathbf{x_3^2+13\cdot x_3}+8, \mathbf{2\cdot x_3\cdot x_4}+x_4^2+11\cdot x_3+7\cdot x_4+4,$$
$$\mathbf{x_4^2}+14\cdot x_4+2].$$

In order to prove Theorem 13, we first consider the case where $m=n$ as an intermediate step.

Theorem 13. *We assume $q=\Omega(n^2)$. Let $(G, \mathbf{s}\times G+\mathbf{e})=(G,\mathbf{c})\in\mathbb{Z}_q^{n\times m}\times\mathbb{Z}_q^m$ be sampled according to $L^{(n)}_{\mathbf{s},\mathcal{U}(\mathbb{F}_2)}$, and let $P(x)=X(X-1)$. For any $m, 1\le m\le n$, the equations*

$$f_1=P\big(c_1-\sum_{j=1}^n x_jG_{j,1}\big),\ \ldots\ , f_m=P\big(c_m-\sum_{j=1}^n x_jG_{j,m}\big),$$

are semigeneric with probability $\geq 1-\frac{2mn}{q}$.

Proof. The strategy is similar to the proof of Theorem 9. Let N be the number of monomials of degree 3 and let Mat_3 be a sub-matrix of size $m\cdot n\times m\cdot n$ of the Macaulay matrix at at degree 3 of $f_1^{\mathrm{H}},\ldots,f_m^{\mathrm{H}}$. We can view $\mathrm{Det}(\mathrm{Mat}_3)$ as a polynomial p of degree $2mn$ whose variables are the components of G. The next result shows that the determinant polynomial considered is not identically zero.

Lemma 10. *Let $P(x)=X(X-1)$. Let $G^*=G_1$ be as defined in Lemma 9. We set $\mathbf{c}=\mathbf{s}\times G^*+\mathbf{e}$, $\mathbf{s}\in\mathbb{Z}_q^n$ chosen uniformly at random, $m=n$ and $\mathbf{e}\in\{0,1\}^m$ sampled uniformly. We define:*

$$f_1=P\big(c_1-\sum_{k=1}^n x_kG^*_{k,1}\big),\ \ldots\ , f_m=P\big(c_m-\sum_{k=1}^n x_kG^*_{k,m}\big).$$

Then, the polynomials $f_1,\ldots,f_m$ are semi-generic.

Proof. We first perform a simple reduction on the f_j, that is:

$$\tilde{f}_2=\tilde{f}_2-f_1,\ldots,\tilde{f}_n=\tilde{f}_n-f_1,\tilde{f}_1=2f_1.$$

From now on, we consider a degree ordering for which $x_1>x_2>\ldots>x_n$. It holds that

$$\mathrm{LT}(\tilde{f}_j)=2\cdot x_1\cdot x_j, \forall j, 1\le j\le n.$$

We can see that the terms of degree 3 in $x_i\cdot\tilde{f}_1$ is equal to $T^{(1)}_{i,1}:=\{2x_1^2x_i\}_{1\le i\le n}$. Similarly, we have that $T^{(1)}_{i,j}:=\{2\cdot x_1x_ix_j, x_ix_j^2\}^{2\le j\le n}_{1\le i\le n}$ are terms of degree 3 in $x_i\cdot\tilde{f}_j$ (with $j\neq 1$).

We consider a matrix $M^{(1)}:=\{M^{(1)}[i,j]=x_i\cdot\tilde{f}_j\}^{1\le j\le m}_{1\le i\le n}$ and define $r^{(1)}_{i,j}$ as the function which returns $\mathrm{LT}(M^{(1)}[i,j])$. For all $(i,j)\in[1,\ldots,n]\times[1,\ldots,n]$, we have: $r^{(1)}_{i,j}=2\cdot x_1\cdot x_i\cdot x_j$. Hence, $r^{(1)}_{i,j}=r^{(1)}_{j,i}$ for all $(i,j)\in[1,\ldots,n]\times[1,\ldots,n]$. So, $M^{(1)}[i,j]$ and $M^{(1)}[j,i]$ have the same leading terms. Our goal is to perform suitable linear combinations on the polynomials of $M^{(1)}[i,j]$ such that all components have distinct leading terms.

We first process the first column and first row of $M^{(1)}$. We define $C_1 := \{(i,1) \mid i \in [1,\dots,n]\}$, and $R_1 := \{(1,j) \mid j \in [2,\dots,n]\}$. For all $(i,j) = (i,1) \in C_1$, we have $\mathrm{LT}(M^{(1)}[i,1]) = 2 \cdot x_1^2 \cdot x_i$. For all $(i,j) = (1,j) \in R_1$, we have also $\mathrm{LT}(M^{(1)}[1,j]) = 2 \cdot x_1^2 \cdot x_j$. Thus, for all $(1,j) \in R_1$, we update $M^{(1)}[1,j]$ as follows:

$$M^{(1)}[1,j] = M^{(1)}[1,j] - M^{(1)}[j,1].$$

After this step, for all $(1,j) \in R_1$, the term of degree 3 in $M^{(1)}[1,j]$ is now $T_{1,j}^{(1)} := \{x_1 x_j^2\}$ and then $r_{1,j}^{(1)} := x_1 x_j^2$. For all $(i,1) \in C_1$, we still have $r_{i,1}^{(1)} := 2\, x_1^2 \cdot x_i$ and $T_{i,1}^{(1)} := \{2\, x_1^2 x_i\}$.

Now, we consider the set $L_1 := \{(i,j) \in [2,\dots,n] \times [2,\dots,n] \mid i - j \geq 0\}$ This is the lower diagonal part. For all $(i,j) \in L_1$, with $i \neq j$, we update the matrix $M^{(1)}$ as follows:

$$M^{(1)}[i,j] = M^{(1)}[i,j] - M^{(1)}[j,i]$$

For $(i,i) \in L_1$, we have that $r_{i,i}^{(1)} = 2\, x_1 x_i^2$. However, $x_1^2 x_i = r_{1,i}^{(1)}/2$, we then update the elements of the diagonal as follows:

$$M^{(1)}[i,i] = M^{(1)}[i,i] - 2 \cdot M^{(1)}[1,i].$$

After this step, for all $(i,j) \in L_1$, with $i \neq j$, the terms of degree 3 in $M^{(1)}[i,j]$ is $T_{i,j}^{(1)} = \{x_j^2 x_i, -x_j x_i^2\}$ and we set $r_{i,j}^{(1)} := x_j^2 x_i$. For $i = j > 1$, $T_{i,i}^{(1)}$ reduces to $\{x_j^3\}$.

For all $(i,j) \in L_1$, the terms $r_{i,j}^{(1)}$ are distinct. Indeed, given $(i,j) \in L_1$, the only solution $(i',j') \in L_1$ to $r_{i,j}^{(1)} = r_{i',j'}^{(1)}$ is trivial, i.e. $(i = i', j = j')$.

Now, let $U_1 := \{(i,j) \in [2,\dots,n] \times [2,\dots,n] \mid i - j < 0\}$. For all $(i,j) \in U_1$, we have $r_{i,j}^{(1)} = 2 x_1 x_i x_j$. For all $(i,j) \in U_1$, the terms $r_{i,j}^{(1)}$ are distinct. Indeed, given $(i,j) \in U_1$, the only non-trivial solution $(i',j') \in U_1$ to $r_{i,j}^{(1)} = r_{i',j'}^{(1)}$ is $(i' = j, j' = i)$. Since $(i,j) \in U_1$, this implies that $i' - j' > 0$ and then $(i',j') \notin U_1$. For all, $(i,j) \in U_1$, the terms of degree 3 of $M^{(1)}[i,j]$ remains $T_{i,j}^{(1)} := \{2\, x_1 x_i x_j, x_i x_j^2\}_{1 \leq i \leq n}^{2 \leq j \leq n}$.

To summarize:

- $r_{i,1}^{(1)} := r_{i,1}^{C_1} = 2\, x_1^2 x_i$, for all $i, 1 \leq i \leq n$,
- $r_{1,j}^{(1)} := r_{1,j}^{R_1} = x_1 x_j^2$, for all $j, 2 \leq j \leq n$,
- $r_{i,j}^{(1)} := r_{i,j}^{U_1} = 2\, x_1 x_i x_j$, for all $(i,j) \in U_1$,
- $r_{i,j}^{(1)} := r_{i,j}^{L_1} = x_j^2 x_i$, for all $(i,j) \in L_1$.

□

Example 3 *For $n = 4$, $G^* = G_1$ is as follows:*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The equations $\tilde{f}_j$ corresponding are:

$$[2 \cdot x_1^2 + 14 \cdot x_1 + 7, 2 \cdot x_1 \cdot x_2 + x_2^2 + 16 \cdot x_1 + 6 \cdot x_2 + 1, 2 \cdot x_1 \cdot x_3 + x_3^2 + 13 \cdot x_1 + 3 \cdot x_3 + 7, \\ 2 \cdot x_1 \cdot x_4 + x_4^2 + 14 \cdot x_1 + 4 \cdot x_4 + 13].$$

By performing the reductions as in the previous lemma, we get:

$$[\mathbf{2\cdot x_1^3} + 14\cdot x_1^2 + 7\cdot x_1, \mathbf{2\cdot x_1^2\cdot x_2} + 14\cdot x_1\cdot x_2 + 7\cdot x_2, \mathbf{2\cdot x_1^2\cdot x_3} + 14\cdot x_1\cdot x_3 + 7\cdot x_3, \mathbf{2\cdot x_1^2\cdot x_4} + 14\cdot x_1\cdot x_4 + 7\cdot x_4,$$
$$\mathbf{x_1\cdot x_2^2} + 16\cdot x_1^2 + 9\cdot x_1\cdot x_2 + x_1 + 10\cdot x_2, \mathbf{2\cdot x_1\cdot x_2\cdot x_3} + x_2\cdot x_3^2 + 13\cdot x_1\cdot x_2 + 3\cdot x_2\cdot x_3 + 7\cdot x_2,$$
$$\mathbf{2\cdot x_1\cdot x_2\cdot x_4} + x_2\cdot x_4^2 + 14\cdot x_1\cdot x_2 + 4\cdot x_2\cdot x_4 + 13\cdot x_2, \mathbf{x_1\cdot x_3^2} + 13\cdot x_1^2 + 6\cdot x_1\cdot x_3 + 7\cdot x_1 + 10\cdot x_3,$$
$$\mathbf{2\cdot x_1\cdot x_3\cdot x_4} + x_3\cdot x_4^2 + 14\cdot x_1\cdot x_3 + 4\cdot x_3\cdot x_4 + 13\cdot x_3,$$
$$\mathbf{x_1\cdot x_4^2} + 14\cdot x_1^2 + 7\cdot x_1\cdot x_4 + 13\cdot x_1 + 10\cdot x_4, \mathbf{x_2^3} + \mathbf{2\cdot x_1^2} + 15\cdot x_1\cdot x_2 + 6\cdot x_2^2 + 15\cdot x_1 + 15\cdot x_2,$$
$$\mathbf{x_2^2\cdot x_3} + 16\cdot x_2\cdot x_3^2 + 4\cdot x_1\cdot x_2 + 16\cdot x_1\cdot x_3 + 3\cdot x_2\cdot x_3 + 10\cdot x_2 + x_3,$$
$$\mathbf{x_2^2\cdot x_4} + 16\cdot x_2\cdot x_4^2 + 3\cdot x_1\cdot x_2 + 16\cdot x_1\cdot x_4 + 2\cdot x_2\cdot x_4 + 4\cdot x_2 + x_4,$$
$$\mathbf{x_3^3} + 8\cdot x_1^2 + x_1\cdot x_3 + 3\cdot x_3^2 + 3\cdot x_1 + 4\cdot x_3, \mathbf{x_3^2\cdot x_4} + 16\cdot x_3\cdot x_4^2 + 3\cdot x_1\cdot x_3 + 13\cdot x_1\cdot x_4 + 16\cdot x_3\cdot x_4 + 4\cdot x_3 + 7\cdot x_4,$$
$$\mathbf{x_4^3} + 6\cdot x_1^2 + 4\cdot x_4^2 + 8\cdot x_1 + 10\cdot x_4].$$

Finally, this is the example used in Theorem 13.

Lemma 11. *Let $P(x) = X(X-1)$. Let G_1 be defined as in Lemma 10. We consider a $n \times m_2$ matrix G_2, with $m_2 = \lfloor \frac{n-2}{2} \rfloor$. The coefficients are zero except for:*

- $G_2[2, j] = 1$, *for all* $j, 1 \le j \le m_2$.
- $G_2[2j+1, j] = G_2[2j+2, j] = 1$, *for all* $j, 1 \le j \le m_2$.

Let $m = n + m_2$, $G^* = G_1 \| G_2$ be a block matrix of size $n \times m$. We set $\mathbf{c} = \mathbf{s} \times G^* + \mathbf{e}$, $\mathbf{s} \in \mathbb{Z}_q^n$ chosen uniformly at random, and $\mathbf{e} \in \{0,1\}^m$ sampled uniformly. We define:

$$f_1 = P\big(c_1 - \sum_{k=1}^{n} x_k G^*_{k,1}\big), \ \ldots \ , f_m = P\big(c_m - \sum_{k=1}^{n} x_k G^*_{k,m}\big).$$

Then, the sequence $f_1, \ldots, f_m$ is semigeneric.

Proof. Let $M^{(1)}$ be the matrix constructed as in Lemma 10. The matrix is such that:

- $\forall (1, j) \in R_1$, the term of degree 3 in $M^{(1)}[1, j]$ is $T^{(1)}_{1,j} := \{x_1 x_j^2\}$.
- $\forall (i, 1) \in C_1$, the term of degree 3 in $M^{(1)}[i, 1]$ is $T^{(1)}_{i,1} := \{2\, x_1^2 x_i\}$.
- $\forall (i, j) \in U_1$, the terms of degree 3 of $M^{(1)}[i, j]$ is $T^{(1)}_{i,j} := \{2 x_1 x_i x_j, x_i x_j^2\}_{1 \le i \le n}^{2 \le j \le n}$.
- $\forall (i, j) \in L_1$, $i \neq j$, the terms of degree 3 in $M^{(1)}_{i,j}$ is $T^{(1)}_{i,j} := \{x_j^2 x_i, -x_j x_i^2\}$.
- The terms of degree 3 in $M^{(1)}_{i,i}$ $(i > 1)$ is $T^{(1)}_{i,i} := \{x_i^3\}$.

The leading terms of the polynomials in $M^{(1)}$ are then divided by a square or divided by x_1. In fact, all the terms of degree 3 divisible by x_1 appear as leading terms.

For all $j, 1 \le j \le m_2$, we denote by $f_{2,j} = f_{m_1+j}$ the equations derived from G_2. This is the polynomial constructed from the jth column of G_2. We define by

$$T^{(2)}_{i,j} := \{x_i x_2^2, x_i x_{2j+1}^2, x_i x_{2j+2}^2, 2\, x_i x_2 x_{2j+1}, 2\, x_i x_2 x_{2j+2}, 2\, x_i x_{2j+1} x_{2j+2}\}_{1 \le i \le n}^{1 \le j \le m_2}$$

the terms of degree 3 in $x_i \cdot f_{2,j}$. We also consider a matrix $M^{(2)} := \{M^{(2)}[i, j] = x_i \cdot f_{2,j}\}_{1 \le i \le n}^{1 \le j \le m_2}$. The first step of the proof is to perform all possible reductions of the polynomials in $M^{(2)}$ modulo the polynomials $M^{(1)}[i, j]$.

The term $x_i x_2^2$ can always be reduced by the leading term of a polynomial in the second column of $M^{(1)}$. For all $(i, j) \in [1, \ldots, n] \times [1, \ldots, m_2]$, we cancel this term as follows:

$$M^{(2)}[i, j] = M^{(2)}[i, j] - M^{(1)}[i, 2].$$

After such reductions, the terms of degree 3 in $M^{(2)}[i,j]$ is updated as:

$$T_{i,j}^{(2)} = \{x_2 x_i^2, x_i x_{2j+1}^2, x_i x_{2j+2}^2, 2\, x_i x_2 x_{2j+1}, 2\, x_i x_2 x_{2j+2}, 2 x_i x_{2j+1} x_{2j+2}\}, \text{ for } i > 2.$$

For $i = 1, 2$ the only difference is that the term $x_2 x_i^2$ not appear in $T_{i,j}^{(2)}$.
We consider the particular case of the first row $M^{(2)}$:

$$T_{1,j}^{(2)} = \{2\, x_1 x_2 x_{2j+1}, 2\, x_1 x_2 x_{2j+2}, x_1 x_{2j+1}^2, 2 x_1 x_{2j+1} x_{2j+2}, x_1 x_{2j+2}^2\}.$$

We can see that all these terms can be reduced by the leading terms of a suitable $M^{(1)}[i,j]$. More precisely, $x_1 x_{2j+1}^2$, and $x_1 x_{2j+2}^2$ can be reduced by $M^{(1)}[1, 2j+1]$ and $M^{(1)}[1, 2j+2]$ respectively. Thus, for all $j, 1 \le j \le m_2$, we update the matrix as follows:

$$M^{(2)}[1,j] = M^{(2)}[1,j] - M^{(1)}[1, 2j+1] - M^{(1)}[1, 2j+2].$$

The corresponding reductions will only yield new terms of degree < 3.
Similarly, $x_1 x_2 x_{2j+1}, x_1 x_2 x_{2j+2}, x_1 x_{2j+1} x_{2j+2}$ can be reduced by a $M^{(1)}[i', j']$ with $(i', j') \in U_1$. Thus, for all $j, 1 \le j \le m_2$, we update the matrix as follows:

$$M^{(2)}[1,j] = -M^{(2)}[1,j] + M^{(1)}[2, 2j+1] + M^{(1)}[2, 2j+2] + M^{(1)}[2j+1, 2j+2].$$

After this step, we have:

$$T_{1,j}^{(2)} = \{x_2 x_{2j+1}^2, x_2 x_{2j+2}^2, x_{2j+1} x_{2j+2}^2\} \text{ and } \mathrm{LT}(M^{(2)}[1,j]) = x_2 x_{2j+1}^2.$$

We consider then the second row of $M^{(2)}$ whose terms of degree 3 are:

$$T_{2,j}^{(2)} = \{2\, x_2^2 x_{2j+1}, 2\, x_2^2 x_{2j+2}, x_2 x_{2j+1}^2, 2\, x_2 x_{2j+1} x_{2j+2}, x_2 x_{2j+2}^2\}.$$

We can again reduce $x_2^2 x_{2j+1}$ and $x_2^2 x_{2j+2}$ by the leading terms of polynomials of $M^{(1)}$. However, the reduction will create new elements $x_2 x_{2j+1}^2$ and $x_2 x_{2j+2}^2$ which are irreducible modulo $\mathrm{LT}(M^{(1)})$. For all $j, 1 \le j \le m_2$, we set:

$$M^{(2)}[2,j] = M^{(2)}[2,j] - 2\, M^{(1)}[2j+1, 2] - 2\, M^{(1)}[2j+2, 2].$$

After this step, we have

$$T_{2,j}^{(2)} = \{3\, x_2 x_{2j+1}^2, 2\, x_2 x_{2j+1} x_{2j+2}, 3\, x_2 x_{2j+2}^2\} \text{ and } \mathrm{LT}(M^{(2)}[1,j]) = 3\, x_2 x_{2j+1}^2.$$

$T_{2,j}^{(2)}$ and $T_{2,j}^{(2)}$ can not be further reduced by $\mathrm{LT}(M^{(1)})$.
Let $U_2 := \{(i,j) \in [4, \ldots, n] \times [1, \ldots, m_2] \mid i > 2j+2\}$. In this case, we can cancel the terms $x_{2j+1}^2 x_i$ and $x_{2j+2}^2 x_i$ by $M^{(1)}$. However, the reduction will create new terms $x_{2j+1} x_i^2$ and $x_{2j+2} x_i^2$ which are irreducible modulo $\mathrm{LT}(M^{(1)})$. More precisely, for all $(i,j) \in U_2$, we update the matrix $M^{(2)}$ as follows:

$$M^{(2)}[i,j] = M^{(2)}[i,j] - M^{(1)}[i, 2j+2] - M^{(1)}[i, 2j+1].$$

The terms of degree 3 of $M^{(2)}[i,j]$ are then:

$$T_{i,j}^{(2)} = \{2\, x_2 x_{2j+1} x_i, 2\, x_2 x_{2j+2} x_i, x_2 x_i^2, 2 x_{2j+1} x_{2j+2} x_i, x_{2j+1} x_i^2, x_{2j+2} x_i^2\}, \ \forall (i,j) \in U_2.$$

The terms of $T_{i,j}^{(2)}$ are clearly no divisible by x_1. We have that $x_2 x_i^2$ is irreducible modulo $\mathrm{LT}(M^{(1)})$. The others terms are divisible by a square only if $i = 2j+2$. However, $x_2 x_{2j+2}^2$ and $x_{2j+1} x_{2j+2}^2$ can not be reduced modulo $\mathrm{LT}(M^{(1)})$. Again, $x_{2j+1} x_i^2$ and $x_{2j+2} x_i^2$ can not be reduced by $\mathrm{LT}(M^{(1)})$. Thus, $\forall (i,j) \in U_2$, the set $T_{i,j}^{(2)}$ is irreducible modulo $\mathrm{LT}(M^{(1)})$.

For $i = 2j+2$, we have a rather similar situation. The only difference is that the reduction of x_{2j+2}^3 will only yield terms of degree < 3. We have then:

$$T_{2j+2,j}^{(2)} = \{2\, x_2 x_{2j+1} x_{2j+2}, 3\, x_2 x_{2j+2}^2, 3\, x_{2j+1} x_{2j+2}^2\}.$$

Similarly, for $i = 2j+1$, we have:

$$T_{i,j}^{(2)} = \{3\, x_2 x_{2j+1}^2, 2\, x_2 x_{2j+1} x_{2j+2}, x_{2j+1}^3, 2\, x_{2j+1}^2 x_{2j+2}, x_{2j+2} x_{2j+1}^2\}.$$

Using $\mathrm{LT}(M^{(1)})$, we can reduce $x_{2j+1}^2 x_{2j+2}$ and x_{2j+1}^3. So, we compute:

$$M^{(2)}[2j+1, j] = M^{(2)}[2j+1, j] - 2\, M^{(1)}[2j+2, 2j+1] - M^{(1)}[2j+1, 2j+1].$$

This yields:

$$T_{2j+1,j}^{(2)} = \{3\, x_2 x_{2j+1}^2, 2\, x_2 x_{2j+1} x_{2j+2}, 3\, x_{2j+1} x_{2j+2}^2\}.$$

Finally, we define $L_2 := \{(i,j) \in [3, \ldots, n] \times [1, \ldots, m_2] \mid 2 < i < 2j+1\}$. For all, $(i,j) \in L_2$, it holds that:

$$T_{i,j}^{(2)} = \{x_2 x_i^2, 2\, x_2 x_i x_{2j+1}, 2\, x_2 x_i x_{2j+2}, x_i x_{2j+1}^2, x_i x_{2j+2}^2, 2\, x_i x_{2j+1} x_{2j+2}\}.$$

All in all, after these steps, no term of $M^{(2)}$ can be reduced by $\mathrm{LT}(M^{(1)})$.

We have then:

- $T_{1,j}^{(2)} = \{x_2 x_{2j+1}^2, x_2 x_{2j+2}^2, x_{2j+1} x_{2j+2}^2\}$, for all $j, 1 \leq j \leq m_2$,
- $T_{2,j}^{(2)} = \{3\, x_2 x_{2j+1}^2, 2\, x_2 x_{2j+1} x_{2j+2}, 3\, x_2 x_{2j+2}^2\}$, for all $j, 1 \leq j \leq m_2$,
- $T_{i,j}^{(2)} = \{x_2 x_i^2, 2\, x_2 x_i x_{2j+1}, 2\, x_2 x_i x_{2j+2}, x_i x_{2j+1}^2, x_i x_{2j+2}^2, 2\, x_i x_{2j+1} x_{2j+2}\}$, $\forall (i,j) \in L_2$,
- $T_{2j+1,j}^{(2)} = \{3\, x_2 x_{2j+1}^2, 2\, x_2 x_{2j+1} x_{2j+2}, 3\, x_{2j+1} x_{2j+2}^2\}$, for all $j, 1 \leq j \leq m_2$,
- $T_{2j+2,j}^{(2)} = \{2\, x_2 x_{2j+1} x_{2j+2}, 3\, x_2 x_{2j+2}^2, 3\, x_{2j+1} x_{2j+2}^2\}$, for all $j, 1 \leq j \leq m_2$,
- $T_{i,j}^{(2)} = \{2\, x_2 x_{2j+1} x_i, 2\, x_2 x_{2j+2} x_i, x_2 x_i^2, 2 x_{2j+1} x_{2j+2} x_i, x_{2j+1} x_i^2, x_{2j+2} x_i^2\}$, $\forall (i,j) \in U_2$.

We now proceed the polynomials $M^{(2)}$ to have distinct leading monomials.

We first reduce polynomials of the second row. That is, for all $j, 1 \leq j \leq m_2$:

$$M^{(2)}[2, j] = M^{(2)}[2, j] - 3\, M^{(2)}[1, j].$$

This gives:

$$T_{2,j}^{(2)} = \{\mathbf{2\, x_2 x_{2j+1} x_{2j+2}}, -3\, x_{2j+1} x_{2j+2}^2\}.$$

Also, for all $j, 1 \leq j \leq m_2$, we compute:

$$M^{(2)}[2j+2, j] = M^{(2)}[2j+2, j] - M^{(2)}[2, j].$$

This gives:

$$T_{2j+2,j}^{(2)} = \{\mathbf{3\, x_2 x_{2j+2}^2}, 6\, x_{2j+1} x_{2j+2}^2\}.$$

For all $j, 1 \leq j \leq m_2$, we compute:

$$M^{(2)}[2j+1, j] = M^{(2)}[2j+1, j] - 3 M^{(2)}[1, j].$$

This gives:

$$T_{2j+1,j}^{(2)} = \{\mathbf{2\, x_2 x_{2j+1} x_{2j+2}}, -3\, x_2 x_{2j+2}^2\}.$$

Finally, for all $j, 1 \leq j \leq m_2$:

$$M^{(2)}[2j+1, j] = M^{(2)}[2j+1, j] - M^{(2)}[2, j].$$

This gives:

$$T^{(2)}_{2j+1,j} = \{-\mathbf{3\,x_2x^2_{2j+2}}, 3\,x_{2j+1}x^2_{2j+2}\}.$$

Finally, for all $j, 1 \leq j \leq m_2$:

$$M^{(2)}[2j+1,j] = M^{(2)}[2j+1,j] + M^{(2)}[2j+2,j].$$

This gives

$$T^{(2)}_{2j+1,j} = \{\mathbf{9\,x_{2j+1}x^2_{2j+2}}\}.$$

To summarize, we have:

- $T^{(2)}_{1,j} = \{\mathbf{x_2x^2_{2j+1}}, x_2x^2_{2j+2}, x_{2j+1}x^2_{2j+2}\}$, and $r^{(2)}_{1,j} = x_2x^2_{2j+1}$, for all $j, 1 \leq j \leq m_2$,
- $T^{(2)}_{2,j} = \{\mathbf{2\,x_2x_{2j+1}x_{2j+2}}, -3\,x_{2j+1}x^2_{2j+2}\}$, and $r^{(2)}_{2,j} = 2\,x_2x_{2j+1}x_{2j+2}$, for all $j, 1 \leq j \leq m_2$,
- $T^{(2)}_{i,j} = \{x_2x_i^2, 2\,x_2x_ix_{2j+1}, 2\,x_2x_ix_{2j+2}, x_ix^2_{2j+1}, x_ix^2_{2j+2}, \mathbf{2\,x_ix_{2j+1}x_{2j+2}}\}$, and $r^{U_2}_{i,j} = 2\,x_ix_{2j+1}x_{2j+2}, \forall (i,j) \in L_2$,
- $T^{(2)}_{2j+2,j} = \{\mathbf{3\,x_2x^2_{2j+2}}, 6\,x_{2j+1}x^2_{2j+2}\}$, and and $r^{(2)}_{2j+2,j} = 3\,x_2x^2_{2j+2}$, for all $j, 1 \leq j \leq m_2$,
- $T^{(2)}_{2j+1,j} = \{\mathbf{9\,x_{2j+1}x^2_{2j+2}}\}$, and $r^{(2)}_{2j+2,1} = 9\,x_{2j+1}x^2_{2j+2}$ for all $j, 1 \leq j \leq m_2$,
- $T^{(2)}_{i,j} = \{\mathbf{2\,x_2x_{2j+1}x_i}, 2\,x_2x_{2j+2}x_i, x_2x_i^2, 2x_{2j+1}x_{2j+2}x_i, x_{2j+1}x_i^2, x_{2j+2}x_i^2\}$, and $r^{U_2}_{i,j} = 2\,x_2x_{2j+1}x_i, \forall (i,j) \in U_2$

By inspecting the terms in bold, it can be noticed that they are all distinct. $\square$

Example 4 *For $n = 5$, $G^* = G_1 \| G_2$ is as follows:*

$$\begin{bmatrix} 1&1&1&1&0 \\ 0&1&0&0&1 \\ 0&0&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&1&0 \end{bmatrix}.$$

The initial system – after a first simple reduction on the equations of G_1 – is:

$$[2\cdot x_1^2 + 16\cdot x_1 + 17, 2\cdot x_1\cdot x_2 + x_2^2 + 5\cdot x_1 + 13\cdot x_2 + 19, 2\cdot x_1\cdot x_3 + x_3^2 + 3\cdot x_1 + 11\cdot x_3 + 7, 2\cdot x_1\cdot x_4 + x_4^2 + 27\cdot x_1 + 6\cdot x_4 + 22,$$
$$2\cdot x_1\cdot x_5 + x_5^2 + 15\cdot x_1 + 23\cdot x_5 + 22, x_2^2 + 2\cdot x_2\cdot x_3 + 2\cdot x_2\cdot x_4 + x_3^2 + 2\cdot x_3\cdot x_4 + x_4^2 + 3\cdot x_2 + 3\cdot x_3 + 3\cdot x_4 + 2].$$

We give below homogeneous components of degree 3 of the polynomials of $M^{(2)}$ after all the operations described in the previous proof.

$$[\mathbf{x_2\cdot x_3^2} + x_2\cdot x_4^2 + x_3\cdot x_4^2, \mathbf{2\cdot x_2\cdot x_3\cdot x_4} + 26\cdot x_3\cdot x_4^2, \mathbf{9\cdot x_3\cdot x_4^2},$$
$$\mathbf{3\cdot x_2\cdot x_4^2} + 6\cdot x_3\cdot x_4^2, \mathbf{2\cdot x_2\cdot x_3\cdot x_5} + 2\cdot x_2\cdot x_4\cdot x_5 + x_2\cdot x_5^2 + 2\cdot x_3\cdot x_4\cdot x_5 + x_3\cdot x_5^2 + x_4\cdot x_5^2].$$

Remark 2. We emphasize that we have found an example which extends Lemma 11. We construct a matrix $G := G_1\|G_2\|\cdots\|G_n$. The matrices G_1 and G_2 are defined as in Lemma 11. Each block $G_b, b > 2$ will be of size $n \times \left\lfloor \frac{n-b}{b+1} \right\rfloor$ and such that all the coefficients are zero except for :

- $G_2[b,j] = 1$, for all $j, 1 \leq j \leq m_2$.
- $G_2[j(b+1), j] = G_2[b\cdot j + 2, j] = 1, \cdots, G_2[j(b+1)+b, j]$, for all $j, 1 \leq j \leq \left\lfloor \frac{n-b}{b+1} \right\rfloor$.

We perform experiments to verify that such G yields semigeneric systems. We have been able to verify the assumption up to $n = 100$. However, we have not been able to prove that such family is semigeneric. This would allow to prove semigenericity for $m \approx n^2/\log n$.

We can generalize for instance Theorem 9 to UniformError-LWE.

Theorem 14. *Let $T > 0$, and $(G, \mathbf{s} \times G + \mathbf{e}) = (G, \mathbf{c}) \in \mathbb{Z}_q^{n \times m} \times \mathbb{Z}_q^m$ be sampled according to $L^{(n)}_{\mathbf{s},\mathcal{U}([-T\ldots,T])}$. Let also $P(x) = X \prod_{i=1}^{T}(X - i)$. We assume that $q > (2T + 1) \cdot m$. For all $1 \leq m \leq \binom{n+1}{2}$, we define:*

$$f_1 = P\big(c_1 - \sum_{j=1}^{n} x_j G_{j,1}\big), \ \ldots \ , f_m = P\big(c_m - \sum_{j=1}^{n} x_j G_{j,m}\big).$$

It holds that $f_1^{\mathrm{H}}, \ldots, f_m^{\mathrm{H}}$ are linearly independent with probability $\geq 1 - \frac{(2T+1)m}{q}$.

Proof. Let N be the number of monomials of degree $\leq 2T + 1$. We define Mac as $m \times N$ matrix whose rows are the coefficients of the f_is. Let $p = \mathrm{Det}(\mathrm{Mat})$ be the determinant of a $m \times m$ sub-matrix Mat of Mac. If p is non-zero, Schwartz-Zippel-DeMillo-Lipton Lemma yields the result stated. The fact that p is non-zero follows from a similar argument than Lemma 9. $\square$