# Asymptotic description of stochastic neural networks. I. Existence of a large deviation principle

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**Abstract**

We study the asymptotic law of a network of interacting neurons when the number of neurons becomes infinite. Given a completely connected network of neurons in which the synaptic weights are Gaussian *c*orrelated random variables, we describe the asymptotic law of the network when the number of neurons goes to infinity. Unlike previous works which made the biologically unplausible assumption that the weights were i.i.d. random variables, we assume that they are correlated. We introduce the process-level empirical measure of the trajectories of the solutions to the equations of the finite network of neurons and the averaged law (with respect to the synaptic weights) of the trajectories of the solutions to the equations of the network of neurons. The result is that the image law through the empirical measure satisfies a large deviation principle with a good rate function. We provide an analytical expression of this rate function.

**Résumé**

**Description asymptotique de réseaux de neurones stochastiques. I - existence d'un principe de grandes déviation**

Étant donné un réseau complètement connecté de neurones dans lequel les poids synaptiques sont des variables aléatoires gaussiennes *c*orrélées, nous caractérisons la loi asymptotique de ce réseau lorsque le nombre de neurones tend vers l'infini. Tous les travaux précédents faisaient l'hypothèse, irréaliste du point de vue de la biologie, de poids indépendants. Nous introduisons la mesure empirique sur l'espace des trajectoires solutions des équations du réseau de neurones de taille finie et la loi moyennée (par rapport aux poids synaptiques) des trajectoires de ces solutions. Le résultat est que la loi image de cette loi par la mesure empirique satisfait un principe de grandes déviations avec une bonne fonction de taux dont nous donnons une expression analytique.

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## Version française abrégée

Nous considérons le problème de décrire la dynamique asymptotique d'un ensemble de $2n+1$ neurones lorsque ce nombre tend vers l'infini. Ce problème est motivé par un désir de parcimonie, par celui de rendre compte de l'apparition de phénomènes émergents, ainsi que par celui de comprendre les effets de taille finie. Nous considérons donc un réseau de $2n+1$ neurones interconnectés dont la dynamique commune (en temps discret) obéit aux équations stochastiques (2). Dans celles-ci apparaissent les poids synaptiques ou coefficients de couplage notés $J_{ij}^n$ qui sont des variables aléatoires gaussiennes corrélées. Pour répondre à la question posée nous considérons la loi notée $Q^{V_n}$ de la solution à (2) moyennée par rapport aux poids synaptiques ou plus précisément l'image $\Pi^n$ de cette loi par la mesure empirique (1). Nous montrons que cette loi satisfait un principe de grande déviations avec une bonne fonction de taux $H$ dont nous donnons une expression analytique dans la définition 3.1 et les équations (9) et (12). Ce travail généralise au cas des poids synaptiques corrélés celui d'auteurs comme Sompolinsky [10] et Moynot et Samuelides [8] qui ont considéré le cas de poids synaptiques indépendants. Dans ce cas, plus simple d'un point de vue mathématique, mais beaucoup moins plausible d'un point de vue biologique, on observe le phénomène de propagation du chaos. Nous montrons dans un second article [4] que la bonne fonction de taux a un minimum unique que nous caractérisons complètement. La propagation du chaos n'a pas lieu mais la représentation est parcimonieuse dans un sens défini dans [4].

## 1 Introduction

### *1.1 Neural networks*

Our goal is to study the asymptotic behaviour and large deviations of a network of interacting neurons when the number of neurons becomes infinite.

Sompolinsky also succesfully explored this particular topic [10] for fully connected networks of neurons. In his study of the continuous time dynamics of networks of rate neurons, Sompolinsky and his colleagues assumed that the synaptic weights in neuroscience, were random variables i.i.d. with zero mean Gaussian laws. The main result obtained by Sompolinsky and his colleagues (using the local chaos hypothesis) under the previous hypotheses is that the averaged law of the neurons dynamics is chaotic in the sense that the averaged law of a finite number of neurons converges to a product measure as the system gets very large.

The next efforts in the direction of understanding the averaged law of neurons are those of Cessac, Moynot and Samuelides [1,7,8,2,9]. From the technical viewpoint, the study of the collective dynamics is done in discrete time. Moynot and Samuelides obtained a large deviation principle and were able to describe in detail the limit averaged law that had been obtained by Cessac using the local chaos hypothesis and to prove rigorously the propagation of chaos property.

One of the next challenges is to incorporate in the network model the fact that the synaptic weights are not independent and in effect often highly correlated.

The problem we solve in this paper is the following. Given a completely connected network of neurons in which the synaptic weights are Gaussian correlated random variables, we describe the asymptotic law of the network when the number of neurons goes to infinity. Like in [7,8] we study a discrete time dynamics but unlike these authors we cope with more complex intrinsic dynamics of the neurons.

### *1.2 Mathematical framework*

For some topological space $\Omega$ equipped with its Borelian $\sigma$-algebra $\mathcal{B}(\Omega)$, we denote the set of all probability measures by $\mathcal{M}(\Omega)$. We equip $M(\Omega)$ with the topology of weak convergence. For some positive integer $n > 0$, we let $V_n = \{j \in \mathbb{Z} : |j| \leq n\}$. Let $\mathcal{T} = \mathbb{R}^{T+1}$, for some positive integer $T$. We equip $\mathcal{T}$ with the Euclidean topology, $\mathcal{T}^{\mathbb{Z}}$ with the cylindrical topology, and denote the Borelian $\sigma$-algebra generated by this topology by $\mathcal{B}(\mathcal{T}^{\mathbb{Z}})$. For some $\mu \in \mathcal{M}(\mathcal{T}^{\mathbb{Z}})$ governing a process $(X^j)_{j\in\mathbb{Z}}$, we let $\mu^{V_n} \in \mathcal{M}(\mathcal{T}^{V_n})$ denote the marginal governing $(X^j)_{j\in V_n}$. For some $j \in \mathbb{Z}$, let the shift operator $\mathcal{S}^j : \mathcal{T}^{\mathbb{Z}} \to \mathcal{T}^{\mathbb{Z}}$ be $S(\omega)^k = \omega^{j+k}$. We let $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ be the set of all stationary probability measures $\mu$ on $(\mathcal{T}^{\mathbb{Z}}, \mathcal{B}(\mathcal{T}^{\mathbb{Z}}))$ such that for all $j \in \mathbb{Z}^d$, $\mu \circ (\mathcal{S}^j)^{-1} = \mu$. Let $p_n : \mathcal{T}^{V_n} \to \mathcal{T}^{\mathbb{Z}}$ be such that $p_n(\omega)^k = \omega^{k \mod V_n}$. Here, and throughout the paper, we take $k \mod V_n$ to be the element $l \in V_n$ such that $l = k \mod (2n+1)$. Define the process-level empirical measure $\hat{\mu}_n : \mathcal{T}^{V_n} \to \mathcal{M}_{\mathcal{S}}\left(\mathcal{T}^{\mathbb{Z}}\right)$ as

$$\hat{\mu}_n(\omega) = \frac{1}{2n+1} \sum_{k\in V_n} \delta_{S^k p_n(\omega)}. \tag{1}$$

Let $(Y^j)$ be a stationary Process on $\mathcal{T}$ such that the $Y^j$s are independent. Each $Y^j$ is governed by a law $P$, and we write the governing law in $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ as $P^{\mathbb{Z}}$. It is clear that the governing law over $V_n$ may be written as $P^{\otimes V_n}$ (that is the product measure of $P$, indexed over $V_n$).

The equation describing the time variation of the membrane potential $U^j$ of the $j$th neuron writes

$$U_t^j = \gamma U_{t-1}^j + \sum_{i \in V_n} J_{ji}^n f(U_{t-1}^i) + \theta^j + B_{t-1}^j, \quad j \in V_n \quad t = 1, \ldots, T. \qquad (2)$$

$f : \mathbb{R} \to ]0, 1[$ is a monotonically increasing Lipschitz continuous bijection. $\gamma$ is in $[0, 1)$ and determines the time scale of the intrinsic dynamics of the neurons. The $B_t^j$s are i.i.d. Gaussian random variables distributed as $\mathcal{N}_1(0, \sigma^2)$ [1]. They represent the fluctuations of the neurons' membrane potentials. The $\theta^j$s are i.i.d. as $\mathcal{N}_1(\bar{\theta}, \theta^2)$. The are independent of the $B_t^i$s and represent the current injected in the neurons. The $U_0^j$s are assumed to be independent random variables with law $\mu_I$.

The $J_{ij}^n$s are the synaptic weights. $J_{ij}^n$ represents the strength with which the 'presynaptic' neuron $j$ influences the 'postsynaptic' neuron $i$. They arise from a stationary Gaussian random field specified by its mean and covariance function

$$\mathbb{E}[J_{ij}^n] = \frac{\bar{J}}{2n+1} \quad cov(J_{ij}^n J_{kl}^n) = \frac{1}{2n+1} \Lambda \left( (k-i) \bmod V_n, (l-j) \bmod V_n \right),$$

$\Lambda$ is positive definite, let $\tilde{\Lambda}$ be the corresponding (positive) Fourier transform. We make the technical assumption that the series $(\Lambda(i,j))_{i,j \in \mathbb{Z}}$ is absolutely convergent to $\Lambda^{asum} > 0$ and convergent to $\Lambda^{sum} > 0$.

We note $J^n$ the $(2n+1) \times (2n+1)$ matrix of the synaptic weights, $J^n = (J_{ij}^n)_{i,j \in V_n}$.

The process $(Y^j)$ defined by

$$Y_t^j = \gamma Y_{t-1}^j + \bar{\theta} + B_{t-1}^j, \quad j \in V_n, \quad t = 1, \cdots T, \quad Y_0^j = U_0^j$$

is stationary and independent. The law of each $Y^j$ is easily found to be given by

$$P = (\mathcal{N}_T(0_T, \sigma^2 \mathrm{Id}_T) \otimes \mu_I) \circ \Psi,$$

where $\Psi : \mathcal{T} \to \mathcal{T}$ is the following affine bijection. Writing $v = \Psi(u)$, we define

$$\begin{cases} v_0 = \Psi_0(u) = u_0 \\ v_s = \Psi_s(u) = u_s - \gamma u_{s-1} - \bar{\theta} \quad s = 1, \cdots, T. \end{cases} \qquad (3)$$

We extend $\Psi$ to a mapping $\mathcal{T}^{\mathbb{Z}} \to \mathcal{T}^{\mathbb{Z}}$ componentwise.

The application $\Psi$ defined in (3) plays a central role in the sequel we introduce the following definition.

1. We note $\mathcal{N}_p(m, \Sigma)$ the law of the $p$-dimensional Gaussian variable with mean $m$ and covariance matrix $\Sigma$.

**Définition** 1.1 *For each measure* $\mu \in \mathcal{M}(\mathcal{T}^{V_n})$ *or* $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ *we define* $\underline{\mu}$ *to be* $\mu \circ \Psi^{-1}$.

We next introduce the following definitions.

**Définition** 1.2 *Let* $\mathcal{E}_2$ *be the subset of* $\mathcal{M}_s(\mathcal{T}^{\mathbb{Z}})$ *defined by*

$$\mathcal{E}_2 = \{\mu \in \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}) \,|\, \mathbb{E}^{\underline{\mu}_{1,T}}[\|v^0\|^2] < \infty\}.$$

*We define the process-level entropy to be, for* $\mu \in \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$

$$I^{(3)}(\mu, P^{\mathbb{Z}}) = \lim_{n\to\infty} \frac{1}{(2n+1)} I^{(2)}\left(\mu^{V_n}, P^{\otimes V_n}\right).$$

*If* $\mu \notin \mathcal{E}_2$, *then* $I^{(3)}(\mu, P^{\mathbb{Z}}) = \infty$. *For further discussion of this rate function, and a proof that* $I^{(3)}$ *is well-defined, see [3].*

We note $Q^{V_n}(J^n)$ the element of $\mathcal{M}(\mathcal{T}^{V_n})$ which is the law of the solution to (2) conditioned on $J^n$. We let $Q^{V_n} = \mathbb{E}^J[Q^{V_n}(J^n)]$ be the law averaged with respect to the weights. The reason for this is that we want to study the empirical measure $\hat{\mu}_n$ on path space. There is no reason for this to be a simple problem since for a fixed interaction $J^n$, the variables $(U^{-n}, \cdots, U^n)$ are not exchangeable. So we first study the law of $\hat{\mu}_n$ averaged over the interactions.

Finally we introduce the image laws in terms of which the principal results of this paper are formulated.

**Définition** 1.3 *Let* $\Pi^n$ *and* $R^n$ *in* $\mathcal{M}(\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}))$ *be the image laws of* $Q^{V_n}$ *and* $P^{\otimes V_n}$ *through the function* $\hat{\mu}_n : \mathcal{T}^{V_n} \to \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ *defined by* (1)*:*

$$\Pi^n = Q^{V_n} \circ \hat{\mu}_n \quad R^n = P^{\otimes V_n} \circ \hat{\mu}_n$$

## 2 The good rate function

We obtain an LDP for the process with correlations $(\Pi^n)$ via the (simpler) process without correlations $(R^n)$. To do this we obtain an expression for the Radon-Nikodym derivative of $\Pi^n$ with respect to $R^n$. This is done in propositions 2.4 and 2.5. In equation (13) there appear certain Gaussian random variables defined from the right handside of the equations of the neuronal dynamics (2). Applying the Gaussian calculus to this expression we obtain equation (14) which expresses the Radon-Nikodym derivative as a function (depending on $n$) of the empirical measure (1). Using the fact that this function is measurable we obtain equation (15). This equation is essential in a) finding the expression for the rate function $H$ of definition 3.1, b) proving the lower-bound for $\Pi^n$ on the open sets, c) proving that the sequence $(\Pi^n)$ is exponentially tight.

The key idea is to associate to every stationary measure $\mu$ a certain stationary Gaussian process $G^\mu$, or equivalently a certain Gaussian measure defined by its mean $c^\mu$ and its covariance operator $K^\mu$.

Given $\mu$ in $\mathcal{M}_\mathcal{S}(\mathcal{T}^\mathbb{Z})$ we define a stationary Gaussian process $G^\mu$, i.e. a measure $\mathcal{Q}^\mu \in \mathcal{M}_\mathcal{S}(\mathcal{T}_{1,T}^\mathbb{Z})$. For all $i$ the mean of $G_t^{\mu,i}$ is given by $c_t^\mu$, where

$$c_t^\mu = \bar{J} \int_{\mathcal{T}^\mathbb{Z}} f(u_{t-1}^i) d\mu(u), \; t = 1, \cdots, T \,, i \in \mathbb{Z}, \tag{4}$$

The covariance between the Gaussian vectors $G^{\mu,i}$ and $G^{\mu,i+k}$ is defined to be

$$K^{\mu,k} = \theta^2 \delta_k 1_T{}^\dagger 1_T + \sum_{l=-\infty}^{\infty} \Lambda(k,l) M^{\mu,l}, \tag{5}$$

where

$$M_{st}^{\mu,k} = \int_{\mathcal{T}^\mathbb{Z}} f(u_{s-1}^0) f(u_{t-1}^k) d\mu(u), \tag{6}$$

The above integrals are well-defined because of the definition of $f$ and that the series in (5) is convergent since the series $(\Lambda(k,l))_{k,\,l\in\mathbb{Z}}$ is absolutely convergent and the elements of $M^{\mu,l}$ are bounded by 1 for all $l \in \mathbb{Z}$.

These definitions imply the existence of a Hermitian-valued spectral representation for the sequence $M^{\mu,k}$ (resp. $K^{\mu,k}$) noted $\tilde{M}^\mu$ (resp. $\tilde{K}^\mu$) which satisfies

$$\tilde{K}^\mu(\theta) = \theta^2 1_T{}^t 1_T + \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\Lambda}(\theta, -\varphi) \tilde{M}^\mu(d\varphi).$$

We also use the partial sums, noted $K_{[n]}^{\mu,k}$, $k \in V_n$, in (5), to define another sequence $A_{[n]}^{\mu,k}$ which is used to define in the limit $n \to \infty$

$$\tilde{A}^\mu(\theta) = \tilde{K}^\mu(\theta)(\sigma^2 \mathrm{Id}_T + \tilde{K}^\mu(\theta))^{-1}. \tag{7}$$

We next define a functional $\Gamma_{[n]} = \Gamma_{[n],1} + \Gamma_{[n],2}$, which we use to characterise the Radon-Nikodym derivative of $\Pi^n$ with respect to $R^n$. Let $\mu \in \mathcal{M}_\mathcal{S}(\mathcal{T}^\mathbb{Z})$ and

$$\Gamma_{[n],1}(\mu) = -\frac{1}{2(2n+1)} \log\left(\det\left(\mathrm{Id}_{(2n+1)T} + \frac{1}{\sigma^2} K_{[n]}^\mu\right)\right), \tag{8}$$

where $K_{[n]}^\mu$ the $((2n+1)T \times (2n+1)T)$ covariance matrix of the law $\mathcal{Q}^{\mu,V_n}$ Because of previous remarks the above expression has a sense. Taking the limit when $N \to \infty$ does not pose any problem and we can define $\Gamma_1(\mu) = \lim_{n\to\infty} \Gamma_{[n],1}(\mu)$. The following lemma whose proof is again straightforward indicates that this is well-defined.

**Lemma 2.1** *When $N$ goes to infinity the limit of* (8) *is given by*

$$\Gamma_1(\mu) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \log\left(\det\left(\mathrm{Id}_T + \frac{1}{\sigma^2} \tilde{K}^\mu(\theta)\right)\right) d\theta \tag{9}$$

*for all* $\mu \in \mathcal{M}_{1,S}^{+}(\mathcal{T}^{\mathbb{Z}})$*.*

It also follows easily from previous remarks that

**Proposition 2.1** $\Gamma_{[n],1}$ *and* $\Gamma_1$ *are bounded below and continuous on* $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$*.*

The definition of $\Gamma_{[n],2}(\mu)$ is slightly more technical but follows directly from proposition 2.4. For $\mu \in \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ let

$$\Gamma_{[n],2}(\mu) = \int_{\mathcal{T}_{1,T}^{V_n}} \phi^n(\mu, v) \underline{\mu}_{1,T}^{V_n}(dv) \tag{10}$$

where $\phi^n : \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}) \times \mathcal{T}_{1,T}^{V_n} \to \mathbb{R}$ is defined by

$$\phi^n(\mu, v) = \frac{1}{2\sigma^2}\left(\frac{1}{2n+1}\sum_{j,k=-n}^{n} {}^{\dagger}(v^j - c^\mu) A_{[n]}^{\mu,k}(v^{k+j} - c^\mu) + \frac{2}{2n+1}\sum_{j=-n}^{n}\langle c^\mu, v^j\rangle - \|c^\mu\|^2\right). \tag{11}$$

$\Gamma_{[n],2}(\mu)$ is finite in the subset $\mathcal{E}_2$ of $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ defined in definition 1.2. If $\mu \notin \mathcal{E}_2$, then we set $\Gamma_{[n],2}(\mu) = \infty$.

We define $\Gamma_2(\mu) = \lim_{N\to\infty} \Gamma_{[n],2}(\mu)$. The following proposition indicates that $\Gamma_2(\mu)$ is well-defined.

**Proposition 2.2** *If the measure* $\mu$ *is in* $\mathcal{E}_2$*, i.e. if* $\mathbb{E}^{\underline{\mu}_{1,T}}[\|v^0\|^2] < \infty$*, then* $\Gamma_2(\mu)$ *is finite and writes*

$$\Gamma_2(\mu) = \frac{1}{2\sigma^2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{A}^\mu(-\theta) : \tilde{v}^\mu(d\theta) + {}^{\dagger}c^\mu(\tilde{A}^\mu(0) - \mathrm{Id}_T)c^\mu + 2\mathbb{E}^{\underline{\mu}_{1,T}}\left[{}^{t}v^0(\mathrm{Id}_T - \tilde{A}^\mu(0))c^\mu\right]\right). \tag{12}$$

*The ":" symbol indicates the double contraction on the indexes.*

It is shown in [5] that $\phi^n(\mu, v)$ defined by (11) is a continuous function of $\mu$ which satisfies

$$\phi^n(\mu, v) \geq -\beta_2, \quad \beta_2 = \frac{T\bar{J}^2}{2\sigma^2\Lambda^{sum}}(\sigma^2 + \theta^2 + \Lambda^{asum})$$

By a standard argument we otain the following proposition.

**Proposition 2.3** $\Gamma_{[n],2}(\mu)$ *is lower-semicontinuous.*

We define $\Gamma_{[n]}(\mu) = \Gamma_{[n],1}(\mu) + \Gamma_{[n],2}(\mu)$. We may conclude from propositions 2.1 and 2.3 that $\Gamma_{[n]}$ is lower-semicontinuous hence measurable.

From these definitions it is relatively easy, and proved in [5], to show that the measure $Q^{V_n}$ is absolutely continuous with respect to $P^{\otimes V_n}$ with a Radon-

Nikodym derivative which can be expressed as a function of the functional $\Gamma$.

**Proposition 2.4** *The Radon-Nikodym derivative of $Q^{V_n}$ with respect to $P^{\otimes V_n}$ is given by the following expression.*

$$\frac{dQ^{V_n}}{dP^{\otimes V_n}}(u) = \mathbb{E}\left[\exp\left(\frac{1}{\sigma^2}\left(\sum_{j\in V_n}\langle \Psi_{1,T}(u^j), G^j\rangle - \frac{1}{2}\|G^j\|^2\right)\right)\right], \tag{13}$$

*the expectation being taken against the $N$ $T$-dimensional Gaussian processes $(G^i)$, $i \in V_n$ given by*

$$G_t^i = \sum_{j\in V_n} J_{ij}^N f(u_{t-1}^j), \quad t = 1, \cdots, T,$$

*and the function $\Psi$ being defined by* (3)*.*

Using standard Gaussian calculus we obtain the following proposition.

**Proposition 2.5** *The Radon-Nikodym derivatives write as*

$$\frac{dQ^{V_n}}{dP^{\otimes V_n}}(u) = \exp(N\Gamma_{[n]}(\hat{\mu}_n(u)), \tag{14}$$

$$\frac{d\Pi^n}{dR^n}(\mu) = \exp((2n+1)\Gamma_{[n]}(\mu)). \tag{15}$$

*Here $\mu \in \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$, $\Gamma_{[n]}(\mu) = \Gamma_{[n],1}(\mu) + \Gamma_{[n],2}(\mu)$ and the expressions for $\Gamma_{[n],1}$ and $\Gamma_{[n],2}$ have been defined in equations* (8) *and* (10)*.*

## 3 The large deviation principle

We define the function $H : (\mathcal{T}^{\mathbb{Z}}) \to [0, +\infty)$ as follows.

**Définition** 3.1 *Let $H$ be the function $\mathcal{M}_{1,S}^+(\mathcal{T}^{\mathbb{Z}}) \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$H(\mu) = \begin{cases} +\infty & \text{if} \quad I^{(3)}(\mu, P^{\mathbb{Z}}) = \infty \\ I^{(3)}(\mu, P^{\mathbb{Z}}) - \Gamma(\mu) & \text{otherwise}, \end{cases}$$

*where $\Gamma = \Gamma_1 + \Gamma_2$.*

We state the following theorem.

**Theorem 3.1** $\Pi^n$ *is governed by a large deviation principle with a good rate function $H$.*

The proof is too long to be reproduced here, see [5]. We only give the general strategy. First we prove the lower bound on the open sets. For the upper bound on the closed sets, we simply avoid it by a) proving that $(\Pi^n)$ is exponentially tight which allows us to b) restrict the proof of the upper bound to compact sets. The proof of b) is long and technical. It is built upon ideas found in [6].

Note that we have found an analytical form for $H$ through equations (9) and (12)

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