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Asymptotic description of stochastic neural networks. II - Characterization of the limit law

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Abstract

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Résumé

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Version française abrégée

1 Introduction

In [2] we started our asymptotic analysis of very large networks of neurons with correlated synaptic weights. We showed that the image Π^n of the averaged law Q^{V_n} through the empirical measure satisfied a large deviation principle with good rate function H. In the same article we provided an analytical expression of this rate function in terms of the spectral representation of certain Gaussian processes. In the next section we recall some definitions given in [2].

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2 Mathematical framework

For some topological space Ω equipped with its Borelian σ-algebra $\mathcal{B}(\Omega)$, we denote the set of all probability measures by $\mathcal{M}(\Omega)$. We equip $\mathcal{M}(\Omega)$ with the topology of weak convergence. For some positive integer $n > 0$, we let $V_n = \{j \in \mathbb{Z} : |j| \leq n\}$, and $|V_n| = 2n+1$. Let $\mathcal{T} = \mathbb{R}^{T+1}$, for some positive integer T. We equip $\mathcal{T}$ with the Euclidean topology, $\mathcal{T}^{\mathbb{Z}}$ with the cylindrical topology, and denote the Borelian σ-algebra generated by this topology by $\mathcal{B}(\mathcal{T}^{\mathbb{Z}})$. For some $\mu \in \mathcal{M}(\mathcal{T}^{\mathbb{Z}})$ governing a process $(X^j)_{j\in\mathbb{Z}}$, we let $\mu^{V_n} \in \mathcal{M}(\mathcal{T}^{V_n})$ denote the marginal governing $(X^j)_{j\in V_n}$. For some $j \in \mathbb{Z}$, let the shift operator $\mathcal{S}^j : \mathcal{T}^{\mathbb{Z}} \to \mathcal{T}^{\mathbb{Z}}$ be $S(\omega)^k = \omega^{j+k}$. We let $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ be the set of all stationary probability measures μ on $(\mathcal{T}^{\mathbb{Z}}, \mathcal{B}(\mathcal{T}^{\mathbb{Z}}))$ such that for all $j \in \mathbb{Z}$, $\mu \circ (\mathcal{S}^j)^{-1} = \mu$. Let $p_n : \mathcal{T}^{V_n} \to \mathcal{T}^{\mathbb{Z}}$ be such that $p_n(\omega)^k = \omega^{k \mod V_n}$. Here, and throughout the paper, we take $k \mod V_n$ to be the element $l \in V_n$ such that $l = k \mod |V_n|$. Define the process-level empirical measure $\hat{\mu}_n : \mathcal{T}^{V_n} \to \mathcal{M}_{\mathcal{S}}\left(\mathcal{T}^{\mathbb{Z}}\right)$ as

$$\hat{\mu}_n(\omega) = \frac{1}{|V_n|} \sum_{k\in V_n} \delta_{S^k p_n(\omega)}. \tag{1}$$

The equation describing the time variation of the membrane potential U^j of the jth neuron writes

$$U_t^j = \gamma U_{t-1}^j + \sum_{i\in V_n} J_{ji}^n f(U_{t-1}^i) + \theta^j + B_{t-1}^j,, \quad U_0^j = u_0^j, \quad j \in V_n \quad t = 1, \dots, T \tag{2}$$

$f : \mathbb{R} \to]0, 1[$ is a monotonically increasing Lipschitz continuous bijection. γ is in $[0, 1)$ and determines the time scale of the intrinsic dynamics of the neurons. The B_t^js are i.i.d. Gaussian random variables distributed as $\mathcal{N}_1(0, \sigma^2)$ [1]. They represent the fluctuations of the neurons' membrane potentials. The θ^js are i.i.d. as $\mathcal{N}_1(\bar{\theta}, \theta^2)$. The are independent of the B_t^is and represent the current injected in the neurons. The u_0^js are i.i.d. random variables each governed by law μ_I.

The J_{ij}^ns are the synaptic weights. J_{ij}^n represents the strength with which the 'presynaptic' neuron j influences the 'postsynaptic' neuron i. They arise from a stationary Gaussian random field specified by its mean and covariance function Λ, see [1,2].

We note J^n the $|V_n| \times |V_n|$ matrix of the synaptic weights, $J^n = (J_{ij}^n)_{i,j\in V_n}$.

1. We note $\mathcal{N}_p(m, \Sigma)$ the law of the p-dimensional Gaussian variable with mean m and covariance matrix Σ.

where $\Psi : \mathcal{T} \to \mathcal{T}$ is the following affine bijection. Writing $v = \Psi(u)$, we define

$$\begin{cases} v_0 = \Psi_0(u) = u_0 \\ v_s = \Psi_s(u) = u_s - \gamma u_{s-1} - \bar{\theta} \quad s = 1, \cdots, T. \end{cases} \tag{3}$$

We extend Ψ to a mapping $\mathcal{T}^{\mathbb{Z}} \to \mathcal{T}^{\mathbb{Z}}$ componentwise and introduce the following notation.

Définition 2.1. *For each measure $\mu \in \mathcal{M}(\mathcal{T}^{V_n})$ or $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ we define $\underline{\mu}$ to be $\mu \circ \Psi^{-1}$.*

We note $Q^{V_n}(J^n)$ the element of $\mathcal{M}(\mathcal{T}^{V_n})$ which is the law of the solution to (2) conditioned on J^n. We let $Q^{V_n} = \mathbb{E}^J[Q^{V_n}(J^n)]$ be the law averaged with respect to the weights.

Finally we introduce the image law in terms of which the principal results of this paper are formulated.

Définition 2.2. *Let $\Pi^n \in \mathcal{M}(\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}))$ be the image law of Q^{V_n} through the function $\hat{\mu}_n : \mathcal{T}^{V_n} \to \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ defined by* (1)*:*

$$\Pi^n = Q^{V_n} \circ \hat{\mu}_n^{-1}$$

3 Characterization of the unique minimum of the rate function

In [1], to each measure $\nu \in \mathcal{M}(\mathcal{T}^{\mathbb{Z}})$ we associate the measure, noted Q^ν of $\mathcal{M}(\mathcal{T}^{\mathbb{Z}})$ defined by $Q^\nu = \mu_I^{\mathbb{Z}} \otimes \underline{Q}^\nu_{1,T}$ where $\underline{Q}^\nu_{1,T}$ is a Gaussian measure on $\mathcal{T}^{\mathbb{Z}}_{1,T}$ with spectral density $\tilde{K}^\mu(\theta)$. We also define the rate function H^ν, which is a linear approximation of the functional Γ defined in [1]. We prove the following lemma in [1].

Lemma 3.1. *For $\mu, \nu \in \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$, $H^\nu(\mu) = 0$ if and only if $\mu = Q^\nu$.*

As stated in the following proposition, proved in [1], there exists a unique minimum μ_e of the rate function. We provide explicit equations for μ_e which would facilitate its numerical simulation.

Proposition 3.1. *There is a unique distribution $\mu_e \in \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ which minimises H. This distribution satisfies $H(\mu_e) = 0$ which is equivalent to $\mu_e = Q^{\mu_e}$.*

Proof. By the previous lemma, it suffices to prove that there is a unique μ_e such that

$$Q^{\mu_e} = \mu_e. \tag{4}$$

We define the mapping $L : \mathcal{M}^+_{1,S}(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{M}^+_{1,S}(\mathcal{T}^{\mathbb{Z}})$ by

$$\mu \to L(\mu) = Q^\mu.$$

By the definition of Q^{μ} we have $Q^{\mu}_0 = \mu_I^{\mathbb{Z}}$, which is independent of μ.

It may be inferred from the definitions in Section 2 of [2] that the marginal of $L(\mu) = Q^{\mu}$ over $\mathcal{F}_{0,t}$ only depends upon the marginal of μ over $\mathcal{F}_{0,t-1}$, $t \geq 1$. This follows from the fact that $\underline{Q}^{\mu}_{1,t}$ (which determines $Q^{\mu}_{0,t}$) is completely determined by the means $\{c^{\mu}_s; s = 1, \ldots, t\}$ and covariances $\{K^{\mu,j}_{uv}; j \in \mathbb{Z}, u, v \in [1,t]\}$. In turn, it may be observed from equations (4-5) in [2] that these variables are determined by $\mu_{0,t-1}$. Thus for any $\mu, \nu \in \mathcal{M}^+_{1,S}(\mathcal{T}^{\mathbb{Z}})$ and $t \in [1,T]$, if

$$\mu_{0,t-1} = \nu_{0,t-1},$$

then

$$L(\mu)_{0,t} = L(\nu)_{0,t}.$$

It follows from repeated application of the above identity that for any ν satisfying $\nu_0 = \mu_I^{\mathbb{Z}}$,

$$L^T(\nu)_{0,T} = L(L^T(\nu))_{0,T}. \tag{5}$$

Defining

$$\mu_e = L^T(\nu), \tag{6}$$

it follows from (5) that μ_e satisfies (4).

Conversely if $\mu = L(\mu)$ for some μ, then we have that $\mu = L^2(\nu)$ for any ν such that $\nu_{0,T-2} = \mu_{0,T-2}$. Continuing this reasoning, we find that $\mu = L^T(\nu)$ for any ν such that $\nu_0 = \mu_0$. But because $Q^{\mu}_0 = \mu_I^{\mathbb{Z}}$, since $Q^{\mu} = \mu$, we have $\mu_0 = \mu_I^{\mathbb{Z}}$. But we have just seen that any μ satisfying $\mu = L^T(\nu)$, where $\nu_0 = \mu_I^{\mathbb{Z}}$, is uniquely defined by (6), which means that $\mu = \mu_e$. □

We may use the proof of proposition 3.1 to characterize the unique measure μ_e such that $\mu_e = Q^{\mu_e}$ in terms of its image $\underline{\mu_e}$. This characterization allows one to directly numerically calculate μ_e. We characterize $\underline{\mu_e}$ recursively (in time), by providing a method of determining $\underline{\mu_e}_{0,t}$ in terms of $\underline{\mu_e}_{0,t-1}$. However we must firstly outline explicitly the bijective correspondence between $\mu_{e0.t}$ and $\underline{\mu_e}_{0,t}$, as follows. For $v \in \mathcal{T}$, we write $\Psi^{-1}(v) = (\Psi^{-1}(v)_0, \ldots, \Psi^{-1}(v)_T)$. The coordinate $\Psi^{-1}(v)_t$ is the affine function of v_s, $s = 0 \cdots t$ obtained from equations (3)

$$\Psi^{-1}(v)_t = \sum_{i=0}^{t} \gamma^i v_{t-i} + \bar{\theta}\frac{\gamma^t - 1}{\gamma - 1}.,\ \ t = 0, \cdots, T.$$

Let $K^{\mu_e,l}_{(t-1,s-1)}$ be the $(t-1) \times (s-1)$ submatrix of $K^{\mu_e,l}$ composed of the rows from times 1 to $(t-1)$ and the columns from times 1 to $(s-1)$, and

$$c^{\mu_e}_{(t-1)} = {}^{\dagger}(c^{\mu_e}_1, \ldots, c^{\mu_e}_{t-1}).$$

Let the measures $\underline{\mu_e}^1_{0,t} \in \mathcal{M}(\mathcal{T}_{0,t})$ and $\underline{\mu_e}^{(0,l)}_{t,s} \in \mathcal{M}(\mathcal{T}_{0,t} \times \mathcal{T}_{0,s})$ be given by

$$\underline{\mu_e}^1_{0,t}(dv) = \mu_I(dv_0) \otimes \mathcal{N}_t\left(c^{\mu_e}_{(t)}, \sigma^2 \mathrm{Id_t} + \mathrm{K}^{\mu_e,0}_{(\mathrm{t,t})}\right) dv_1 \cdots dv_t.$$
$$\underline{\mu_e}^{(0,l)}_{(t,s)}(dv^0 dv^l) = \mu_I(dv^0_0) \otimes \mu_I(dv^l_0) \otimes \mathcal{N}_{t+s}((c^{\mu_e}_{(t)}, c^{\mu_e}_{(s)}), \sigma^2 \mathrm{Id}_{t+s} + K^{\mu_e,(0,l)}_{(t,s)}) dv^0_1 \cdots dv^0_t dv^l_1 \cdots dv^l_s,$$

where

$$K^{\mu_e,(0,l)}_{(t,s)} = \begin{bmatrix} K^{\mu_e,0}_{(t,t)} & K^{\mu_e,l}_{(t,s)} \\ {}^\dagger K^{\mu_e,l}_{(t,s)} & K^{\mu_e,0}_{(s,s)} \end{bmatrix}.$$

The inductive method for calculating $\underline{\mu_e}$ is outlined in the theorem below.

Theorem 3.2. *We may characterise $\underline{\mu_e}$ inductively as follows. Initially $\underline{\mu_e}_0 = \mu_I^{\mathbb{Z}}$. Given that we have a complete characterisation of $\left\{\underline{\mu_e}^{(0,l)}_{(0,t-1)}, \underline{\mu_e}^1_{0,t-1} : l \in \mathbb{Z}\right\}$,we may characterise $\left\{\underline{\mu_e}^{(0,l)}_{(0,t)}, \underline{\mu_e}^1_{0,t} : l \in \mathbb{Z}\right\}$ according to the following identities. For $s \in [1, t]$,*

$$c^{\mu_e}_s = \bar{J} \int_{\mathbb{R}^t} \left(f\left(\Psi^{-1}(v)_{s-1}\right)\right) \underline{\mu_e}^1_{0,s-1}(dv). \tag{7}$$

For $1 \leq r, s \leq t$, $K^{\mu_e,k}_{rs} = \theta^2 \delta_k 1_T {}^\dagger 1_T + \sum_{l=-\infty}^{\infty} \Lambda(k,l) M^{\mu,l}_{rs}$. Here, for $p =$ $\max(r-1, s-1)$,

$$M^{\mu_e,0}_{rs} = \int_{\mathbb{R}^{p+1}} \left(f(\Psi^{-1}(v)_{r-1})\right) \times \left(f(\Psi^{-1}(v)_{s-1})\right) \underline{\mu_e}^1_{0,p}(dv), \tag{8}$$

and for $l \neq 0$

$$M^{\mu_e,l}_{rs} = \int_{\mathbb{R}^r \times \mathbb{R}^s} \left(f(\Psi^{-1}(v^0)_{r-1})\right) \times \left(f(\Psi^{-1}(v^l)_{s-1})\right) \underline{\mu_e}^{(0,l)}_{(r-1,s-1)}(dv^0 dv^l). \tag{9}$$

Remark 1. *From a numerical point of view the t-dimensional integral in* (7) *and the* $\max(r,s)$*-dimensional integral in* (8) *can be reduced by a change of variable to at most two dimensions. Similarly the $r+s$-dimensional integral in* (9) *can be reduced to at most four dimensions.*

Remark 2. *If we make the biologically reaslistic assumption that the synaptic weights are not correlated beyond a certain correlation distance $d \geq 0$, $\Lambda(k,l) = 0$ if k or l does not belong to V_d it is seen that the matrixes K^{μ_e} are 0 as soon as $k \notin V_d$: the asymptotic description of the network of neurons is sparse.*

4 Convergence results

We use the Large Deviation Principle proved in [2,1] to establish convergence results for the measures Π^n, Q^{V_n} and $Q^{V_n}(J^n)$.

Theorem 4.1. Π^n *converges weakly to δ_{μ_e}, i.e., for all $\Phi \in \mathcal{C}_b(\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}))$,*

$$\lim_{n\to\infty} \int_{\mathcal{T}^{V_n}} \Phi(\hat{\mu}_n(u))\, Q^{V_n}(du) = \Phi(\mu_e).$$

Similarly,

$$\lim_{n\to\infty}\int_{\mathcal{T}^{V_n}}\Phi(\hat{\mu}_n(u))\,Q^{V_n}(J^n)(du)=\Phi(\mu_e)\quad J\quad \textit{almost surely}$$

Proof. The proof of the first result follows directly from the existence of an LDP for the measure Π^n, see theorem 3.1 in [2], and is a straightforward adaptation of the one in [3, Theorem 2.5.1]. The proof of the second result uses the same method, making use of theorem 4.2 below. □

We can in fact obtain the following quenched convergence analogue of the usual lower bound inequality in the definition of a Large Deviation Principle.
Theorem 4.2. *For each closed set F of $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}})$ and for almost all J*

$$\overline{\lim_{n\to\infty}}\frac{1}{|V_n|}\log\left[Q^{V_n}(J^n)(\hat{\mu}_n\in F)\right]\le-\inf_{\mu\in F}H(\mu).$$

Proof. The proof is a combination of Tchebyshev's inequality and the Borel-Cantelli lemma and is an adaptation of the one in [3, Theorem 2.5.4, Corollary 2.5.6]. □

We define $\check{Q}^{V_n}(J^n)=\frac{1}{|V_n|}\sum_{j\in V_n}Q^{V_n}(J^n)\circ\mathcal{S}^{-j}$, where we recall the shift operator $\mathcal{S}$. Clearly $\check{Q}^{V_n}(J^n)$ is in $\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{V_n})$.
Corollary 4.3. *Fix m and let $n>m$. For almost every J and all $h\in\mathcal{C}_b(\mathcal{T}^{V_m})$,*

$$\lim_{n\to\infty}\int_{\mathcal{T}^{V_m}}h(u)\,\check{Q}^{V_n,V_m}(J^n)(du)=\int_{\mathcal{T}^{V_m}}h(u)\,\mu_e^{V_m}(du).$$
$$\lim_{n\to\infty}\int_{\mathcal{T}^{V_m}}h(u)\,Q^{V_n,V_m}(du)=\int_{\mathcal{T}^{V_m}}h(u)\,\mu_e^{V_m}(du).$$

That is, the V_m^{th} marginals $\check{Q}^{V_n,V_m}(J^N)$ and Q^{V_n,V_m} converge weakly to $\mu_e^{V_m}$ as $n\to\infty$.

Proof. It is sufficient to apply theorem 4.1 in the case where Φ in $\mathcal{C}_b(\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}))$ is defined by

$$\Phi(\mu)=\int_{\mathcal{T}^{V_m}}h\,d\mu^{V_m}$$

and to use the fact that $Q^{V_n},\check{Q}^{V_n}(J)\in\mathcal{M}_{\mathcal{S}}(\mathcal{T}^{V_n})$. □

We now prove the following ergodic-type theorem. We may represent the ambient probability space by $\mathfrak{W}$, where $\omega\in\mathfrak{W}$ is such that $\omega=(J_{ij},B_t^j,u_0^j)$, where $i,j\in\mathbb{Z}$ and $0\le t\le T-1$, recall (2). We denote the probability measure governing ω by $\mathfrak{P}$. Let $u^{(n)}(\omega)\in\mathcal{T}^{V_n}$ be defined by (2). As an aside, we may then understand $Q^{V_n}(J^n)$ to be the conditional law of $\mathfrak{P}$ on $u^{(n)}(\omega)$, for given J^n.

Theorem 4.4. *Fix $m > 0$ and let $h \in C_b(\mathcal{T}^{V_m})$. For $u^{(n)}(\omega) \in \mathcal{T}^{V_n}$ (where $n > m$) $\mathfrak{P}$ almost surely,*

$$\lim_{n\to\infty} \frac{1}{|V_n|} \sum_{j\in V_n} h\left(\pi^{V_m}(\mathcal{S}^j u^{(n)}(\omega))\right) = \int_{\mathcal{T}^{V_m}} h(u) d\mu_e^{V_m}(u), \tag{10}$$

Hence $\hat{\mu}_n(u^{(n)}(\omega))$ converges $\mathfrak{P}$-almost-surely to μ_e.

Proof. Our proof is an adaptation of [3]. We may suppose without loss of generality that $\int_{\mathcal{T}^{V_m}} h(u)d\mu_e^{V_m}(u) = 0$. For $p > 1$ let

$$F_p = \left\{\mu \in \mathcal{M}_{\mathcal{S}}(\mathcal{T}^{\mathbb{Z}}) | \left| \int_{\mathcal{T}^{V_m}} h(u)\, \mu^{V_m}(du) \right| \geq \frac{1}{p} \right\}.$$

Since $\mu_e \notin F_p$, but it is the unique zero of H, it follows that $\inf_{F_p} H = m > 0$. Thus by theorem 3.1 in [2] there exists an n_0, such that for all $n > n_0$,

$$Q^{V_n}\left(\hat{\mu}_n \in F_p\right) \leq \exp\left(-m|V_n|\right).$$

However

$$\mathfrak{P}\left(\omega | \hat{\mu}_n(u^{(n)}(\omega)) \in F_p\right) = Q^{V_n}\left(u|\hat{\mu}_n(u) \in F_p\right).$$

Thus

$$\sum_{n=0}^{\infty} \mathfrak{P}\left(\omega|\hat{\mu}_n(u^{(n)}(\omega)) \in F_p\right) < \infty.$$

We may thus conclude from the Borel-Cantelli Lemma that $\mathfrak{P}$ almost surely, for every $\omega \in \mathfrak{W}$, there exists n_p such that for all $n \geq n_p$,

$$\left| \frac{1}{|V_n|} \sum_{j\in V_n} h\left(\pi^{V_m} \mathcal{S}^j u^{(n)}(\omega)\right) \right| \leq \frac{1}{p}.$$

This yields (10) because p is arbitrary. The convergence of $\hat{\mu}_n(u^{(n)}(\omega))$ is a direct consequence of (10), since this means that each of the V_m^{th} marginals converge. □

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