Some singular sample path properties of a multiparameter fractional Brownian motion

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SOME SINGULAR SAMPLE PATH PROPERTIES OF A MULTIPARAMETER FRACTIONAL BROWNIAN MOTION

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Abstract

We prove a Chung-type law of the iterated logarithm for a multiparameter extension of the fractional Brownian motion which is not increment stationary. This multiparameter fractional Brownian motion behaves very differently at the origin and away from the axes, which also appears in the Hausdorff dimension of its range and in the measure of its pointwise Hölder exponents. A functional version of this Chung-type law is also provided.

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1 Introduction

In the 1920's, Khinchine introduced for the first time a law of the iterated logarithm for sums of independent and identically distributed random variables. Thereafter, many works extended this result and in particular Chung [5] presented a new law of the iterated logarithm for Brownian motion of the *lim inf* type, thus capturing the slowest local oscillations. This law was generalized over the last few decades in numerous ways to Gaussian [20, 28, 35] and non-Gaussian processes [3, 18, 19], Gaussian samples [23] and empirical processes [8], and Gaussian random fields [25, 26, 32], for a non-exhaustive list of works. A key step in establishing Chung-type laws of the iterated logarithm (abbreviated as LIL) is usually to determine the small ball probabilities, together with a good decomposition of the process into independent processes. Finding a Chung-type law seems generally more difficult than finding a standard LIL, as can be seen from the case of the fractional Brownian sheet, for which a LIL is given in [36], but a Chung-type law is far from clear, except in special cases [26, 32].

We propose to determine a Chung-type law of the iterated logarithm for a multiparameter extension of the fractional Brownian motion (multiparameter fBm for short), which is neither increment stationary (see Monrad and Rootzén [28], Talagrand [33], Xiao [37] and for a more general theory, Luan and Xiao [25] where this is an assumption) nor has an immediate decomposition as a sum of independent processes (as for the Brownian sheet [32]). This natural extension is a centred Gaussian process on $\mathbb{R}_+^\nu$, $\nu \in \mathbb{N}^*$, with covariance defined for any Hurst parameter $h \in (0, 1/2]$ by:

$$k_h^{(\nu)}(s,t) = \frac{1}{2}\left(\lambda([0,s])^{2h} + \lambda([0,t])^{2h} - \lambda([0,s] \triangle [0,t])^{2h}\right) \ , \ s,t \in \mathbb{R}_+^\nu \ , \tag{1.1}$$

where λ denotes the Lebesgue measure in $\mathbb{R}^\nu$, $[0,t]$ is the rectangle with vertices at 0 and t, and $\triangle$ is the symmetric difference of sets. This is a special case of a family of covariance on sets introduced by Herbin and Merzbach [15] to define the set-indexed fractional Brownian motion, and extended in [29] to its most general expression as a covariance on $L^2(T,m)$. This process differs from the other extensions that are the Lévy fractional Brownian motion, and the fractional Brownian sheet, although it shares several properties with them (see [29] for a more thorough discussion on the links between these processes). Besides, $h = 1/2$ in (1.1) yields the usual Brownian sheet. However, our results hold for $h < 1/2$ and cannot be extended to $h = 1/2$.

The distance induced by the Lévy fBm X^h and the multiparameter fBm $\boldsymbol{B}^h$, which are defined respectively as $d_{X^h}(s,t) = \sqrt{\mathbb{E}(X_s^h - X_t^h)^2} = \|s-t\|^h$ and $d_{\boldsymbol{B}^h}(s,t) = \lambda([0,s] \bigtriangleup [0,t])^h$, are in fact equivalent on a domain of $\mathbb{R}_+^\nu$ that does not approach the axes. Thus, it is expected that these processes will share certain sample path properties, at least away from the axes. This is the purpose of the work of Herbin and Xiao [16], where the authors propose a modulus of continuity, laws of the iterated logarithm and the Hausdorff dimension of the level sets of $\boldsymbol{B}^h$. These results coincide with their analogue for the Lévy fractional Brownian motion, but for the law of the iterated logarithm, which is a local result, this is only true away from the axes. The lack of stationarity forbids here to conclude that these modulus are the same whatever the point we choose. Note here that the same happens for the fractional Brownian sheet: its LIL is known away from 0 (see [27]), but not in the neighbourhood of the origin (except for particular increments as in [36]). If t_0 is not on the axes, the Chung-type law of the iterated logarithm given in [16] for the multiparameter fBm reads:

$$\liminf_{r\to 0^+} \left(\log\log(r^{-1})\right)^{h/\nu} \frac{\sup_{\|t\|\leq r} |\boldsymbol{B}^h_{t_0+t} - \boldsymbol{B}^h_{t_0}|}{r^h} = c \quad \text{a.s.}\ ,$$

for some deterministic c that may depend on t_0. Near 0, we will show that the local modulus is in fact of order $r^{\nu h}\ \widetilde{\Psi}_h(r)$, where $\widetilde{\Psi}_h$ is a logarithmic correcting term, which differs significantly from r^h (as soon as $\nu \geq 2$) and justifies this notion of singularity at the origin. The main new ingredients are a sharp estimate of the small ball probabilities, and a spectral representation of the multiparameter fBm, in an abstract Wiener space. Besides this singularity, the aforementioned techniques apply to the more general class of L^2-indexed fractional Brownian motion (which includes e.g. the Lévy fBm [29]), and could yield more general results.

Statement of the main results. The spectral representation we obtain is related to stable measures in Banach spaces: we prove that for H the reproducing kernel Hilbert space (RKHS, see definition below) of the ν-dimensional Brownian sheet, there exists an abstract Wiener space (H, E, μ) such that for any $h \in (0, 1/2)$, there is a strictly stable measure Γ^h on E whose characteristic function is given by $\exp\{-\|S\xi\|_H^{4h}/2\}$, $\xi \in E^*$. This measure has a Lévy-Khintchine decomposition, with Lévy measure Δ^h. We let $\mathscr{I}$ denote the Paley-Wiener map (defined further), and let $\mathbb{B}^h$ be the white noise on the Borel sets of E with control measure Δ^h. Then, letting $\varphi_t(\cdot) = \lambda(\mathbf{1}_{[0,t]}\cdot) \in H$, the multiparameter fractional Brownian motion has the following representation (Proposition 2.4):

$$\boldsymbol{B}^h_t = \int_E \left(1 - e^{i\langle \mathscr{I}(\varphi_t), x\rangle}\right)\ \mathrm{d}\mathbb{B}^h_x\ , \quad t \in [0,1]^\nu\ .$$

Using this representation and the sharp estimate of the small deviations given in Proposition 2.9, we will provide a lower and an upper bound in Chung-type law. The lower bound is given by

$$\Psi_h^{(\ell)}(r) = r^{\nu h}\ \widetilde{\Psi}_h^{(\ell)}(r) = r^{\nu h}\ (\log\log r^{-1})^{-h/\nu}\ ,$$

and the modulus for the upper bound is $\Psi_h^{(u)} = r^{\nu h}\ \widetilde{\Psi}_h^{(u)}$, where $\widetilde{\Psi}_h^{(u)}$ is an increasing function started at 0 and such that $\widetilde{\Psi}_h^{(u)} \geq \widetilde{\Psi}_h^{(\ell)}$, so that in particular, $\widetilde{\Psi}_h^{(u)}(r)^{-1} = o(r^{-\nu h})$ as $r \to 0$. The existence of $\widetilde{\Psi}_h^{(u)}$ is proven in Section 3, and related implicitly to the following decay function of Δ^h:

$$\mathbf{F}(\mathbf{x}) = \sup_{\varphi\in A(1)} \int_{\|x\|_E<\mathbf{x}} \left(1 - \cos\langle \mathscr{I}(\varphi), x\rangle\right)\ \Delta^h(\mathrm{d}x)\ , \tag{1.2}$$

where $A(1)$ is a compact subset of H, defined in the sequel. Note that $\mathbf{F}$ depends a priori on ν, hence so does $\widetilde{\Psi}_h^{(u)}$. For every $h \in (0,1/2)$, let us finally define $M^h(r) = \sup_{t\in[0,r]^\nu} |\boldsymbol{B}_t^h|,\ r \in [0,1]$.

Theorem 1.1. *Let $h \in (0,1/2)$ and let M^h, $\Psi_h^{(\ell)}$ and $\Psi_h^{(u)}$ be as above. Then we have almost surely:*

$$\liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(\ell)}(r)} \geq \kappa_1^{h/\nu} \quad \textit{and} \quad \liminf_{r\to 0^+} \frac{M^h(r)}{r^{\nu h}\widetilde{\Psi}_h^{(u)}(r)} \leq \kappa_2^{h/\nu} \ ,$$

where $0 < \kappa_1 \leq \kappa_2 < \infty$ are the constants appearing in the small deviations (see Equation (2.8)).

This result is not sharp *a priori*, and depends on the rate of decay of $\mathbf{F}$. We discuss how this gap could be filled at the end of Section 3.

In Strassen [30], while looking for an invariance principle for scaled random walks, the author obtained the fact that the same scaling on a Brownian motion gives a family of processes which is almost surely relatively compact in the unit ball of H_0^1, the Sobolev space of continuous functions started at 0 with square-integrable weak derivative. Functional laws of the iterated logarithm have now been widely studied in the literature: Csáki [6] was the first to get a rate of convergence for certain functions in this unit ball, and this result was extended by de Acosta [7] to scaled random walks, for any function of the unit ball of the RKHS (with radius strictly smaller than 1). Then, we can mention the contributions to this problem of Goodman and Kuelbs [11], Grill [12] and finally Kuelbs et al. [23], where the authors bring a new understanding of the rate of convergence towards the unit sphere in the general frame of Gaussian samples in Banach spaces. Similarly to the standard LIL, the functional result for fractional Brownian motion was also given by Monrad and Rootzén [28]. So for the multiparameter fBm, let us define, for $r \in (0,1)$,

$$\eta_r^{(h,\ell)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\sqrt{\log\log(r^{-1})}} \ , \forall t \in [0,1]^\nu$$

and

$$\eta_r^{(h,u)}(t) = \frac{\boldsymbol{B}^h(rt)}{r^{\nu h}\left(\widetilde{\Psi}_h^{(u)}(r)\right)^{-\nu/2h}} \ , \forall t \in [0,1]^\nu$$

the lower and upper rescaled multiparameter fBm for which we seek an invariance principle.

Theorem 1.2. *Let $h \in (0,1/2)$ and let H_h^ν denote the reproducing kernel Hilbert space of $k_h^{(\nu)}$. Let $\varphi \in H_h^\nu$ having norm strictly smaller than 1. Then, there exist two positive and finite constants $\gamma^{(\ell)}(\varphi)$ and $\gamma^{(u)}(\varphi)$ such that, almost surely,*

$$\liminf_{r\to 0^+}\ \widetilde{\Psi}_h^{(\ell)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,\ell)}(t) - \varphi(t)| \geq \gamma^{(\ell)}(\varphi)$$
$$\liminf_{r\to 0^+}\ \widetilde{\Psi}_h^{(u)}(r)^{-1-\nu/2h} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,u)}(t) - \varphi(t)| \leq \gamma^{(u)}(\varphi) \ .$$

As usual, taking $\varphi = 0$ yields the standard law of the iterated logarithm.

Finally, observing the techniques of [33, 37], it appears that for the Lévy fBm and similar processes, the same estimates usually allow to obtain both a Chung-type law and the exact Hausdorff measure of the range of these processes. However, we will prove that the Hausdorff dimension does not grasp the slow oscillations of the multiparameter fBm at the origin (Proposition 5.1). The cause is again the non-increment stationarity. Since this result is related to the values of the pointwise and local Hölder exponents of the multiparameter fBm, these exponents are computed and appear to behave differently: the pointwise exponent has a different value at the origin, while the local exponent is the same at any point.

Organisation of the paper. In section 2, we prove some preliminary results. The main new tools and ideas essentially lie in this section. They concern the small deviations around the origin 0, which are obtained using the local nondeterminism of [29], and a spectral representation on the Wiener space of this process. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. Finally, the Hausdorff dimension of the graph is computed in Section 5, where a remark on the local Hölder regularity at the origin of the multiparameter fBm is also formulated.

2 Preliminaries

We recall a few notions about Gaussian measures on Banach spaces and abstract Wiener spaces (see also [22, Lemma 2.1]). Let E be a separable Banach space and μ a Gaussian measure on E. Let H be the completion of E^* by the action of the covariance operator S of μ, defined by:

$$S\xi = \int_E x \ \langle \xi, x\rangle \ \mu(\mathrm{d}x) \ , \xi \in E^* \ ,$$

which maps E^* into a subspace of E, and the completion is with respect to the scalar product:

$$\left(S\xi, S\xi'\right)_H = \int_E \langle \xi, x\rangle \ \langle \xi', x\rangle \ \mu(\mathrm{d}x) \ .$$

This permits to define a sequence $\{\xi_n, n\in\mathbb{N}\}$ in E^*, such that $\{S\xi_n, n\in\mathbb{N}\}$ is a complete orthonormal system (CONS) in H. We recall that the Paley-Wiener map $\mathscr{I}$ is defined as the isometric extension of the map $\xi\in E^* \mapsto \langle \xi, \cdot\rangle$ to a map from H to $L^2(\mu)$. Conversely, it is also possible to start from a separable Hilbert space and to construct an embedding into a larger Banach space, on which there exists a Gaussian measure whose covariance will be related to the inner product on H. This is the abstract Wiener space (AWS) approach [13, 31].

Definition 2.1 (Reproducing Kernel Hilbert Space)**.** *Let (T,m) be a separable and complete metric space and R a continuous covariance function on $T\times T$. R determines a unique Hilbert space $H(R)$ satisfying the following properties:* i) *$H(R)$ is a space of functions mapping T to $\mathbb{R}$;* ii) *for all $t\in T$, $R(\cdot,t)\in H(R)$;* iii) *for all $t\in T$, $\forall f\in H(R)$, $\left(f, R(\cdot,t)\right)_{H(R)} = f(t)$.*

In [29], starting from the reproducing kernel Hilbert space (RKHS) of the $L^2([0,1]^\nu,\lambda)$-indexed fBm, denoted by $H(k_h)$ and built upon the kernel $k_h(f,g) = 1/2\left(\lambda(f^2)^{2h} + \lambda(g^2)^{2h} - \lambda((f-g)^2)^{2h}\right)$ for $f,g\in L^2([0,1]^\nu,\lambda)$, the following integral representation for the multiparameter fBm was obtained:

$$\mathbf{W}^h_t = \int_E \langle \mathscr{I}(k_h(\mathbf{1}_{[0,t]},\cdot)), x\rangle \ \mathrm{d}\mathbb{W}^h_x \ , \tag{2.1}$$

where $\mathbb{W}^h$ is a white noise on some Gaussian measure space (E_h,μ_h) with RKHS $H(k_h)$. Note the link with $k_h^{(\nu)}$ defined in the introduction, as for any $s,t\in[0,1]^\nu$, $k_h(\mathbf{1}_{[0,s]},\mathbf{1}_{[0,t]}) = k_h^{(\nu)}(s,t)$.

In general, the embedding between H and E is continuous. We will need it to be Hilbert-Schmidt for an extension of Bochner's theorem to be valid. The following lemma states that starting from a separable Hilbert space H, it is possible to find E and μ satisfying this property and such that (H,E,μ) is an AWS.

Lemma 2.2. *Let H be a separable Hilbert space. There is a separable Hilbert space $(E,\|\cdot\|)$ and a Gaussian measure μ on E such that (H,E,μ) is an abstract Wiener space and the embedding $H\subset E$ is Hilbert-Schmidt.*

Proof. Let us assume that there exists separable Hilbert spaces H_0 and E_0 such that H_0 is densely embedded into E_0 by an operator R which is Hilbert-Schmidt, and that there exists a Gaussian measure μ_0 such that (H_0, E_0, μ_0) is an abstract Wiener space. In that case, R is the covariance operator. Let u be any linear isometry between H_0 and H, and denote by (H, E, μ) the AWS given by $E = \tilde{u}(E_0)$ and $\mu = \tilde{u}_*\mu_0$, where $\tilde{u}$ is the isometric extension of u (see [31, p.317]). Since E_0 is a Hilbert space, E is also a Hilbert space and the operator $R' = \tilde{u} \circ R \circ u^{-1}$ is the natural embedding from H into E, and is of Hilbert-Schmidt type.
The existence of such a (H_0, E_0, μ_0) triple follows either from examples as in sections 6 and 7 of [23], or by the construction of the next paragraph. □

Let us detail the Wiener space structure of (H, E, μ) when E is a Hilbert space. Let $\{x_n, n \in \mathbb{N}\}$ be a complete orthonormal system of $\big(E, (\cdot,\cdot)_E\big)$. For each n, let λ_n^2 be the variance of $(x_n, \cdot)_E \in E^*$ under μ. Note that $\sum_{n\geq 1} \lambda_n^2 < \infty$, which follows from the fact that:

$$\begin{aligned}\sum_{n\in\mathbb{N}} \lambda_n^2 &= \sum_{n\in\mathbb{N}} \int_E (x_n, x)_E^2 \, \mu(\mathrm{d}x) \\ &= \int_E \|x\|_E^2 \, \mu(\mathrm{d}x)\,,\end{aligned}$$

and this quantity is finite (we know from Fernique [10] that μ has exponential moments). Then H is given by:

$$H = \left\{ x \in E : \sum_{n=1}^{\infty} \left(\frac{(x, x_n)_E}{\lambda_n} \right)^2 < \infty \right\} . \tag{2.2}$$

$\{h_n = \lambda_n x_n,\ n \in \mathbb{N}\}$ defines a CONS of H for the scalar product given by $(x, h_n)_H = \lambda_n^{-1}(x, x_n)_E$, for any $x \in H$, and any $n \in \mathbb{N}$. Then, one can check that H is densely and continuously embedded into E.

2.1 Spectral representation of the multiparameter fBm

The multiparameter fBm does not have independent increments, hence there is no spectral measure in the sense of Yaglom [38]. Such a situation already appeared for the bifractional Brownian motion [35], but the difficulty was overcome due to the equivalence of the distance induced by the bifractional Brownian motion with the Euclidean distance, even near 0. This could not work here. We shall use instead the L^2-increment stationarity in the Wiener space, to produce independent processes.

Now, we address the spectral decomposition itself. For any $\alpha \in (0, 2]$, the application $\xi \in E^* \mapsto \|S\xi\|_H^\alpha$ is continuous (because of the inequality $\|\cdot\|_H \leq C\,\|\cdot\|_{E^*}$) and negative definite (by an argument on Bernstein functions, see for instance the introduction of [29]). Thus, according to Schoenberg's theorem, $\xi \mapsto \exp(-t\|S\xi\|_H^\alpha)$ is positive definite for any $t \in \mathbb{R}_+^*$. It follows from Lemma 2.2 and Sazonov's theorem, according to which a Hilbert-Schmidt map is γ-radonifying[1], that since $\xi \mapsto \exp(-\frac{1}{2}\|S\xi\|_H^\alpha)$ is continuous on H, it is the Fourier transform of a measure Γ_α on E, i.e:

$$e^{-\frac{1}{2}\|S\xi\|_H^\alpha} = \int_E e^{i\langle \xi, x\rangle} \, \mathrm{d}\Gamma_\alpha(x) \,.$$

The measure Γ_α is a strictly stable and symmetric measure on E of index α, since it satisfies (we denote by $\widehat{\Gamma}_\alpha$ the Fourier transform of Γ_α), for any integer k, and any $\xi \in E^*$:

$$\left(\widehat{\Gamma}_\alpha(\xi)\right)^k = \widehat{\Gamma}_\alpha(k^{1/\alpha}\xi) \quad \text{and} \quad \widehat{\Gamma}_\alpha(-\xi) = \widehat{\Gamma}_\alpha(\xi)\,.$$

[1] see for instance [39] for Sazonov's theorem, and [4] for its use in a similar context, as well as the references therein.

In particular, we see that Γ_α is infinitely divisible. Kuelbs [21] extended the spectral decomposition of α-stable measures on $\mathbb{R}$ to the Hilbert space setting. Thus, when $\alpha \in (0,2)$, Γ_α has a Lévy measure Δ_α and can be written:

$$\int_E e^{i\langle \xi, x\rangle} \, \mathrm{d}\Gamma_\alpha(x) = \exp\left\{ \int_E \left(e^{i\langle \xi, x\rangle} - 1 - i \frac{\langle \xi, x\rangle}{1 + \|x\|_E} \right) \Delta_\alpha(\mathrm{d}x) \right\} ,$$

with Δ_α satisfying $\int_E (1 \wedge \|x\|_E^2) \, \Delta_\alpha(\mathrm{d}x) < \infty$ and $\Delta_\alpha(\{0\}) = 0$. That α is strictly smaller than 2 is essential, and this will be assumed implicitly throughout the rest of this article. It follows, cancelling the imaginary part (by symmetry of Δ_α), that:

$$\forall \xi \in E^*, \qquad -\|S\xi\|_H^\alpha = 2 \int_E (\cos\langle \xi, x\rangle - 1) \, \Delta_\alpha(\,\mathrm{d}x) \tag{2.3}$$

In the finite-dimensional setup, Δ_α is known explicitely and appears in the spectral representation of the Lévy fractional Brownian motion, as in [33]. In fact, Lemma 2.1 and 2.2 of [21] give a radial decomposition of Δ_α in terms of a finite measure σ_α defined on the Borel sets of the unit ball $\mathscr{S} = \{x \in E : \|x\|_E = 1\}$, such that for any borel set B of E:

$$\Delta_\alpha(B) = \int_0^\infty \frac{\mathrm{d}r}{r^{1+\alpha}} \int_{\mathscr{S}} \mathbf{1}_B(ry) \, \sigma_\alpha(\mathrm{d}y) \,. \tag{2.4}$$

Besides, $\sigma_\alpha(\mathrm{d}y) = \Delta_\alpha\left(\{x \in E : \|x\|_E \geq 1 \text{ and } x/\|x\|_E \in \mathrm{d}y\}\right)$. The previous discussion is summarized in the following proposition.

Proposition 2.3. *Let (H, E, μ) be any abstract Wiener space such that E is a Hilbert space and the embedding $H \subset E$ is Hilbert-Schmidt. Let $\alpha \in (0,2)$. Then there exists a non-trivial Lévy measure Δ_α on E such that Equation (2.3) is satisfied, and that can be radially decomposed as in (2.4).*

In the sequel, H will be specifically the RKHS of the Brownian sheet in $\mathbb{R}^\nu$, that is, the Hilbert space with kernel $\{\lambda(g\cdot), g \in L^2([0,1]^\nu)\}$, where λ is the Lebesgue measure of $\mathbb{R}^\nu$ and for $g \in L^2([0,1]^\nu)$, $\lambda(g\cdot)$ is the mapping:

$$f \in L^2([0,1]^\nu) \mapsto \int_{[0,1]^\nu} f g \, \mathrm{d}\lambda \,.$$

H is a separable Hilbert space and we endow it with E and μ chosen as in Lemma 2.2 to get an AWS. For coherence with the definition of the multiparameter fBm, we now use the notations $\Delta^h \equiv \Delta_{4h}$, $h \in (0, 1/2)$, for the measures defined in the previous paragraph.

Proposition 2.4 (Spectral representation of the multiparameter fBm)**.** *Let $\mathbb{B}^h$ be the (Gaussian) white noise on E with control measure Δ^h, and define the stochastic process $\{\mathscr{B}^h(\xi),\ \xi \in E^*\}$ by:*

$$\mathscr{B}^h(\xi) = \int_E \left(1 - e^{i\langle \xi, x\rangle}\right) \, d\mathbb{B}_x^h \,.$$

Then the domain of definition of $\mathscr{B}$ extends to H, and the following process is a multiparameter fBm:

$$\left\{ \mathscr{B}^h\left(\lambda(\mathbf{1}_{[0,t]}\cdot)\right),\ t \in [0,1]^\nu \right\} \overset{(d)}{=} \boldsymbol{B}^h \,.$$

Proof. The variance of the increments of $\mathscr{B}^h$ reads ($\overline{(\cdot)}$ denotes complex conjugation):

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(\mathscr{B}^h_\xi - \mathscr{B}^h_{\xi'}\right) &= \mathbb{E}\left((\mathscr{B}^h_\xi - \mathscr{B}^h_{\xi'})\overline{(\mathscr{B}^h_\xi - \mathscr{B}^h_{\xi'})}\right)\\ &= \int_E \left(e^{i\langle\xi,x\rangle} - e^{i\langle\xi',x\rangle}\right)\left(e^{-i\langle\xi,x\rangle} - e^{-i\langle\xi',x\rangle}\right)\ \Delta^h(\mathrm{d}x)\\ &= 2\int_E (1-\cos\langle\xi-\xi',x\rangle)\ \Delta^h(\mathrm{d}x) = \|S(\xi-\xi')\|_H^{4h} .\end{aligned}$$

Hence this process has the following covariance:

$$\mathbb{E}\left(\mathscr{B}^h_\xi\ \mathscr{B}^h_{\xi'}\right) = \frac{1}{2}\left(\|S\xi\|_H^{4h} + \|S\xi'\|_H^{4h} - \|S(\xi-\xi')\|_H^{4h}\right) .$$

By analogy with the Paley-Wiener map $\mathscr{I}$ that maps H to $L^2(\mu)$, let $\mathscr{I}_h$ be the mapping from E^* to $L^2(\Delta^h)$ such that $\mathscr{I}_h(\xi) = 1-e^{i\langle\xi,\cdot\rangle}$, and extend it to H using $\mathscr{I}$ by simply putting $\mathscr{I}_h(\varphi) = 1-e^{i\langle\mathscr{I}(\varphi),\cdot\rangle}$, for any $\varphi\in H$. Similarly to $\mathscr{I}$ in $L^2(\mu)$, $\mathscr{I}_h$ is a well-defined isometry from $(H,\|\cdot\|_H^{2h})$ to $L^2(\Delta^h)$. Thus, $\mathscr{B}^h$ is a well-defined process on H.
For any $f,g\in L^2([0,1]^\nu)$, $\lambda(f\cdot)$ and $\lambda(g\cdot)$ are in H, which yields:

$$\mathbb{E}\left(\mathscr{B}^h(\lambda(f\cdot))\ \mathscr{B}^h(\lambda(g\cdot))\right) = \frac{1}{2}\left(\lambda(f^2)^{2h} + \lambda(g^2)^{2h} - \lambda((f-g)^2)^{2h}\right) ,$$

and for $f = \mathbf{1}_{[0,s]}$ and $g = \mathbf{1}_{[0,t]}$, this is a multiparameter fBm of index h, as defined in equation (1.1). □

Remark 2.5. $\mathbf{W}^h$ *defined by (2.1) on $H(k_h)$ and $\boldsymbol{B}^h$ on H are different processes: they are not defined on the same spaces, and the first one is a linear application for fixed ω, which is not true for the second. Nevertheless, we have just seen in the previous example that $\{\mathbf{W}^h_{k_h(f,\cdot)},\ f\in L^2([0,1]^\nu)\}$ and $\{\boldsymbol{B}^h_{\lambda(f\cdot)},\ f\in L^2([0,1]^\nu)\}$ are equal in distribution. This implies that they have the same RKHS. In particular, we will be interested only in the multiparameter process, which means that the RKHS is given by*

$$H_h^\nu = \overline{Span\left\{k_h^{(\nu)}(t,\cdot),\ t\in[0,1]^\nu\right\}} ,$$

where $k_h^{(\nu)}(t,\cdot) = k_h(\mathbf{1}_{[0,t]},\mathbf{1}_{[0,\cdot]})$ and the completion is with respect to the scalar product given by:

$$\left(k_h^{(\nu)}(t,\cdot), k_h^{(\nu)}(s,\cdot)\right)_{h,\nu} = k_h^{(\nu)}(t,s) .$$

To conclude this section, we present inequalities on Δ^h that will be useful in the proof of the LIL. These are extensions of the truncation inequalities of Loève [24, p.209]. For any $r\in[0,1]$, let us define the subset $A(r)$ of H:

$$A(r) = \left\{\varphi_{s,t}\ ;\ s,t\in[0,r]^\nu\right\} , \tag{2.5}$$

where $\varphi_{s,t} = \lambda\left(\mathbf{1}_{[0,t]\triangle[0,s]}\cdot\right)$. Note that $A(r)$ is a subset of H.

Lemma 2.6. *For any $a>0$ and $\varphi\in A(1)$, we have:*

$$\int_{\|x\|_E<a} \left(1-\cos\langle\mathscr{I}(\varphi),x\rangle\right)\ \Delta^h(dx) \le \|\varphi\|_H^{4h}\ \mathbf{F}(a\Phi) , \tag{2.6}$$

where $\mathbf{F}$ *is the function defined in Equation (1.2), and* $\mathbf{F}$ *continuously decreases to* 0*. Besides, there is a constant* $C(h) > 0$ *such that for any* $b > 0$ *and* $\varphi \in H$*,*

$$\int_{\|x\|_E > b} \left(1 - \cos\langle \mathscr{I}(\varphi), x\rangle\right)\ \Delta^h(dx) \leq C(h) b^{-4h} . \tag{2.7}$$

Proof. We start with the first inequality, that we prove by approximation of φ by elements of E^*. Let Φ denote the norm of φ. Let $(\zeta'_n)_{n\in\mathbb{N}} = \{(z'_n, \cdot)_E,\ n \in \mathbb{N}\}$ be a sequence of E^* such that $S\zeta'_n$ belongs to the H-sphere of radius Φ and converges to φ in H. For all n, let ζ_n and z_n be the associated normalized (in E^* and E) vectors and $(\lambda'_n)_{n\in\mathbb{N}}$ be the family of norms in H: $\lambda'_n = \|S\zeta_n\|_H$. Note that if $\varphi \in H \setminus S(E^*)$, $\lambda'_n \to 0$ as $n \to \infty$. By construction, $\lambda'_n = \Phi \|\zeta'_n\|_{E^*}^{-1} > 0$. Then, the radial decomposition (2.4) of σ^h yields:

$$\begin{aligned}\int_{\|x\|_E < a} (1 - \cos\langle \zeta'_n, x\rangle)\ \Delta^h(\mathrm{d}x) &= \int_0^a \frac{\mathrm{d}r}{r^{1+4h}} \int_{\mathscr{S}} \left(1 - \cos\left\{\frac{r\Phi}{\lambda'_n}(z_n, y)_E\right\}\right)\ \sigma^h(\mathrm{d}y) \\ &= \Phi^{4h} \int_0^{a\Phi} \frac{\mathrm{d}u}{u^{1+4h}} \int_{\mathscr{S}} \left(1 - \cos\left\{\frac{u}{\lambda'_n}(z_n, y)_E\right\}\right)\ \sigma^h(\mathrm{d}y) ,\end{aligned}$$

where we applied the change of variable $u = \Phi r$. The last integral converges in $L^2(\sigma^h)$, so this reads:

$$\begin{aligned}\int_{\|x\|_E < a} (1 - \cos\langle \mathscr{I}(\varphi), x\rangle)\ \Delta^h(\mathrm{d}x) &= \Phi^{4h} \int_0^{a\Phi} \frac{\mathrm{d}u}{u^{1+4h}} \int_{\mathscr{S}} \left(1 - \cos\langle u\mathscr{I}(\varphi/\|\varphi\|_H), y\rangle\right)\ \sigma^h(\mathrm{d}y) \\ &\leq \Phi^{4h} \mathbf{F}(a\Phi) ,\end{aligned}$$

which gives (2.6). Finally, $\mathbf{F}$ decreases continuously to 0 since the mapping:

$$(\varphi, \mathbf{x}) \in A(1) \times [0,1] \mapsto \int_{\|x\|_E < \mathbf{x}} (1 - \cos\langle \mathscr{I}(\varphi/\|\varphi\|_H), x\rangle)\ \Delta^h(\mathrm{d}x)$$

is continuous on a compact (being the image of $[0,1]^\nu \times [0,1]^\nu$ by $\varphi_{\cdot,\cdot}$, $A(1)$ is compact).

To show (2.7) holds, we use a simple inequality on the cosine function:

$$\begin{aligned}\int_{\|x\|_E > b} (1 - \cos\langle \xi, x\rangle)\ \Delta^h(\mathrm{d}x) &\leq 2 \int_{\|x\|_E > b} \Delta^h(\mathrm{d}x) \\ &\leq 2 \int_b^\infty \frac{\mathrm{d}r}{r^{1+4h}} \sigma^h(\mathscr{S}) \\ &\leq \frac{2\sigma(\mathscr{S})}{2 - 4h} b^{-4h} .\end{aligned}$$

This concludes the proof of this lemma. $\square$

2.2 Small deviations of the multiparameter fBm

Let us state the following observation on the metric induced by the multiparameter fBm:

Lemma 2.7. *For any* $a \in (0,1)$*, any* $b > a$*, there exist* $m_{a,b}$ *depending on* a *and* b *and* M_b *depending on* b *only, such that for any* $s, t \in [a,b]^\nu$*,*

$$m_{a,b}\|s - t\| \leq \lambda\left([0,s] \triangle [0,t]\right) \leq M_b \|s - t\|$$

In particular, the upper bound holds even if $s,t \in [0,b]$. However, for any given $\alpha \in (0,1)$, we have that for all $\epsilon > 0$, there exist $s,t \in [0,1]^\nu$ such that $\lambda\left([0,s] \triangle [0,t]\right) \leq \varepsilon$ but $\|s-t\| \geq \alpha$.

Proof. The upper and lower bounds on $\lambda\left([0,s] \triangle [0,t]\right)$ are stated in Lemma 3.1 of [17] (up to equivalence of l^1 and l^∞ distances with the Euclidean distance), except that there, the constant in the upper bound is said to depend also on a. From the proof of [17], it is clear that this is not necessary. To prove the last statement, let $s_n = (2^{-n}, b, \dots, b) \in [0,b]^\nu$ and $t_n = (b, 2^{-n}, b, \dots, b) \in [0,b]^\nu$. It appears that $\lambda\left([0,s_n] \triangle [0,t_n]\right) \to 0$ as $n \to \infty$, while $\|s_n - t_n\|$ increases to $\sqrt{2}b$. $\square$

Concerning notations, we will have to compare several distances, so d_E will denote the Euclidean distance in $\mathbb{R}^\nu$, and for any $h \in (0,1]$, d_h is the following distance:

$$\text{for } s,t \in [0,1]^\nu, \quad d_h(s,t) = \lambda\left([0,s] \triangle [0,t]\right)^h .$$

When $h = 1$, we will prefer the notation d_λ. Note that we will only consider results for $h \leq 1/2$ because of the definition of $\boldsymbol{B}^h$, but d_h is still a distance for $h \in (1/2, 1]$ (but no longer negative definite which prevents the definition of a multiparameter fBm for such values). Accordingly, $B_h(t,r)$ is the ball of d_h-radius r centred at t. If no subscript is written, this will be the Euclidean ball. The notation $\asymp$ between two functions f and g means that near a point a, $f(x) = O(g(x))$ and $g(x) = O(f(x))$. We recall that on a (pre-)compact metric space (T,d), the metric entropy $N(T,d,\varepsilon)$ gives, for any $\varepsilon > 0$, the minimal number of balls of radius ε that are necessary to cover T.

Lemma 2.8. *Let $\nu \in \mathbb{N}$, then the d_λ-metric entropy of $[0,1]^\nu$ is, for ε small enough:*

$$N([0,1]^\nu, d_\lambda, \varepsilon) \asymp \varepsilon^{-\nu} .$$

Proof. Let us remark that due to Lemma 2.7, $d_\lambda(s,t) \leq M_1 d_E(s,t)$, for any $s,t \in [0,1]^\nu$. Thus, for any ball one has $B_\lambda(t_0, r) \supseteq B(t_0, M_1^{-1} r)$. We can assert that:

$$N\left([0,1]^\nu, d_\lambda, \varepsilon\right) \leq N\left([0,1]^\nu, d_E, M_1^{-1}\varepsilon\right) \asymp \left(M_1^{-1}\varepsilon\right)^{-\nu} ,$$

as $\varepsilon \to 0$. Conversely,

$$\begin{aligned} N\left([0,1]^\nu, d_\lambda, \varepsilon\right) &\geq N\left([1/2,1]^\nu, d_\lambda, \varepsilon\right) \\ &\geq N\left([1/2,1]^\nu, d_E, m_{1/2,1}^{-1}\varepsilon\right) \\ &\asymp \left(2 m_{1/2,1}^{-1}\varepsilon\right)^{-\nu} , \end{aligned}$$

so both inequalities give the expected result. $\square$

Proposition 2.9. *For $h < 1/2$, there are constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that for any fixed $r \in (0,1)$ and any ε small enough (compared to r),*

$$\exp\left\{-\kappa_2 \frac{r^{\nu^2}}{\varepsilon^{\nu/h}}\right\} \leq \mathbb{P}\left(\sup_{t \in [0,r]^\nu} |\boldsymbol{B}^h_t| \leq \varepsilon\right) \leq \exp\left\{-\kappa_1 \frac{r^{\nu^2}}{\varepsilon^{\nu/h}}\right\} \tag{2.8}$$

Proof. First, notice the isometry between the metric space $([0,1]^\nu, d_h)$ and the subset of $H(k_h)$ defined by $\{k_h(\mathbf{1}_{[0,t]}, \cdot),\ t \in [0,1]^\nu\}$ with the metric induced by the fBm indexed on $L^2([0,1]^\nu, \lambda)$ (see Eq. (2.1)). Hence, we can apply Theorem 4.6 of [29], which states that (for $h < 1/2$), there are positive constants $k_1 \leq k_2$ such that for any $\varepsilon > 0$,

$$k_1\, N([0,1]^\nu, d_h, \varepsilon) \leq -\log \mathbb{P}\left(\sup_{t \in [0,1]^\nu} |\boldsymbol{B}^h_t| \leq \varepsilon\right) \leq k_2\, N([0,1]^\nu, d_h, \varepsilon) .$$

For any $\varepsilon > 0$, any $t \in [0,1]^\nu$, the ball $B_h(t,\varepsilon)$ is the same as $B_\lambda(t,\varepsilon^{1/h})$. A direct consequence is that $N([0,1]^\nu, d_h, \varepsilon) = N([0,1]^\nu, d_\lambda, \varepsilon^{1/h})$. Hence it suffices to calculate the d_λ-entropy to obtain the result for any h. Besides, $\boldsymbol{B}^h$ satisfies the subsequent self-similarity property: for any $r > 0$,

$$\left\{\boldsymbol{B}^h_t, t \in [0,1]^\nu\right\} \stackrel{(d)}{=} \left\{r^{-\nu h}\boldsymbol{B}^h_{rt}, t \in [0,1]^\nu\right\} .$$

Therefore, $\mathbb{P}\left(\sup_{t\in[0,r]^\nu} |\boldsymbol{B}^h_t| \leq \varepsilon\right) = \mathbb{P}\left(\sup_{t\in[0,1]^\nu} |\boldsymbol{B}^h_t| \leq r^{-\nu h}\varepsilon\right)$ and so:

$$k_1\, N\left([0,1]^\nu, d_\lambda, r^{-\nu}\varepsilon^{1/h}\right) \leq -\log \mathbb{P}\left(\sup_{t\in[0,r]^\nu} |\boldsymbol{B}^h_t| \leq \varepsilon\right) \leq k_2\, N\left([0,1]^\nu, d_\lambda, r^{-\nu}\varepsilon^{1/h}\right) .$$

Lemma 2.8 permits to conclude, with $\kappa_1 \leq \kappa_2$ derived from k_1, k_2 and the approximation on the metric entropy. □

Remark 2.10. *This is different from the Lévy fBm X^h, for which the above* log-probability *is of the order $r^\nu \varepsilon^{-\nu/h}$ (see [33]). In fact, the small deviations of the multiparameter fBm away from the axes are also different of those at* 0*, and similar to the Lévy fBm. Indeed, if t_0 is not on the axes and r is such that $B(t_0,r) \subset (0,\infty)^\nu$, the equivalence between distances d_λ and d_E yields, as $\varepsilon \to 0$:*

$$-\log \mathbb{P}\left(\sup_{t\in B(t_0,r)} |\boldsymbol{B}^h_t| \leq \varepsilon\right) \asymp N(B(t_0,r), d_E, \varepsilon^{1/h}) \asymp \left(\frac{r}{\varepsilon^{1/h}}\right)^\nu .$$

3 A Chung-type law of the iterated logarithm

In this section, we prove Theorem 1.1. The abstract Wiener space is the same as in Proposition 2.4 and below. We fix h and simply write Δ for Δ^h in the sequel.

Remark 3.1. *The case $h = 1/2$ is special since it corresponds to the Brownian sheet. Its behaviour differs very much from the h-multiparameter fBm, $h < 1/2$, although we recall that the* 1/2*-multiparameter fBm is the Brownian sheet. This difference is due to the loss of the property of local nondeterminism, which the multiparameter fBm possesses when $h < 1/2$ only. For more information on small deviations and Chung-type law of the iterated logarithm of the Brownian sheet, we refer to [32].*

Proof of Theorem 1.1. The proof will be carried out in three steps. In the first, we obtain the lower bound for some constant β_1. In the second and third steps, we follow the scheme proposed in [28], but with the addition of methods related to the infinite dimensional setting described above.

1) Let $\gamma > 1$, $r_k = \gamma^{-k}$ and $\beta_1 = \left(\kappa_1/(1+\epsilon)\right)^{h/\nu}$, where κ_1 is the constant in the upper bound of the small deviation probability of $\boldsymbol{B}^h$. The upper bound in the small deviations (2.8) implies:

$$\begin{aligned}\sum^\infty \mathbb{P}\left(M^h(r_k) \leq \beta_1 \Psi^{(\ell)}_h(r_k)\right) &\leq \sum^\infty \exp\left\{-\kappa_1 \beta_1^{-\nu/h} \log\log(r_k^{-1})\right\} \\ &\leq \sum^\infty (\log \gamma^k)^{-(1+\epsilon)} < \infty ,\end{aligned}$$

where the sums start at k large enough (i.e. so that $\beta_1(\log\log\gamma^k)^{-h}$ is small enough, as in Proposition 2.9). Then, the Borel-Cantelli lemma gives:

$$\liminf_{k\to\infty} M^h(r_k)/\Psi^{(\ell)}_h(r_k) \geq \beta_1 \quad \text{a.s.}$$

So for $r_{k+1} < r \leq r_k$:

$$M^h(r)/\Psi^{(\ell)}_h(r) \geq M^h(r_{k+1})/\Psi^{(\ell)}_h(r_k) \geq \beta_1 \frac{\Psi^{(\ell)}_h(r_{k+1})}{\Psi^{(\ell)}_h(r_k)} \geq \left(\kappa_1/(1+\epsilon)\right)^{h/\nu} \gamma^{-\nu h} \frac{\widetilde{\Psi}^{(\ell)}_h(r_{k+1})}{\widetilde{\Psi}^{(\ell)}_h(r_k)} .$$

This is true for any $\epsilon > 0, \gamma > 1$, hence we get the following lower bound:

$$\mathbb{P}\left(\liminf_{r\to 0} \frac{M^h(r)}{\Psi_h^{(\ell)}(r)} \geq \kappa_1^{h/\nu}\right) = 1 \ . \tag{3.1}$$

2) Now, recall that κ_2 is the constant in the lower bound of the small balls, and define $\beta_2 = \kappa_2^{h/\nu}$. For some small (fixed) $\eta > 0$, we define the sequence $(\epsilon_k)_{k\in\mathbb{N}^*}$ by:

$$\epsilon_k = \mathbf{F}^{-1}\left((\log k)^{-2h/\nu - 2\eta}\right) \ . \tag{3.2}$$

By Lemma 2.6, $\mathbf{F}$ is a continuous increasing function on any interval $[0,T]$ such that $\mathbf{F}(0) = 0$. Thus, ϵ_k is a well-defined sequence which converges to 0 and satisfies:

$$\frac{(\log k)^{h/\nu}}{\sqrt{-\mathbf{F}(\epsilon_k)\log \mathbf{F}(\epsilon_k)}} \to \infty \quad \text{ as } k \to \infty \ .$$

Let's define another sequence $(r_k)_{k\in\mathbb{N}^*}$ by the following induction:

$$r_1 = 1 \quad \text{ and } \quad \forall k \geq 2,\ r_{k+1} = r_k\ \mathbf{F}(\epsilon_k)^{1/(2\nu h)}\ \epsilon_{k+1}^{2/\nu} \ . \tag{3.3}$$

One can now choose $\widetilde{\Psi}_h^{(u)}$ to be any increasing continuous function on $[0,1]$, satisfying the following set of conditions: for any $k\in\mathbb{N}^*$,

$$\widetilde{\Psi}_h^{(u)}(r_k) = \left(\log k\right)^{-h/\nu} \ . \tag{3.4}$$

We recall that for a given $\widetilde{\Psi}_h^{(u)}$, chosen as above, $\Psi_h^{(u)}$ is defined by $\Psi_h^{(u)}(r) = r^{\nu h}\ \widetilde{\Psi}_h^{(u)}(r)\ ,\ r\in[0,1]$.

For these parameters, the lower bound in the small deviations of $\boldsymbol{B}^h$ implies:

$$\begin{aligned}\sum^{\infty} \mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}_t^h|/\Psi_h^{(u)}(r_k) \leq \beta_2\right) &\geq \sum^{\infty} \exp\left\{-\kappa_2(\beta_2 \widetilde{\Psi}_h^{(u)}(r_k))^{-\nu/h}\right\} \\ &\geq \sum^{\infty} \frac{1}{k} = \infty \ ,\end{aligned} \tag{3.5}$$

where the sums start at k large enough (i.e. so the hypothesis of Proposition 2.9 is satisfied). This is not enough to prove the expected result, because these events are not independent. We will fix this using an idea that appeared in [33] and [28], to create independence by means of increment stationarity. Since this last property is not satisfied by the multiparameter fractional Brownian motion, we shall rely instead on the spectral representation obtained in the previous section.

We recall that $\varphi_t = \lambda(\mathbf{1}_{[0,t]}\cdot), t \in [0,1]^\nu$ is an element of H. For a family of disjoint intervals $\{I_k = (a_k, a_{k+1}],\ k\in\mathbb{N}\}$, where $(a_k)_{k\in\mathbb{N}}$ is an increasing sequence of $\mathbb{R}_+$ such that $a_k \to \infty$ (a_k will be specified later), we define the following processes:

$$\boldsymbol{B}_t^{h,k} = \int_{\|x\|_E \in I_k} \left(1 - e^{i\langle \mathscr{I}(\varphi_t), x\rangle}\right)\ \mathrm{d}\mathbb{B}_x^\Delta \ ,\ t\in[0,1]^\nu \tag{3.6}$$

$$\tilde{\boldsymbol{B}}_t^{h,k} = \boldsymbol{B}_t^h - \boldsymbol{B}_t^{h,k} = \int_{\|x\|_E \notin I_k} \left(1 - e^{i\langle \mathscr{I}(\varphi_t), x\rangle}\right)\ \mathrm{d}\mathbb{B}_x^\Delta \ ,\ t\in[0,1]^\nu \ . \tag{3.7}$$

Let Σ denote the covariance operator of $\boldsymbol{B}^h$ and Σ_k denote the covariance operator of $\boldsymbol{B}^{h,k}$. It is clear that $\Sigma - \Sigma_k$ is a positive semi-definite operator. Hence, Anderson's correlation inequality [2] applies and we get, for all $k\in\mathbb{N}$:

$$\mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}_t^{h,k}|/\Psi_h^{(u)}(r_k) \leq \beta_2\right) \geq \mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}_t^h|/\Psi_h^{(u)}(r_k) \leq \beta_2\right) \ .$$

As a consequence of Equation (3.5), we see that:

$$\sum_{k\geq 1}\mathbb{P}\left(\sup_{t\in[0,r_k]^\nu}|\boldsymbol{B}^{h,k}_t|/\Psi^{(u)}_h(r_k)\leq\beta_2\right)=\infty\ .$$

Since the events $\left\{\sup_{t\in[0,r_k]^\nu}|\boldsymbol{B}^{h,k}_t|/\Psi^{(u)}_h(r_k)\leq\beta_2\right\}$, $k\in\mathbb{N}$, are independent, the reciprocal of Borel-Cantelli lemma yields that almost surely,

$$\liminf_{k\to\infty}\sup_{t\in[0,r_k]^\nu}|\boldsymbol{B}^{h,k}_t|/\Psi^{(u)}_h(r_k)\leq\beta_2\ . \tag{3.8}$$

3) For any $k\in\mathbb{N}^*$, let $a_k=r_k^{-\nu/2}\epsilon_k$. Note that (3.3) implies that:

$$a_{k+1}r_k^{\nu/2}=r_{k+1}\ \mathbf{F}(\epsilon_k)^{-1/4h}\ \epsilon_{k+1}^{-1}\geq\mathbf{F}(\epsilon_k)^{-1/4h}\ . \tag{3.9}$$

In particular, $a_{k+1}r_k^{\nu/2}$ goes to infinity. Now, Lemma 2.6 acts on the incremental variance of $\tilde{\boldsymbol{B}}^{h,k}$ as follows: for any $s,t\in[0,r_k]^\nu$, letting $\varphi_{s,t}=\varphi_s-\varphi_t$,

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(\tilde{\boldsymbol{B}}^{h,k}_s-\tilde{\boldsymbol{B}}^{h,k}_t\right)&=\mathbb{V}\mathrm{ar}\left(\tilde{\boldsymbol{B}}^{h,k}_{\varphi_{s,t}}\right)\\&=\int_{\|x\|_E<a_k}\left(1-\cos\langle\mathscr{I}(\varphi_{s,t}),x\rangle\right)\ \Delta(\mathrm{d}x)+\int_{\|x\|_E\geq a_{k+1}}\left(1-\cos\langle\mathscr{I}(\varphi_{s,t}),x\rangle\right)\ \Delta(\mathrm{d}x)\\&\leq C\left(\|\varphi_{s,t}\|_H^{4h}\ \mathbf{F}\left(a_k\ \|\varphi_{s,t}\|_H\right)+a_{k+1}^{-4h}\right)\\&\leq C\left(r_k^{2\nu h}\ \mathbf{F}(a_kr_k^{\nu/2})+a_{k+1}^{-4h}\right)\\&\leq Cr_k^{2\nu h}\left(\mathbf{F}(\epsilon_k)+(a_{k+1}\ r_k^{\nu/2})^{-4h}\right)\ ,\end{aligned} \tag{3.10}$$

for some positive constant C, where $\|\varphi_{s,t}\|_H^2=\lambda([0,s]\bigtriangleup[0,t])\leq\lambda([0,r_k]^\nu)=r_k^\nu$. Thus, for this choice of r_k and a_k, letting D_k^2 denote this incremental variance, the previous equation and (3.9) give:

$$\begin{aligned}D_k^2&=\sup_{s,t\in[0,r_k]^\nu}\mathbb{V}\mathrm{ar}\left(\tilde{\boldsymbol{B}}^{h,k}_s-\tilde{\boldsymbol{B}}^{h,k}_t\right)\\&\leq 2C\ r_k^{2\nu h}\ \mathbf{F}(\epsilon_k)\ ,\end{aligned}$$

which decreases faster than $\sup_{s,t\in[0,r_k]^\nu}\mathbb{V}\mathrm{ar}\left(\boldsymbol{B}^h_s-\boldsymbol{B}^h_t\right)$ (as $k\to\infty$). By a Gaussian concentration result, we will see that D_k will permit us to obtain an upper bound for the large deviations of $\tilde{\boldsymbol{B}}^{h,k}$. Let $\tilde{d}_{h,k}$ be the distance induced by this process. We have just seen that $\tilde{d}_{h,k}\leq d_h$. Thus, the metric entropy of a set computed with $\tilde{d}_h$ is smaller than the one computed with d_h.

$$\begin{aligned}\int_0^{D_k}\sqrt{\log N([0,r_k]^\nu,\tilde{d}_{h,k},\varepsilon)}\ \mathrm{d}\varepsilon&\leq\int_0^{D_k}\sqrt{\log N([0,r_k]^\nu,d_h,\varepsilon)}\ \mathrm{d}\varepsilon\\&\leq\int_0^{D_k}\sqrt{\log\left(\kappa\frac{r_k^{\nu^2}}{\varepsilon^{\nu/h}}\right)}\ \mathrm{d}\varepsilon\ ,\end{aligned}$$

where the upper bound on $N([0,r_k]^\nu,d_h,\varepsilon)$ is due to the link with the small balls of $\boldsymbol{B}^h$ as in Proposition 2.9, with some $\kappa>0$ that comes from the asymptotics of $N([0,1]^\nu,d_\lambda,\varepsilon)$ in Lemma 2.8.

$$\begin{aligned}\int_0^{D_k}\sqrt{\log N([0,r_k]^\nu,\tilde{d}_{h,k},\varepsilon)}\ \mathrm{d}\varepsilon&\leq\sqrt{\frac{\nu}{h}}\kappa^{h/\nu}r_k^{\nu h}\int_0^{\sqrt{2C}\kappa^{-h/\nu}\sqrt{\mathbf{F}(\epsilon_k)}}\sqrt{\log x^{-1}}\ \mathrm{d}x\\&\leq C_1r_k^{\nu h}\sqrt{-\mathbf{F}(\epsilon_k)\ \log(C_2\mathbf{F}(\epsilon_k))}\ ,\end{aligned}$$

where we made the change of variable $x = \varepsilon\, \kappa^{-h/\nu} r_k^{-\nu h}$, and C_1 and C_2 are given by:

$$C_1 = \sqrt{\frac{C\nu}{h}}\,, \qquad C_2 = 2C\kappa^{-2h/\nu}\,.$$

By Talagrand's lemma [33], if $u > u_0(k) := C_1 r_k^{\nu h}\sqrt{-\mathbf{F}(\epsilon_k)\,\log(C_2\mathbf{F}(\epsilon_k))}$,

$$\mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\tilde{\boldsymbol{B}}^{h,k}_t| \geq u\right) \leq \exp\left(-\frac{(u-u_0(k))^2}{D_k^2}\right)\,.$$

Let $\epsilon > 0$. In order to replace u by $\epsilon\beta_2\Psi^{(u)}_h(r_k)$, one notices that:

$$\frac{\Psi^{(u)}_h(r_k)}{u_0(k)} = \frac{\widetilde{\Psi}^{(u)}_h(r_k)}{C_1\sqrt{-\mathbf{F}(\epsilon_k)\,\log(C_2\mathbf{F}(\epsilon_k))}}\,,$$

and this quantity goes to infinity, by definition of ϵ_k in (3.2) and $\widetilde{\Psi}^{(u)}_h$ in (3.4). Thus, replacing u with $\epsilon\beta_2\Psi^{(u)}_h(r_k)$ for k big enough, reads:

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,r_k]^\nu} |\tilde{\boldsymbol{B}}^{h,k}_t| \geq \epsilon\beta_2\Psi^{(u)}_h(r_k)\right) &\leq \exp\left(-\frac{(\epsilon\beta_2\Psi^{(u)}_h(r_k)-u_0(k))^2}{D_k^2}\right)\\ &\leq \exp\left(-\frac{u_0(k)^2}{D_k^2}\ \left(\epsilon\beta_2\Psi^{(u)}_h(r_k)\,u_0(k)^{-1}-1\right)^2\right)\\ &\leq \exp\left(-\frac{C_1\sqrt{-\log(C_2\mathbf{F}(\epsilon_k))}}{2C}\ \left(\epsilon\beta_2\Psi^{(u)}_h(r_k)\,u_0(k)^{-1}-1\right)^2\right)\,,\end{aligned}$$

whose sum is finite, since $\Psi^{(u)}_h(r_k)u_0(k)^{-1}$ diverges and $\mathbf{F}(\epsilon_k)$ goes to 0. Hence, applying once again the Borel-Cantelli lemma, we have almost surely,

$$\liminf_{k\to\infty}\ \sup_{t\in[0,r_k]^\nu} |\tilde{\boldsymbol{B}}^{h,k}_t|/\Psi^{(u)}_h(r_k) \leq \epsilon\beta_2\,.$$

Therefore, combining this with (3.8), we see that almost surely:

$$\liminf_{k\to\infty}\ \sup_{t\in[0,r_k]^\nu} |\boldsymbol{B}^h_t|/\Psi^{(u)}_h(r_k) \leq (1+\epsilon)\beta_2\,.$$

Since this is true for any $\epsilon > 0$, we obtain the expected upper bound:

$$\mathbb{P}\left(\liminf_{r\to 0}\frac{M^h(r)}{\Psi^{(u)}_h(r)} \leq \kappa_2^{h/\nu}\right) = 1\,. \tag{3.11}$$

□

Remark 3.2. *Let $(r_k)_{k\in\mathbb{N}}$ be defined as in (3.3) and $\beta_1 = \left(\kappa_1/(1+\epsilon)\right)^{h/\nu}$. Using also the relation (3.4) and proceding as in* **1)***, we obtain:*

$$\mathbb{P}\left(\liminf_{k\to\infty}\frac{M^h(r_k)}{\Psi^{(u)}_h(r_k)} \geq \kappa_1^{h/\nu}\right) = 1\,,$$

but note that this is not sufficient to conclude that $\Psi^{(u)}_h$ is the good modulus.

We end this part with a discussion on the consequences of the rate of decay of $\mathbf{F}$. To make this *lim inf* result precise, one would need to find $\widetilde{\Psi}_h^{(u)}$ explicitely, which depends only on the rate of decay of $\mathbf{F}$ near 0. For instance, if we were able to prove that $\mathbf{F}(\mathbf{x}) \leq \mathbf{x}^\gamma$ for some $\gamma > 0$, as $\mathbf{x} \to 0$, then some computations lead to the conclusion that $\widetilde{\Psi}_h^{(u)}(r) = \left(\log\log(r^C)\right)^{-h/\nu}$, where $C = -\nu^{-2}(1+4h/\gamma)$, is a function for which (3.4) holds. Since in that case $\widetilde{\Psi}_h^{(u)}(r) \sim \widetilde{\Psi}_h^{(\ell)}(r)$ as $r \to 0$, we would get

$$\mathbb{P}\left(\liminf_{r\to 0} \frac{M^h(r)}{r^{\nu h}(\log\log(r^{-1}))^{-h/\nu}} \in [\kappa_1^{h/\nu}, \kappa_2^{h/\nu}]\right) = 1 \ .$$

Note that in this situation, a $0-1$ law (which is explained in Remark 3.3 below) implies that the above limit is constant almost surely. A faster rate would yield the same conlusion, while a slower rate for $\mathbf{F}$ would certainly mean that $\widetilde{\Psi}_h^{(\ell)}$ converges to 0 too quickly.

Remark 3.3. ($0-1$ law of the multiparameter fBm if $\mathbf{F}(\mathbf{x}) \leq \mathbf{x}^\gamma$.) *The following is very similar to the $0-1$ law presented in [25]. Let $\mathscr{F}_n$ be the σ-algebra generated by the process $\sum_{k=n}^{\infty} \boldsymbol{B}^{h,k}$ and let $\mathscr{F}^\infty = \cap_{n\geq 1}\mathscr{F}_n$ be the tail σ-algebra. According to Kolmogorov's $0-1$ law, any event A in $\mathscr{F}^\infty$ is trivial, i.e. $\mathbb{P}(A) = 0$ or 1. Thus, if the event:*

$$A = \left\{\liminf_{r\to 0} M^h(r)/\Psi_h(r) = constant\right\}$$

belongs to $\mathscr{F}^\infty$, we will have found an exact modulus in the Chung-type law.

Fix $n \in \mathbb{N}^*$. We know from Lemma 2.6 and the first part of Equation (3.10) that

$$\begin{aligned}\sup_{s,t\in[0,1]^\nu} \mathbb{V}ar\left(\sum_{k=1}^{n-1}\boldsymbol{B}_s^{h,k} - \sum_{k=1}^{n-1}\boldsymbol{B}_t^{h,k}\right) &\leq K d_\lambda(s,t)^{2h}\, \mathbf{F}(a_n d_\lambda(s,t)^{1/2}) \\ &\leq K a_n^\gamma\, d_\lambda(s,t)^{2h+\gamma/2} \ .\end{aligned}$$

Thus, Kolmogorov's continuity criterion for multiparameter Gaussian processes implies that $\sum_{k=1}^{n-1} \boldsymbol{B}^{h,k}$ is almost surely (d_λ-)Hölder-continuous of order $h+\gamma/4-\epsilon$, for any $\epsilon \in (0,1)$. For $\epsilon < \gamma/4$, this implies that almost surely:

$$\liminf_{r\to 0} \sup_{t\in[0,r]^\nu} \frac{\left|\sum_{k=1}^{n-1} \boldsymbol{B}_t^{h,k}\right|}{\Psi_h(r)} \leq \liminf_{r\to 0} \frac{\lambda([0,r]^\nu)^{h+\gamma/4-\epsilon}}{r^{\nu h}\tilde{\Psi}_h(r)} = 0 \ .$$

Here $\Psi_h = \Psi_h^{(u)}$, since we have seen that if $\mathbf{F}(\mathbf{x}) \leq \mathbf{x}^\gamma$, we would obtain the same upper and lower modulus. As a consequence, we have almost surely that:

$$\liminf_{r\to 0} M^h(r)/\Psi_h(r) = \liminf_{r\to 0} \sup_{t\in[0,r]^\nu} \Big|\sum_{k=n}^{\infty} \boldsymbol{B}_t^{h,k}\Big|/\Psi_h(r) \ ,$$

which is a $\mathscr{F}_n$-measurable random variable. Hence A is a tail event.

4 Functional law of the iterated logarithm

We prove Theorem 1.2. As in the previous part, we also have values for $\gamma^{(\ell)}(\varphi)$ and $\gamma^{(u)}(\varphi)$:

$$\gamma^{(\ell)}(\varphi) = \frac{1}{\sqrt{2}}\kappa_1^{h/\nu}(1-\|\varphi\|_{h,\nu}^2)^{-h/\nu} \ \text{ and } \ \gamma^{(u)}(\varphi) = \frac{1}{\sqrt{2}}\kappa_2^{h/\nu}(1-\|\varphi\|_{h,\nu}^2)^{-h/\nu} \ .$$

With the preliminary results of Section 2, almost all the ingredients are available to follow the proofs of [7, 28], and the following technical lemma is adapted from these papers. The norm of H_h^ν (see Remark 2.5) is denoted by $\|\cdot\|_{h,\nu}$ and we will also abbreviate $\sup_{t\in[0,1]^\nu} |f(t)| = \|f\|_\infty$.

Lemma 4.1. *For $0 < s < r < u < e^{-1}$ and $\varphi \in H_h^\nu$,*

$$\left(\log\log r^{-1}\right)^{h/\nu+1/2} \|\eta_r^{(h,\ell)} - \varphi\|_\infty \geq \left(\frac{s \log\log u^{-1}}{u \log\log s^{-1}}\right)^{h\nu} \left(\log\log s^{-1}\right)^{h/\nu+1/2} \|\eta_s^{(h,\ell)} - \varphi\|_\infty$$
$$- M_1 \left(\log\log u^{-1}\right)^{h/\nu+1/2} \left(\frac{u-s}{u}\right)^h \|\varphi\|_{h,\nu}$$
$$- \left(\log\log u^{-1}\right)^{h/\nu+1/2} \sqrt{\left(1 - \left(\frac{s}{u}\right)^{2\nu h} \frac{\log\log s^{-1}}{\log\log u^{-1}}\right)} \|\varphi\|_\infty ,$$

where M_1 is the constant in Lemma 2.7 which corresponds to $b = 1$.

For the proof of this lemma, one can refer to appendix A .

We recall the following nice proposition from [28], concerning the Gaussian measure of shifted convex sets[2]:

Proposition 4.2. *Let μ be a Gaussian measure on a separable Banach space E. For any convex, symmetric, bounded and measurable subset V of E of positive measure, if φ belongs to the RKHS of μ, then*

$$\lim_{t\to\infty} t^{-2} \left(\log \mu(V + t\varphi) - \log \mu(V)\right) = -\frac{1}{2}\|\varphi\|_\mu^2 .$$

Proof of Theorem 1.2. This proof is divided into two parts: the first one to give the lower bound on $\gamma(\varphi)$, and the second one for the upper bound.

I) Proof of the lower bound

Let $\epsilon > 0$ and γ_1 defined by:

$$\gamma_1 = \left(\frac{\kappa_1}{(1+\epsilon)}\right)^{h/\nu} (1 - \|\varphi\|_{h,\nu}^2)^{-h/\nu} .$$

Recall that $\widetilde{\Psi}_h^{(\ell)}(r) = (\log\log r^{-1})^{-h/\nu}$, so that the following events, defined for $k \in \mathbb{N}$ by:

$$A_k = \left\{\widetilde{\Psi}_h^{(\ell)}(r_k)^{-1-\nu/2h} \|\eta_{r_k}^{(h,\ell)} - \varphi\|_\infty \leq \gamma_1\right\}$$

for some decreasing sequence r_k (explicited later), will be written:

$$A_k = \left\{\left\| r_k^{-\nu h} \boldsymbol{B}^h(r_k \cdot) - \sqrt{2\log\log r_k^{-1}}\,\varphi \right\|_\infty \leq \gamma_1 (\log\log r_k^{-1})^{-h/\nu}\right\} .$$

Let $\delta > 0$ and $\delta < \epsilon(1 - \|\varphi\|_{h,\nu}^2)$. By Proposition 4.2, and then by the small deviations of $\boldsymbol{B}^h$, we have for k large enough (depending on δ),

$$\log \mathbb{P}(A_k) \leq \log \mathbb{P}\left(\sup_{t\in[0,1]^\nu} |\boldsymbol{B}^h(r_k t)| \leq \gamma_1 r_k^{h\nu} (\log\log r_k^{-1})^{-h/\nu}\right) - (\log\log r_k^{-1})(\|\varphi\|_{h,\nu}^2 - \delta)$$
$$\leq -(1+\epsilon)(1 - \|\varphi\|_{h,\nu}^2)(\log\log r_k^{-1}) - (\log\log r_k^{-1})(\|\varphi\|_{h,\nu}^2 - \delta) .$$

This implies that

$$\mathbb{P}(A_k) \leq \exp\left\{-\left(1 + \epsilon(1 - \|\varphi\|_{h,\nu}^2) - \delta\right) \log\log r_k^{-1}\right) .$$

[2]It existed before in the literature, in a more general form. See the references therein.

Now put:

$$r_k = \exp\left\{-ky(k)\right\} ,$$

where

$$y(k) = \frac{\log\log k}{(\log k)^{h^{-1}+1}} .$$

Since δ was chosen appropriately, $\epsilon(1-\|\varphi\|_{h,\nu}^2)-\delta$ is positive, and

$$\sum^{\infty}\mathbb{P}(A_k) < \infty ,$$

where the sum is over k large enough, according to the previous remarks. Therefore, almost surely,

$$\liminf_{k\to\infty}\left(\log\log r_k^{-1}\right)^{h/\nu+1/2}\sup_{t\in[0,1]^\nu}|\eta_{r_k}(t)-\varphi(t)| \geq \frac{1}{\sqrt{2}}\gamma_1 .$$

To obtain the result for $r\to 0$, we use Lemma 4.1 with $u=r_k$, $s=r_{k+1}$ and r in between. Then

$$\begin{aligned}\left(\log\log r^{-1}\right)^{h/\nu+1/2}\|\eta_r-\varphi\|_\infty \geq{}& \left(\frac{r_{k+1}\log\log r_k^{-1}}{r_k\log\log r_{k+1}^{-1}}\right)^{h\nu}\left(\log\log r_{k+1}^{-1}\right)^{h/\nu+1/2}\|\eta_{r_{k+1}}-\varphi\|_\infty && (\star)\\ &- M_1\left(\log\log r_k^{-1}\right)^{h/\nu+1/2}\left(\frac{r_k-r_{k+1}}{r_k}\right)^h\|\varphi\|_{h,\nu} && (\star\star)\\ &- \left(\log\log r_k^{-1}\right)^{h/\nu+1/2}\sqrt{\left(1-\left(\frac{r_{k+1}}{r_k}\right)^{2\nu h}\frac{\log\log r_{k+1}^{-1}}{\log\log r_k^{-1}}\right)}\|\varphi\|_\infty . && (\star\star\star)\end{aligned}$$

Note that by the inequality $e^{-x}\geq 1-x$, and the decrease of $y(k)$ (for k large),

$$\begin{aligned}\frac{r_{k+1}}{r_k} &\geq 1-\left\{y(k+1)\log\left(y(k+1)\right)-y(k)\log\left(y(k)\right)\right\}\\ &\geq 1-y(k+1) .\end{aligned}$$

Thus, the ratio in Equation ($\star$) converges to 1. Likewise, the ratio in ($\star\star$) is smaller than $y(k+1)^h$, so that:

$$\begin{aligned}\left(\frac{r_k-r_{k+1}}{r_k}\right)^h\left(\log\log r_k^{-1}\right)^{h/\nu+1/2} &\leq y(k+1)^h\left(\log\left(ky(k)\right)\right)^{h/\nu+1/2}\\ &\leq \frac{\left(\log\log(k+1)\right)^h}{\left(\log(k+1)\right)^{h+1}}(\log k)^{h/\nu+1/2}\left(1+\frac{\log y(k)}{\log k}\right)^{h/\nu+1/2} ,\end{aligned}$$

which clearly goes to 0. For the last term ($\star\star\star$),

$$\begin{aligned}
&\left(\log\log r_k^{-1}\right)^{2h/\nu+1}\left(1-\left(\frac{r_{k+1}}{r_k}\right)^{2\nu h}\frac{\log\log r_{k+1}^{-1}}{\log\log r_k^{-1}}\right)\\
&\qquad\le \left(\log\log r_k^{-1}\right)^{2h/\nu}\Big\{\log(ky(k))-\log\left((k+1)y(k+1)\right)\\
&\qquad\qquad\qquad\qquad+\log\left((k+1)y(k+1)\right)\left[1-(1-y(k+1))^{2h\nu}\right]\Big\}\\
&\qquad\le \left(\log\log r_k^{-1}\right)^{2h/\nu}\Big\{\log(ky(k))-\log\left((k+1)y(k+1)\right)\\
&\qquad\qquad\qquad\qquad+2\nu h\ y(k+1)\ \log\left((k+1)y(k+1)\right)\Big\}\ .
\end{aligned}$$

One can show that $\log\left(ky(k)\right)-\log\left((k+1)y(k+1)\right)\sim -k^{-1}$, thus

$$\left(\log\log r_k^{-1}\right)^{2h/\nu}\left\{\log(ky(k))-\log\left((k+1)y(k+1)\right)\right\}$$

converges to 0, and so does the remaining term, since:

$$\left(\log\log r_k^{-1}\right)^{2h/\nu}\ \ y(k+1)\ \log\left((k+1)y(k+1)\right)\sim\left(\log k\right)^{2h/\nu-1-h^{-1}+1}\ \log\log(k+1)$$

and the sum of the exponents $1+2h/\nu-h^{-1}-1$ is strictly negative ($\nu\ge 1$ and $h<1/2$).

II) Proof of the upper bound

The proof of Theorem 1.1 and Proposition 4.2 allow to make a quick proof for this bound. Let us define $\gamma_2=\kappa_2^{h/\nu}\left(1-\|\varphi\|_{h,\nu}^2\right)^{-h/\nu}$, and put r_k and a_k as in steps 2) and 3) of the proof of the LIL. Again, let $\boldsymbol{B}^{h,k}$ and $\tilde{\boldsymbol{B}}^{h,k}$ be the processes defined by (3.6) and (3.7). As in [28], we define the following events, for any $\epsilon>0$:

$$\begin{aligned}
A_k(\epsilon)&=\left\{\left\|r_k^{-\nu h}\boldsymbol{B}^h(r_k\cdot)-\sqrt{2}\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/2h}\varphi\right\|_\infty\le\gamma_2(1+\epsilon)\ \widetilde{\Psi}_h^{(u)}(r_k)\right\}\\
B_k(\epsilon)&=\left\{\left\|r_k^{-\nu h}\boldsymbol{B}^{h,k}(r_k\cdot)-\sqrt{2}\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/2h}\varphi\right\|_\infty\le\gamma_2(1+\epsilon)\ \widetilde{\Psi}_h^{(u)}(r_k)\right\}\\
C_k(\epsilon)&=\left\{\left\|r_k^{-\nu h}\tilde{\boldsymbol{B}}^{h,k}(r_k\cdot)\right\|_\infty\ge\gamma_2\epsilon\ \widetilde{\Psi}_h^{(u)}(r_k)\right\}\ .
\end{aligned}$$

This time, apply Proposition 2.9 and Proposition 4.2 to deduce the existence of a small $\delta>0$ such that for k large enough, the following lower bound on the probability of the event $A_k(\epsilon)$ holds:

$$\begin{aligned}
\log\mathbb{P}(A_k(\epsilon))&\ge\log\mathbb{P}\left(\sup_{t\in[0,1]^\nu}|\boldsymbol{B}^h(r_kt)|\le\gamma_2(1+\epsilon)r_k^{h\nu}\widetilde{\Psi}_h^{(u)}(r_k)\right)-\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/h}\left(\|\varphi\|_{h,\nu}^2+\delta\right)\\
&\ge-(1+\epsilon)^{-\nu/h}\left(1-\|\varphi\|_{h,\nu}^2\right)\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/h}-\left(\widetilde{\Psi}_h^{(u)}(r_k)\right)^{-\nu/h}\left(\|\varphi\|_{h,\nu}^2+\delta\right)\\
&\ge-\log k\ \left((1+\epsilon)^{-\nu/h}\left(1-\|\varphi\|_{h,\nu}^2\right)-\left(\|\varphi\|_{h,\nu}^2+\delta\right)\right)\ .
\end{aligned}$$

Therefore, choosing δ small enough to ensure that $-(1+\epsilon)^{-\nu/h}\left(1-\|\varphi\|_{h,\nu}^2\right)-\left(\|\varphi\|_{h,\nu}^2+\delta\right)$ is greater than -1 implies that:

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k(\epsilon)) \geq \sum_{k=1}^{\infty} k^{-(1+\epsilon)^{-\nu/h}\left(1-\|\varphi\|_{h,\nu}^2\right)-\left(\|\varphi\|_{h,\nu}^2+\delta\right)}$$
$$= \infty \; .$$

All that remains to notice is that:

$$A_k(\epsilon) \subset B_k(2\epsilon) \; \cup \; C_k(\epsilon) \subset A_k(3\epsilon) \; \cup \; C_k(\epsilon) \; ,$$

and that the choice of a_k and r_k implies that $\sum \mathbb{P}(C_k(\epsilon)) < \infty$ (as in the proof of Theorem 1.1). The rest follows strictly the proof of [28]. □

As in Remark 3.3, if **F** were proven to have fast decay, the same $0-1$ law that we used for the Chung law would give the same conclusion, i.e. that there is a constant between $\gamma^{(\ell)}(\varphi)$ and $\gamma^{(u)}(\varphi)$ such that almost surely:

$$\liminf_{r\to 0^+} (\log\log(r^{-1}))^{h/\nu+1/2} \sup_{t\in[0,1]^\nu} |\eta_r^{(h,\ell)}(t) - \varphi(t)| = \gamma(\varphi) \; .$$

We end this part on laws of the iterated logarithm with a remark concerning the previous result when $\|\varphi\|_{h,\nu} = 1$. This case was studied a lot in the literature, as it yields a different rate of convergence. In fact, for $\|\varphi\|_{h,\nu} = 1$, part **I)** of the previous proof can be directly adapted to give:

$$\liminf_{r\to 0^+} (\log\log(r^{-1}))^{h/\nu+1/2} \sup_{t\in[0,1]^\nu} |\eta_r(t) - \varphi(t)| = \infty \quad \text{a.s.} \; .$$

The exact rate was computed in many situations and it is likely that standard techniques (as in [7, 28]) and the present spectral representation and small deviations will permit to compute the exact rate in the functional law of the iterated logarithm on the unit sphere for the multiparameter fBm.

5 Hausdorff dimension of the range of the multiparameter fBm

Let us first recall the definition of a Hausdorff measure and of Hausdorff dimension. Following [9], let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and increasing function, which we call *gauge function*. Let the diameter be the set function defined by $\mathrm{diam}(A) = \sup\{\|s-t\| : s,t \in A\}$ for any subset A of $\mathbb{R}^\nu$. For any $\delta > 0$, we call δ–cover of A a family $\{A_i\}$ of subsets of $\mathbb{R}^\nu$ of diameter smaller than δ and such that $A \subseteq \cup_i A_i$. Then for any $A \subset \mathbb{R}^\nu$,

$$\mu_\Phi(A) = \lim_{\delta\to 0} \inf \left\{ \sum_i \Phi(\mathrm{diam}(A_i)) : \{A_i\} \text{ is a } \delta - \text{covering of } A \right\}$$

defines the Hausdorff measure on $\mathbb{R}^\nu$ with gauge function Φ. A case of special interest is when the gauge function is $\Phi(x) = x^s$, and we denote by μ_s the Hausdorff measure. Then the Hausdorff dimension of a set $A \subset \mathbb{R}^\nu$ is:

$$\dim_H(A) = \inf\{s > 0 : \mu_s(A) = 0\} = \sup\{s > 0 : \mu_s(A) = \infty\} \; .$$

In general, it can happen that a set A has Hausdorff dimension s but that $\mu_s(A)$ is either 0 or ∞. So choosing a special gauge function such that $\mu_\Phi(A) \in (0,\infty)$ gives more information on A. A measure having this property will be called an *exact Hausdorff measure* on A. For more on Hausdorff measure of random fractals, one can refer for instance to the survey [34].

The method to prove a Chung-type LIL often relies on the same estimates than the method to compute the exact Hausdorff measure of the range of a Gaussian process with stationary increments. We refer to [37, Prop. 3.1], which extends the work of [33, Prop. 4.1] on the Lévy fractional Brownian motion. Let d be an integer and assume in the sequel that $\nu < hd$. Taking d independent copies of the multiparameter fBm, we still let $\boldsymbol{B}^h$ be the $\mathbb{R}^d$-valued process with one of these copies for each coordinate. Hence, if $\boldsymbol{B}^h$ was increment stationary, we would obtain the following, almost surely:

$$\mu_{\Phi^{(l)}}\left(\boldsymbol{B}^h([0,1]^\nu)\right) > 0 \quad \text{and} \quad \mu_{\Phi^{(u)}}\left(\boldsymbol{B}^h([0,1]^\nu)\right) < \infty \ ,$$

where

$$\Phi^{(l)}(x) = x^{1/h} \log\log(x^{-1})^\nu \quad \text{and} \quad \Phi^{(u)} = x^{1/h} \left(\widetilde{\Psi}_h^{(u)}(x)\right)^{-1/h} .$$

This would then imply that the Hausdorff dimension of $\boldsymbol{B}^h([0,1]^\nu)$ is equal to $1/h$. However, this is not the case, as seen in the following result: in some sense, the Hausdorff measure of the range of the multiparameter fBm on a rectangle containing 0 does not "see" the singularity.

Proposition 5.1. *Let $\boldsymbol{B}^h$ be a d-dimensional multiparameter fBm and assume that $\nu \le hd$. Then,*

$$\dim_H\left(\boldsymbol{B}^h([0,1]^\nu)\right) = \frac{\nu}{h} \ \ a.s.$$

Proof. Let I be a subset of $[0,1]^\nu$ that does not intersect the axis. The Hausdorff dimension of the image of I by a multiparameter fBm has been computed in [17], and when $\nu \le hd$, $\dim_H(\boldsymbol{B}^h(I)) = \nu/h$. Since I is included in $[0,1]^\nu$, this implies that:

$$\dim_H\left(\boldsymbol{B}^h(I)\right) \le \dim_H\left(\boldsymbol{B}^h([0,1]^\nu)\right) \ ,$$

so it remains to prove the upper bound. A result of Adler [1] relates the Hausdorff dimension of the range of an increment stationary Gaussian process to its Hölder regularity. An adaptation of this result (see [17, Corollary 2.7]) gives an upper bound for the Hausdorff dimension of the range of a Gaussian process (not necessarily increment stationary), in terms of the infimum of its deterministic local Hölder exponent, defined at any point $t_0 \in \mathbb{R}_+^\nu$ by:

$$\widetilde{\alpha}_{\boldsymbol{B}^h}(t_0) = \sup\left\{\alpha \in (0,1) : \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{\mathbb{E}\left(\boldsymbol{B}^h(t) - \boldsymbol{B}^h(s)\right)^2}{\|t-s\|^{2\alpha}} < \infty\right\} , \tag{5.1}$$

where $B(t_0,\rho)$ refers to the Euclidean ball. If t_0 is not on the axes, the equivalence between the Euclidean distance and d_λ (Lemma 2.7) implies that $\widetilde{\alpha}_{\boldsymbol{B}^h}(t_0) = h$. Let us prove that the same happens on the axes, and since this is enough to prove it at 0, let $t_0 = 0$. Let $\rho > 0$ and $s,t \in B_+(0,\rho) \equiv B(0,\rho) \cap \mathbb{R}_+^\nu$. Lemma 2.7 implies:

$$\frac{\lambda\left([0,t] \triangle [0,s]\right)^{2h}}{\|t-s\|^{2\alpha}} \le \frac{\lambda\left([0,t] \triangle [0,s]\right)^{2h}}{M_1^{2\alpha}\lambda\left([0,t] \triangle [0,s]\right)^{2\alpha}}$$

where M_1 is the constant of Lemma 2.7. The last expression is bounded on $B_+(0,\rho)$ whenever $\alpha \le h$. Therefore, $\widetilde{\alpha}_{\boldsymbol{B}^h}(0) \ge h$. In the opposite situation, let $u \in B_+(0,\rho/2)$ with positive coordinates and

$s_n, t_n \in B_+(0,\rho)$ such that $s_n = (u_1 + 1/n, u_2, \dots, u_\nu)$ and $t_n = (u_1, u_2 + 1/n, u_3, \dots, u_\nu)$ defined for any $n \in \mathbb{N}^*$. Then, for n large enough, $s_n, t_n \in B(0,\rho)$ and

$$\frac{\lambda\left([0,t_n] \triangle [0,s_n]\right)^{2h}}{\|t_n - s_n\|^{2\alpha}} = \frac{n^{-2h}(u_1 u_3 \dots u_\nu + u_2 \dots u_\nu)^{2h}}{2^\alpha n^{-2\alpha}} .$$

This is unbounded as $n \to \infty$ (if $\alpha > h$), which proves that $\widetilde{\alpha}_{\boldsymbol{B}^h}(0) \le h$.
Now, by Corollary 2.7 of [17],

$$\dim_H \left(\boldsymbol{B}^h([0,1]^\nu)\right) \le \frac{\nu}{\inf_{t_0 \in [0,1]^\nu} \widetilde{\alpha}_{\boldsymbol{B}^h}(t_0)} = \frac{\nu}{h} \;\; a.s. ,$$

which completes the proof. □

Remark 5.2. *For the local Hölder exponent defined in (5.1), we have seen that $\widetilde{\alpha}_{\boldsymbol{B}^h}(0) = h$. Another aspect of the singular behaviour of $\boldsymbol{B}^h$ at 0 comes from the fact that the pointwise Hölder exponent,*

$$\alpha_{\boldsymbol{B}^h}(t_0) = \sup\left\{ \alpha \in (0,1) : \limsup_{\rho \to 0} \sup_{s,t \in B_+(t_0,\rho)} \frac{\mathbb{E}\left(\boldsymbol{B}^h(t) - \boldsymbol{B}^h(s)\right)^2}{\rho^{2\alpha}} < \infty \right\}$$

at $t_0 = 0$, is different from $\widetilde{\alpha}_{\boldsymbol{B}^h}(0)$, since $\alpha_{\boldsymbol{B}^h}(0) = \nu h$. Indeed,

$$\sup_{s,t \in B_+(0,\rho)} \lambda\left([0,t] \triangle [0,s]\right) \le \lambda\left([0,\rho]^\nu\right) = \rho^\nu$$

and

$$\sup_{s,t \in B(0,\rho) \cap [0,1]^\nu} \lambda\left([0,t] \triangle [0,s]\right) \ge \sup_{t \in B(0,\rho) \cap [0,1]^\nu} \lambda\left([0,t]\right) = \frac{\rho^\nu}{\nu^{\nu/2}} ,$$

so that

$$\alpha_{\boldsymbol{B}^h}(0) = \sup\left\{ \alpha > 0 : \lim_{\rho \to 0} \frac{(\rho^\nu)^{2h}}{\rho^{2\alpha}} < \infty \right\} = \nu h .$$

For Gaussian processes, these exponents characterize the local and pointwise Hölder regularity of the sample paths [14]. Hence, on the sample paths of the multiparameter fBm, the local and pointwise random exponent (computed without expectations) differ almost surely. For a similar behaviour on functions, see the chirp function in [14].
However, these two exponents are equal when t_0 is not on the axes (this is a simple consequence of Lemma 2.7). It is possible that the pointwise Hölder exponent captures the local oscillations in Chung-type LILs, while the local exponent cannot. This could provide a Chung-type law for a wide class of Gaussian processes having the LND property and a spectral representation similar to the multiparameter fBm.

Appendix A Proof of Lemma 4.1

For the original proof, see Lemma 5.3 of [7]. We make here the necessary modifications.

$$\begin{aligned}
\left(\log\log r^{-1}\right)^{h/\nu+1/2} \|\eta_r^{(h,\ell)} - f\|_\infty &= \frac{(\log\log r^{-1})^{h/\nu}}{r^{\nu h}} \left\| \boldsymbol{B}^h(r\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}} f \right\|_\infty \\
&\ge \frac{(\log\log r^{-1})^{h/\nu}}{r^{\nu h}} \left\| \boldsymbol{B}^h(s\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}} f\left(\frac{s}{r}\cdot\right) \right\|_\infty \\
&\ge \frac{(\log\log u^{-1})^{h/\nu}}{u^{\nu h}} \left\| \boldsymbol{B}^h(s\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}} f\left(\frac{s}{r}\cdot\right) \right\|_\infty .
\end{aligned}$$

Now choosing $a = s^{\nu h}\sqrt{\log\log s^{-1}}$ and $b = u^{\nu h}\sqrt{\log\log u^{-1}}$,

$$\left\| \boldsymbol{B}^h(s\cdot) - r^{\nu h}\sqrt{\log\log r^{-1}} f\left(\frac{s}{r}\cdot\right)\right\|_\infty \geq \left\|\boldsymbol{B}^h(s\cdot) - af\right\|_\infty - b\left\|f - f\left(\frac{s}{r}\cdot\right)\right\|_\infty - (b-a)\|f\|_\infty$$

and we find a bound for each of the last two terms (the first one is exactly the one given in the Lemma). We need the following inequality for $f \in H_h^\nu$, $s,t \in [0,1]^\nu$:

$$|f(s) - f(t)|^2 \leq M_1 \|s-t\|^{2h}\|f\|_{h,\nu}^2 ,$$

which follows from approximation of f by linear combinations of simple functions of the form $\lambda(\mathbf{1}_{[0,t_i]}\mathbf{1}_{[0,\cdot]})$ and the upper bound in Lemma 2.7 (where the constant M_1 comes from). Thus,

$$-b\frac{(\log\log u^{-1})^{h/\nu}}{u^{\nu h}}\left\|f - f\left(\frac{s}{r}\cdot\right)\right\|_\infty \geq -M_1\left(\log\log u^{-1}\right)^{h/\nu+1/2}\left(1-\frac{s}{u}\right)^h \|f\|_{h,\nu} .$$

For the last term, we use the fact that:

$$b - a \leq \sqrt{u^{2\nu h}\log\log u^{-1} - s^{2\nu h}\log\log s^{-1}} ,$$

which ends the proof of this lemma.

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