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***Abstract*—A method to carry out the state estimation is proposed for a class of nonlinear systems with unknown inputs whose dynamics is governed by differential-algebraic equations (DAE). We achieve, under suitable conditions, to replace the original DAE for a system with differential equations only by using a zeroing manifold algorithm inducing a state space dimension reduction. Observability conditions can be checked using the original system parameters. The state estimation is done using a sliding mode high order differentiator.**

***Index Terms*—Differential-Algebraic systems, Observer design, sliding mode high order differentiation.**

I. Introduction

Differential-algebraic equations (DAE) arise in applications such as electrical networks, chemical engineering, semidiscretized Stokes equations, cellular biology and so on (see, e.g. [1]). The problem of state estimation of DAE systems has been tackled in the last two decades [2], [3], [4]. For the case of systems when all the inputs are known, the algebraic observability of DAE time varying systems has been studied in [5] (a system is called observable in the differential-algebraic framework if the state can be expressed in terms of the output and the known inputs and a finite number of their time derivatives [6]). In [7] and [8], observers are proposed for singular systems using an LMI approach. Using also LMI's, a reduced order observer for a class of Lipschitz nonlinear singular systems is presented in [9]. Asymptotic observers for systems having index one were proposed in [10] and [11].

Here, we consider that the system contains unknown inputs and the DAE are given in an explicit form. Under suitable conditions, we achieve to replace the DAE of the system by ODE on a manifold of reduced dimension. This is done by searching for an invariant submanifold (called zeroing submanifold) where the DAE are satisfied during an interval of time. Then observability conditions are found in terms of the original system parameters. Finally, the state estimation is carried out by using a sliding mode high order differentiator (SMHOD). The formulation of the problem is stated in Section II. The procedure to replace the DAE by ODE on a reduced dimension manifold is described in section III. This is done by means of a *zeroing manifold algorithm* (conceived from the zero dynamics concept in [12]). In section IV, we tackle the state estimation. Section V has an example with simulations, supporting the theoretical results. We use $L_f h$ to denote the Lie derivative of the function h along the vector field f, i.e. $L_f h = \langle dh(x), f(x)\rangle = \frac{\partial h(x)}{\partial x} f(x)$. For a smooth manifold M, $T_x M$ is the tangent space to M at x. By $\dim f(x)$, we denote the number of rows of the vector $f(x)$. $\mathrm{Im}A$ is the image (range space) of A matrix. $\mathrm{col}(A,B)$ is the matrix obtained by concatenating matrices A and B in vertical direction. A^T and A^+ are the transpose and pseudo-inverse of A, respectively.

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II. System Description and Problem Formulation

We consider systems described by the following equations:

$$\dot{x}(t) = f(x(t)) + g(x(t))\mu(t) \tag{1a}$$

$$0 = F(x(t)) + G(x(t))\mu(t) \tag{1b}$$

$$y(t) = h(x(t)) \tag{1c}$$

where the state $x(t)$ belongs to an open set $\mathbb{U} \subset \mathbb{R}^n$. The maps $f : \mathbb{U} \to \mathbb{R}^n$, $g : \mathbb{U} \to \mathbb{R}^{n\times m}$, $F : \mathbb{U} \to \mathbb{R}^q$, $G : \mathbb{U} \to \mathbb{R}^{q\times m}$, and $h : \mathbb{U} \to \mathbb{R}^p$ are all smooth maps. The input vector $\mu(t) \in \mathbb{R}^m$ is unknown a priori; however it should be noted that $\mu(t)$ has to be so that a solution for (1a)-(1b) exists. The aim is the estimation of $x(t)$ by means of the system output $y(t)$. Let N be a set defined as

$$N = \{x \in \mathbb{U} : \exists \mu_x \in \mathbb{R}^m \text{ s.t. } F(x) + G(x)\mu_x = 0\} \tag{2}$$

In what follows we will do our study around an $x_0 \in N$ for which $x(t;x_0)$ satisfies (1a)-(1b) in a neighborhood of $t = 0$.

III. Searching for a maximal zeroing submanifold

The procedure pursued here to estimate $x(t)$ lies into two main steps. Firstly, we look for a maximal zeroing submanifold (w.r.t. $F(x)$ and $G(x)$), which is a submanifold such that if $x(0)$ belongs to it then there exists an input function $\mu(t)$ such that $x(t;x(0))$ satisfies (1a)-(1b) for all t in a neighborhood of $t = 0$. The first part yields a coordinates transformation so that some terms of the state in the new coordinates are equal to zero (the same number as the dimension of the zeroing submanifold). This also allows for expressing the input vector as a function of the state vector. The second part consists in using a (reduced order) observer for a system without unknown inputs. Now, we proceed to give a formal definition of a *zeroing submanifold*. For it, we will need to define invariant and locally invariant submanifolds (see [13]). Let M be a *smooth submanifold* of $\mathbb{R}^n$.

Definition 1: Let $V \subset M$ be a smooth submanifold, and f a vector field on M. Then, V is an **Invariant submanifold** (**ISM**) w.r.t. f if, for all $v \in V$, $f(v) \in T_v V \subset T_v M$.

Definition 2: M is a **locally ISM** at x_0 w.r.t. f if there exists a neighborhood U_0 of x_0 such that $M \cap U_0$ is an ISM w.r.t. f.

Next definition is in its essence a definition found in Chapter 6 of [12]; however, we have adapted it using the previous definitions of ISM and also clause i) has been slightly modified to consider the effect of $G(x)$ of (1b).

Definition 3: A **zeroing submanifold** (**ZSM**) at x_0 is a smooth submanifold $M \subset \mathbb{U}$ containing x_0 that satisfies i) $M \subset N$ and ii) there exists a smooth mapping $\mu : M \to \mathbb{R}^m$ so that M is a *locally ISM* at x_0 w.r.t. the vector field $\hat{f}(x) := f(x) + g(x)\mu(x)$.

Remark 1: Clause ii) means that there exists a neighborhood U_0 of x_0 such that if $x(0) \in M \cap U_0$ then $x(t) \in M \cap U_0$ for all t in a neighborhood of $t = 0$ (see, e.g., [13]).

Proposition 1: If a ZSM M is such that the ISM $M \cap U_0$ (w.r.t. $\hat{f}(x)$) is a closed set of $\mathbb{U}$, then $x(t)$ stays within $M \cap U_0$ for all t, provided $x(0)$ belongs to $M \cap U_0$.

Proof. In case $x(0) \in M \cap U_0$, there exists an interval $[0, t_1)$ for which $x(t)$ remains in $M \cap U_0$ (see Remark 1). Let us suppose that $x(t)$ leaves $M \cap U_0$ after a time. Then the set $\Upsilon = \{t : x(t) \notin M \cap U_0\}$ is not empty. Since Υ is lower bounded, we may define $t_{\inf} = \inf(\Upsilon)$. We claim that $t_{\inf}$ does not belong to Υ. Indeed, if $t_{\inf}$ is an element of Υ, then $x(t_{\inf})$ belongs to the complement of $M \cap U_0$ (denoted by $(M \cap U_0)^c$), which by hypothesis is an open set. Hence, there exists an open set Λ contained in $(M \cap U_0)^c$ and having $x(t_{\inf})$ as an element. By the continuity of $x(t)$, the inverse image of Λ is open, that is, $t_{\inf}$ is a point of an open interval contained in Υ, which is a contradiction. Thus, the only remaining option is that $t_{\inf}$ belongs to the complement of Υ, or, what is the same, $t_{\inf}$ is so that $x(t_{\inf}) \in M \cap U_0$. However, in that case, since $M \cap U_0$ is invariant, then there exists an open interval that contains $t_{\inf}$ and on which $x(t)$ belongs to $M \cap U_0$, which implies that $t_{\inf}$ is not the infimum of Υ, a contradiction again by the definition of $t_{\inf}$. Thus, finally, we conclude that Υ is the empty set. ■

Definition 4: A **ZSM** M is **locally maximal** if, for any other ZSM $\bar{M}$, there exists a neighborhood U of x_0 such that the inclusion $M \cap U \supset \bar{M} \cap U$ is satisfied.

We will seek for a locally maximal ZSM. The proposed method is similar to the one given in [12], pp. 299-301. However, we **do not assume that** $q = m$ and we **include the input explicitly in the algebraic equation**. The proposed algorithm is a nonlinear version of the algorithm used to find the weakly unobservable subspace in linear systems with inputs appearing explicitly in the differential equations and in the system output (see, e.g., [14]). The following is our step-by-step algorithm to find a locally maximal ZSM.

Step 1. It is assumed that there exists a neighborhood U_0 containing x_0 such that the $\operatorname{rank} G(x) = r_0$ for all $x \in U_0$, for some r_0. Let us define $M_0 = U_0$. For if, there exists a full row rank matrix $R_0(x)$ with terms being smooth functions of x in a neighborhood U_0' of x_0 such that

$$\operatorname{rank} R_0(x) = q - r_0 \text{ and } R_0(x) G(x) = 0 \text{ for all } x \in U_0' \tag{3}$$

Thus, the maps $\Phi_0(x)$ and $H_1(x)$ are defined as $H_1(x) = \Phi_0(x) := R_0(x) F(x)$. Let us assume that the rank of $dH_1(x)$ is constant in a neighborhood $U_1 \subset U_0'$ of x_0. Then, the set $M_1 := \{x \in U_1 : H_1(x) = 0\}$ is a smooth submanifold.

Proposition 2: M_1 satisfies the identity $M_1 = N \cap U_1$.

Proof. By its construction $M_1 \subset N \cap U_1$. Indeed, if $H_1(x) = 0$, then $F(x) \in \operatorname{Im} G(x)$ (i.e. $x \in N$) because of $R_0(x) G(x) = 0$ and $R_0(x) F(x) = 0$ and $\operatorname{rank} R_0(x) = q - \operatorname{rank} G(x)$. For the converse, if $x \in N \cap U_1$, then, from (2), $F(x) \in \operatorname{Im} G(x)$. Hence, in view of (3) and since $H_1(x) = R_0(x) F(x)$, we obtain that $H_1(x) = 0$. That is, we conclude that $N \cap U_1 \subset M_1$. ■

Step 2. Let us assume that rank of $\operatorname{col}(G(x), L_g H_1(x))$ is equal to a constant r_1 for all x in M_1. Then there exists a matrix $R_1(x)$ with terms being smooth functions of x in a neighborhood U_1' of x_0 such that, for all $x \in M_1 \cap U_1'$, $R_1(x) \begin{pmatrix} G(x) \\ L_g H_1(x) \end{pmatrix} = 0$ and $\operatorname{rank} R_1(x) = q + \dim H_1(x) - r_1$. Thus, we define $H_2(x) = \operatorname{col}(H_1(x), \Phi_1(x))$ where $\Phi_1(x) = R_1(x) \operatorname{col}(F(x), L_f H_1(x))$. Again, let us assume that $dH_2(x)$ has constant rank in a neighborhood $U_2 \subset U_1'$. Thus, the set $M_2 := \{x \in U_2 : H_2(x) = 0\}$ is a smooth submanifold also.

Step k. Assuming that $\operatorname{rank} \operatorname{col}(G(x), L_g H_{k-1}(x)) = r_{k-1}$ for all $x \in M_{k-1}$, then there exists a neighborhood U_{k-1}' of x_0 and a matrix $R_{k-1}(x)$ of smooth functions on U_{k-1}' such that

$$\begin{gathered} \operatorname{rank} R_{k-1}(x) = q + \dim H_{k-1}(x) - r_{k-1} \\ R_{k-1}(x) \begin{pmatrix} G(x) \\ L_g H_{k-1}(x) \end{pmatrix} = 0 \end{gathered} \tag{4}$$

for all $x \in M_{k-1} \cap U_{k-1}'$. By defining $H_k(x)$ as

$$H_k(x) = \begin{pmatrix} H_{k-1}(x) \\ \Phi_{k-1}(x) \end{pmatrix}, \ \Phi_{k-1}(x) = R_{k-1}(x) \begin{pmatrix} F(x) \\ L_f H_{k-1}(x) \end{pmatrix} \tag{5}$$

and if $dH_k(x)$ has constant rank on $U_k \subset U_{k-1}'$ around x_0, we obtain the smooth manifold M_k:

$$M_k = \{x \in U_k : H_k(x) = 0\}$$

Lemma 1: Assume that there exist nested neighborhoods[1] $U_{k+1} \subset U_k$ and $U_{k+1}' \subset U_k'$ ($k \in \overline{1,n}$) of x_0 such that, for every k, $dH_k(x)$ has constant rank in U_k and $\operatorname{col}(G(x), L_g H_k(x))$ has constant rank for all x on the smooth manifold

$$M_k := \{x \in U_k : H_k(x) = 0\} \tag{6}$$

for all $k \in \overline{1,n}$, and $H_k(x)$ and $R_k(x)$ satisfy (5) and (4), respectively, on U_k'. Then, there exists a $k^* \le n$ and a neighborhood $\bar{U}_{k^*}$ so that $M_{k^*} \cap \bar{U}_{k^*} = M_{k^*+j} \cap \bar{U}_{k^*}$ for all $j \ge 1$.

The proof of Lemma 1 rests upon the following proposition.

Proposition 3: Under assumptions of Lemma 1, we obtain that $M_{k+1} \subset M_k$, for $k \ge 1$.

Proof. For if $x \in M_{k+1}$, by (6), we have $x \in U_{k+1}$ and $H_{k+1}(x) = 0$. Thus, $H_{k+1}(x) = 0$ implies, by (5), that $H_k(x) = 0$, which in turns implies, again by (6), that $x \in M_k$. ■

Proof. [Lemma 1] Since $M_{k+1} \subset M_k$, the dimension of M_k is not increasing w.r.t. k. Then, there exists a $k^* \le n$ such that the dimension of M_{k^*+1} must be equal to that of M_{k^*}, i.e., for

[1] Those neighborhoods should be calculated as large as possible in order to calculate an M_{k^*} as large as possible.

a neighborhood W, we have $M_{k^*}\cap W=M_{k^*+1}\cap W$. Now, by (4) and (5), $H_k(x)=0$ only if

$$\begin{pmatrix} F(x) \\ \langle dH_{k-1}(x), f(x)\rangle \end{pmatrix} \in \operatorname{Im}\begin{pmatrix} G(x) \\ \langle dH_{k-1}(x), g(x)\rangle \end{pmatrix} \quad (7)$$

where $H_0(x):=0$. Since rank of dH_{k-1} is constant on U_{k-1}, $\ker dH_{k-1}(x)=T_xM_{k-1}$ for $x\in M_{k-1}\cap U_{k-1}$ (for all $k\geq 1$). Therefore, the submanifold M_k can be rewritten as follows,

$$M_k=\left\{x\in U_k\cap M_{k-1}:\begin{pmatrix} F(x) \\ f(x)\end{pmatrix}\in \operatorname{Im}\begin{pmatrix} G(x) \\ g(x)\end{pmatrix}+\begin{pmatrix} 0 \\ T_xM_{k-1}\end{pmatrix}\right\} \quad (8)$$

Furthermore, $T_xM_{k^*+1}=T_xM_{k^*}$ $(x\in M_{k^*+1}\cap W)$ since $M_{k^*}\cap W=M_{k^*+1}\cap W$. Hence, by (8), we obtain the identities $M_{k^*+2}\cap W\cap U_{k^*+2}=M_{k^*+1}\cap W\cap U_{k^*+2}=M_{k^*}\cap W\cap U_{k^*+2}$. Inductively, we obtain that $M_{k^*+j}\cap \bar{U}_{k^*}=M_{k^*}\cap \bar{U}_{k^*}$ for all $j\geq 1$, for a neighborhood $\bar{U}_{k^*}$. ■

Remark 2: Lemma 1 implies that the algorithm will stop at k^* step. Moreover, k^* will be the first integer k satisfying $\operatorname{rank} dH_k(x)=\operatorname{rank} dH_{k+1}(x)$. This is true because of the dimension of M_k is $n-\operatorname{rank} dH_k(x)$ (for $k\geq 1$) and M_{k^*} and M_{k^*+1} have the same dimension.

Proposition 4: If conditions of Lemma 1 are satisfied with a set of matrices $R_0(x)$, $R_1(x)$, $\ldots, R_{k^*-1}(x)$, then those conditions remain valid for other choice of such a set of matrices.

Proof. Let us define $H_0(x)=0$, then we may define the manifold $M_0=\{x\in U_0: H_k(x)=0\}$. Let $\tilde{R}_0(x)$, $\tilde{R}_1(x),\ldots,\tilde{R}_{k^*-1}(x)$ be other choice of matrices. We are to prove by induction that the maps generated by this set of matrices satisfy the following equations on M_k, (for $k=0,1,\ldots,k^*$):

$$d\tilde{H}_k(x)=S_k(x)\,dH_k(x) \quad (9)$$

$$\tilde{\Phi}_k(x)=T_k(x)\Phi_k(x)+V_k(x) \quad (10)$$

where $V_k(x)$ vanishes at M_k.

Obviously, $\tilde{H}_0(x)=H_0(x)$ and so $d\tilde{H}_0(x)=dH_0(x)$. Now, since the rows of $\tilde{R}_0(x)$ form a basis of the set of solutions of $\gamma(x)G(x)=0$ for all $x\in M_0$, then this implies that $\tilde{R}_0(x)=T_0(x)R(x)+L_0(x)$ where $T_0(x)$ is nonsingular on M_1 and $L_0(x)$ vanishes at M_0. Thus,

$$\tilde{H}_1(x)=\tilde{R}_0(x)F(x)=T_0(x)H_1(x)+V_1(x) \quad (11)$$

where $V_1(x):=L_0(x)F(x)$. Thus, since by definition $H_1=\Phi_0(x)$, and taking into account (11), then (10) is satisfied for $k=0$. Now, let us suppose that (9) and (10) are satisfied for $k=j$. Then, taking into account (5), (9) and (10), we obtain the equation

$$d\tilde{H}_{j+1}=\begin{pmatrix} d\left(S_j(x)H_j(x)\right) \\ d\left(T_j(x)\Phi_j(x)+V_j(x)\right)\end{pmatrix}$$

which in view that $H_j(x)$ and $\Phi_j(x)$ vanish on M_{j+1}, and since $dV_j(x)=P(x)\,dH_j(x)$ at each $x\in M_{j+1}$, for suitable $P(x)$, then the identities

$$\begin{aligned} d\tilde{H}_{j+1} &= \begin{pmatrix} S_j(x)\,dH_k(x) \\ d\left(T_j(x)\Phi_j(x)+V_j(x)\right)\end{pmatrix} \\ &= \begin{pmatrix} S_j(x) & 0 \\ P(x) & T_j(x)\end{pmatrix} dH_{j+1}(x)=S_{j+1}dH_{j+1}(x) \end{aligned} \quad (12)$$

are valid at each $x\in M_{j+1}$. Now, since at each $x\in M_{j+1}$, the rows of $\tilde{R}_{j+1}(x)$ form a basis of the set of solutions of the equation $\gamma(x)\operatorname{col}\left(G, L_g\tilde{H}_{k+1}\right)=0$, then, by (12), the following equation is obtained

$$\tilde{R}_{j+1}(x)=T_{j+1}R_{j+1}(x)\begin{pmatrix} I & 0 \\ 0 & S_{j+1}^{-1}\end{pmatrix}+L_{j+1}(x)$$

where $T_{j+1}(x)$ is nonsingular at each $x\in M_{j+1}$ and $L_{k+1}(x)$ vanishes on M_{j+1}. Thus, taking into account (5) and (12), the following equation is straightforwardly obtained

$$\begin{aligned} \tilde{\Phi}_{j+1}(x) &= T_{j+1}\Phi_{j+1}+L_{j+1}(x)\begin{pmatrix} F(x) \\ L_f\tilde{H}_{j+1}(x)\end{pmatrix} \\ &= T_{j+1}\Phi_{j+1}+V_{j+1}(x) \end{aligned}$$

where $V_{j+1}(x)$ is implicitly defined and it vanishes on M_{j+1}.

Hence, since (9) is true for $k=1,2,\ldots,k^*$, then the conditions of Lemma 1 are still valid. ■

Theorem 1: $Z^*:=M_{k^*}$ is a locally maximal **ZSM**.

Proof. Since rank of $\operatorname{col}\left(G(x), L_gH_{k^*}(x)\right)$ is constant, by (7) and the Implicit Function Theorem, there exists $\mu^*: M_{k^*+1}\to\mathbb{R}^m$, smooth on a neighborhood $U_{k^*}^{'}$ of x_0, such that

$$F(x)+G(x)\mu^*(x)=0 \text{ and } L_fH_{k^*}(x)+L_gH_{k^*}(x)\mu^*(x)=0 \quad (13)$$

Second equation implies that $f(x)+g(x)\mu(x)\in T_xM_{k^*}$ for $x\in M_{k^*+1}$. However, $\hat{f}(x)\in T_xZ^*$ $(\hat{f}(x):=f(x)+g(x)\mu(x))$, for $x\in Z^*\cap\left(U_{k^*}^{'}\cap\bar{U}_{k^*}\right)$, since $Z^*\cap\bar{U}_{k^*}=M_{k^*+1}\cap\bar{U}_{k^*}$. Hence, the restriction of $\hat{f}(x)$ to $Z^*\cap\left(U_{k^*}^{'}\cap\bar{U}_{k^*}\right)$ maps onto T_xZ^*, i.e. Z^* is a locally ISM w.r.t. $\hat{f}(x)$.

Let us suppose that there exists a manifold Z such that $x(t)\in Z$ for all t on $[0,T]$, provided that $x(0)\in Z$. Thus, if $x\in Z\cap\bar{U}_{k^*}$, then $H_1(x)=0$ and $x\in M_1$. By induction, we obtain that $H_k(x)=0$ for all $k\geq 1$. Hence $x\in M_k\cap\bar{U}_{k^*}$, that is $x\in Z^*\cap\bar{U}_{k^*}$. Therefore, $Z\cap\bar{U}_{k^*}\subset Z^*\cap\bar{U}_{k^*}$. ■

Proposition 5: Assuming that rank of $\operatorname{col}\left(G(x), L_gH_{k^*}(x)\right)$ is equal to m for $x\in Z^*$, there exists a unique (locally) smooth mapping $\mu^*: Z^*\to\mathbb{R}^m$, such that $F(x)+G(x)\mu^*(x)=0$ and $\hat{f}(x)\in T_xZ^*$ $(\hat{f}(x):=f(x)+g(x)\mu^*(x))$. That is, the equation

$$\begin{pmatrix} F(x) \\ L_fH_{k^*}(x)\end{pmatrix}+\begin{pmatrix} G(x) \\ L_gH_{k^*}(x)\end{pmatrix}\mu^*(x)=0 \quad (14)$$

has a unique solution around x_0.

Proof. From the proof of Theorem 1, there is a smooth mapping $\mu^*: M_{k^*+1}\to\mathbb{R}^m$ satisfying (13) for all $x\in M_{k^*+1}\cap U_{k^*}^{'}$, for a neighborhood $U_{k^*}^{'}$ of x_0. We have proved also that (13) implies that $F(x)+G(x)\mu^*(x)=0$ and $\hat{f}(x)\in T_xZ^*$, for all $x\in Z^*\cap\left(U_{k^*}^{'}\cap\bar{U}_{k^*}\right)$. Now, taking into account that $Z^*\cap\bar{U}_{k^*}=M_{k^*+1}\cap\bar{U}_{k^*}$, and since rank of $\operatorname{col}\left(G(x), L_gH_{k^*}(x)\right)$ is equal to m for $x\in Z^*$, then we conclude that there exists a unique smooth mapping $\mu^*: Z^*\to\mathbb{R}^m$ such that $F(x)+G(x)\mu^*(x)=0$ and $\hat{f}(x)\in T_xZ^*$, for $x\in Z^*\cap\left(U_{k^*}^{'}\cap\bar{U}_{k^*}\right)$. ■

Remark 3: Under the assumption of the previous proposition, the differential index of the DAE will be equal to k^* . This is due to the fact that the algorithm followed to calculate the zeroing submanifold introduces intrinsically a procedure with which, after k^* time derivatives of the algebraic equation, we may obtain an ODE around x_0.

IV. State Reconstruction

Let f^* be the restriction of $\hat{f}(x) = f(x) + g(x)\mu^*(x)$ to Z^* (assuming that rank $\begin{pmatrix} G(x) \\ L_g H_{k^*}(x) \end{pmatrix} = m$ for any $x \in Z^*$). Thus on Z^*, the dynamics of system (1) is governed by

$$\dot{x} = f^*(x) \text{ and } y = h(x) \tag{15}$$

Let us recall the definition of local weak observability.

Definition 5: [[15], [16]] A pair $(x_0, x_0') \in \mathbb{U} \times \mathbb{U}$ ($x_0 \neq x_0'$) is indistinguishable for system (1) if, for all μ and for all $t \geq 0$, $h(x_\mu(t, x_0)) = h(x_\mu(t, x_0'))$.

Definition 6: [[16]] System (1) is locally weakly observable (LWO) at x_0 if there exists a neighborhood U of x_0 such that for any neighborhood $V \subset U$ of x_0 there is no indistinguishable pair (x_0, x_0') in V when considering time intervals for which trajectories remain in V.

Lemma 2: Under the assumptions of Lemma 1, system (1) is LWO at x_0 if, and only if, (15) is LWO at x_0.

Proof. It is clear that if (1) is LWO at $x_0 \in Z^*$, (15) is LWO at x_0. Indeed, if (15) is not LWO at x_0, then for any neighborhood $\bar{U} \subset Z^*$, there exists a neighborhood $\bar{V} \subset \bar{U}$ such that (x_0, x_0') is indistinguishable on $Z^* \times Z^*$, for a $x_0' \in \bar{V}$. Therefore, since $Z^* \subset \mathbb{U}$, then (x_0, x_0') is indistinguishable also on $\mathbb{U} \times \mathbb{U}$ and, therefore, (1) is not LWO at x_0 either.

Now, let us consider that $x_0 \in Z^*$ and that (1) is not LWO at x_0. Then by Definition 6, in every neighborhood $U \in \mathbb{R}^n$ of x_0 there exists a pair $\left(x_0, x_0^{'}\right) \in U \times U$ that is indistinguishable on (1). Hence, in particular, the neighborhood $Z^* \cap \left(U_{k^*}^{'} \cap \bar{U}_{k^*}\right)$ ($U_{k^*}^{'}$ and $\bar{U}_{k^*}$ as in the proof of Theorem 1) contains the vector x_0^* such that (x_0, x_0^*) is indistinguishable. Thus, (x_0, x_0^*) is indistinguishable on (15) also because of both x_0 and x_0^* belong to Z^*. Therefore, we can infer that if (15) is LWO at x_0, then (1) is LWO at x_0. ■

Let n^* be the dimension of Z^*. Since $\operatorname{rank} dH_{k^*}(x) = n - n^*$ for all $x \in Z^*$, we can arrange a vector function $\bar{H}^*(x) \in \mathbb{R}^{n-n^* \times n}$ whose terms are taken from $H_{k^*}(x)$ so that $\operatorname{rank} d\bar{H}^*(x_0) = n - n^*$. Thus, there exists a diffeomorphism $\Psi(x) = \begin{pmatrix} \bar{H}^*(x) \\ \phi(x) \end{pmatrix}$ with which, defining $z = \Psi(x)$, we obtain

$$z_1(t) = \bar{H}^*(x(t)) = 0,\ \dot{z}_2(t) = \tilde{f}_2(z_2(t)),\text{ and } y(t) = \tilde{h}_2(z_2)$$

where $z_1(t) \in \mathbb{R}^{n-n^*}$ and $z_2(t) \in \mathbb{R}^{n^*}$. There, $\tilde{f}_2(z_2(t))$ and $\tilde{h}_2(z_2)$ are given by the formulas

$$\begin{aligned} \tilde{f}_2(z_2) &= \left[\tfrac{\partial \phi(x)}{\partial x} f^*(x)\right]_{x=\Psi^{-1}(z)} \\ \tilde{h}_2(z_2) &= \tilde{h}(z)|_{z_1=0} \ \left(\tilde{h}(z) = h\left(\Psi^{-1}(z)\right)\right) \end{aligned} \tag{16}$$

Thereby, the original problem is reduced to the estimation of z_2 from the knowledge of $y(t)$. However, since LWO is not enough for the design of an observer, below we will assume that (15) is **uniformly observable** (see, e.g., [17]), i.e., we assume that on

$$Z_0^* = \left\{z_2 \in \mathbb{R}^{n^*} : z_2 = \phi(x) \text{ for } x \in Z^* \text{ s.t. } \bar{H}^*(x) = 0\right\}$$

the rank condition (17) is satisfied

$$\operatorname{rank}\operatorname{col}\left(d\tilde{h}_2(z_2), dL_{\tilde{f}_2}\tilde{h}_2(z_2), \ldots, dL_{\tilde{f}_2}^{n^*-1}\tilde{h}_2(z_2)\right) = n^* \tag{17}$$

There are several observers that may be used to carry out the estimation of z_2 provided that (17) is satisfied, like high gain observers [17] or finite time observers [18]. Below, in Theorem 2, we show that condition (17) can be checked in the original coordinates.

Theorem 2: Under the assumptions of Lemma 1, (1) is **uniformly observable** on Z^* if (18) is satisfied for all $x \in Z^*$.

$$\operatorname{rank}\operatorname{col}\left(dH_{k^*}(x), dh(x), dL_{f^*}h(x), \ldots, dL_{f^*}^{n^*-1}h(x)\right) = n \tag{18}$$

Proof. Let us define $\tilde{f}(z) = \left[\frac{\partial \Psi(x)}{\partial x} f^*(x)\right]_{x=\Psi^{-1}(z)}$. Thus, from the fact that $L_{f^*}^k h(x) = L_{\tilde{f}}^k \tilde{h}(z)|_{z=\Psi(x)}$, it is clear that

$$\frac{\partial}{\partial x} L_{f^*}^k h(x) = \frac{\partial}{\partial z}\left(L_{\tilde{f}}^k \tilde{h}(z)\right) \frac{\partial \Psi(x)}{\partial x}\bigg|_{z=\Psi(x)} \tag{19}$$

Hence, the identity

$$\frac{\partial}{\partial z_2} L_{\tilde{f}}^k \tilde{h}(z) = \frac{\partial}{\partial z_2} L_{\tilde{f}_2}^k \tilde{h}_2(z_2) \tag{20}$$

is satisfied on Z^*, for $k \geq 0$ (this may be verified by using induction, taking into account that $z_1 = 0$). Thus, for $x \in Z^*$, by (19) and (20), we obtain the identities, for $i \geq 0$,

$$\begin{aligned} d\bar{H}^*(x) &= \begin{bmatrix} I_{n^*} & 0 \end{bmatrix} \tfrac{\partial \Psi(x)}{\partial x} \\ dL_{f^*}^i h(x) &= \begin{bmatrix} * & dL_{\tilde{f}_2}^i \tilde{h}_2(z_2) \end{bmatrix} \tfrac{\partial \Psi(x)}{\partial x} \end{aligned}$$

In view of the previous identities and since $\operatorname{rank} d\bar{H}^*(x) = \operatorname{rank} dH_{k^*}(x) = n - n^*$, we conclude that the rank condition in (17) is satisfied if (18) is satisfied. ■

Condition (17) implies that z_2 can be locally expressed as a function of $\left(y, \dot{y}, \ldots, y^{(n^*-1)}\right)$ which is known as **algebraic observability [6]**. In fact, it was shown in [19] that for analytic systems the fulfillment of (17) is equivalent to the algebraic observability of z_2. Thus, the following corollary is an immediate consequence of Theorem 2.

Corollary 1: Under the assumptions of Lemma 1, $\operatorname{rank}\operatorname{col}(G(x_0), L_g H_{k^*}(x_0)) = m$ and (18), there exists a function Γ such that $x(t) = \Gamma\left(y, \dot{y}, \ldots, y^{(n^*-1)}\right)$.

Proof. The function Γ might be found in the following manner. Let us consider the diffeomorphism $\Psi(x)$ defined in above, where the terms of $\phi(x)$ are chosen to be equal to some of the terms of $\operatorname{col}\left(h(x), L_{f^*}h(x), \ldots, L_{f^*}^{n^*-1}h(x)\right)$ so that $\operatorname{rank}\operatorname{col}(dH_{k^*}(x), \phi(x)) = n$ (this choice is possible due to Theorem 2). Thus, the dynamics of z_2 turns out to be a set of chains of integrators (maybe after a rearrange of the coordinates). Therefore, considering that $z_1 = 0$, we obtain straightly an explicit function Γ_z such that $z = \Gamma_z\left(y, \dot{y}, \ldots, y^{n^*-1}\right)$. Hence, finally we obtain that $\Gamma\left(y, \dot{y}, \ldots, y^{n^*-1}\right) = \Psi^{-1}\left(\Gamma_z\left(y, \dot{y}, \ldots, y^{n^*-1}\right)\right)$. ■

Two real-time differentiators that could be used to estimate the required derivatives of the output are described in [20] and [21]. The former is a sliding mode high order differentiator (SMHOD), which is used in the example given further.

Remark 4: [Further generalization] For the case when the map μ^* is not unique, i.e. that $\operatorname{rank}\operatorname{col}(G(x_0), L_g H_{k^*}(x_0)) = r < m$, the state estimation may still be done. As if rank of $\operatorname{col}(G(x), L_g H_{k^*}(x))$ is constant in a neighborhood of

x_0, locally there exist matrices $D_1(x)$ and $D_2(x)$ of rank r and $m-r$, respectively, whose entries are smooth functions of x, such that $\operatorname{rank}(\operatorname{col}(G(x), L_g H_{k^*}(x)) D_1(x)) = r$ and $\operatorname{col}(G(x), L_g H_{k^*}(x)) D_2(x) = 0$ for all x in a neighborhood of x_0. Thus, with $D(x) := (D_1(x), D_2(x))$, and a partition of its inverse as $(D(x))^{-1} = \operatorname{col}(\bar{D}_1(x), \bar{D}_2(x))$, we obtain that

$$\operatorname{col}\left(F(x), L_f H_{k^*}(x)\right) + \operatorname{col}(G(x), L_g H_{k^*}(x)) D_1(x) \bar{D}_1(x) \mu = 0 \tag{21}$$

Let us define $\alpha_1 = \bar{D}_1(x)\mu$ and $\alpha_2 = \bar{D}_2(x)\mu$. Then, (21) has a unique solution for $\alpha_1 = \alpha_1(x)$. Thus, defining $f^*(x) = f(x) + g(x) D_1(x) \alpha_1(x)$, we can rewrite, locally on the manifold Z^*, the dynamic equations of the system as follows

$$\dot{x} = f^*(x) + g(x) D_2(x) \alpha_2 \text{ and } y = h(x)$$

Thereby, the state estimation may be carried out by using an unknown input observer (α_2 is the UI). In particular a reduced order observer may be designed. Indeed, using the diffeomorphism defined at the beginning of this section and the change of coordinates given by $z = \Psi(x)$, we obtain the sub-vector $z_1 = \bar{H}^*(x) = 0$ and z_2 being the state of the system $\dot{z}_2 = \tilde{f}_2(z_2) + \tilde{g}_2(z_2)\alpha_2$ and $y = \tilde{h}_2(z_2)$ where

$$\begin{aligned}
\tilde{f}_2(z_2) &= \left[\frac{\partial \phi(x)}{\partial x} f^*(x)\right]_{x=\Psi^{-1}(z)} \\
\tilde{g}_2(z_2) &= \left[\frac{\partial \phi(x)}{\partial x} g(x) D_2(x)\right]_{x=\Psi^{-1}(z)} \\
\tilde{h}_2(z_2) &= \tilde{h}(z)|_{z_1=0} \text{ with } \tilde{h}(z) = h\left(\Psi^{-1}(z)\right)
\end{aligned}$$

Hence, an UI observer for $z_2(t) \in \mathbb{R}^{n-n^*}$ could be designed. Just to mention two of various approaches that could be followed, we refer the reader to [22], [23].

V. Example

Let us consider an example with the following functions:

$$f(x) = \begin{pmatrix} x_2 & x_4 + x_6 x_1 & x_1 x_4 & x_5 & x_3 & x_7 & -x_5 x_2 \end{pmatrix}^T$$

$$g(x) = \begin{pmatrix} 0 & x_2 & 1 & x_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 x_7 & 1 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 \end{pmatrix}^T, \; h(x) = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$F(x) = \begin{pmatrix} 0 \\ \frac{-\cos\left(\frac{x_5}{2}\right)}{10} \\ x_6 \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0 & 0 & 2x_5 - \sin x_3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

By (2), $N = \left\{x \in \mathbb{R}^7 : x_6 = 0,\ x_5 = \sin(x_3) \text{ or } x_5 = \pi\right\}$. Thus, the observability is around $x_0 = 0$.

Step 1. Since $\operatorname{rank} G(x) = 1$ for all $x \in \mathbb{R}^7$,

$$\begin{aligned}
R_0(x) &= \begin{pmatrix} -1 & 2x_5 - \sin x_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
H_1 &= R_0(x) F(x) = \begin{pmatrix} -\frac{1}{10}(2x_5 - \sin x_3)\cos(\frac{x_5}{2}) \\ x_6 \end{pmatrix}
\end{aligned}$$

We see that rank of the matrix dH_1 is equal to 2 for all $x \in U_1 = \left\{x \in \mathbb{R}^7 : |x_5| < \pi \text{ and } |x_3| < \frac{\pi}{2}\right\}$. Hence, $M_1 = \left\{x \in U_1 : x_6 = 0,\ 2x_5 = \sin x_3\right\}$ is a 5-dimension manifold.

Step 2. For $\operatorname{col}(G(x), L_g H_1(x))$, we obtain

$$\begin{pmatrix} G(x) \\ L_g H_1(x) \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & x_6 x_7 \\ 2x_5 - \sin x_3 & 1 & 0 & \gamma & 0 \end{pmatrix} \tag{22}$$

where $\gamma = \frac{1}{10} x_1 \cos\left(\frac{x_5}{2}\right)\cos(x_3)$ and $\delta = \frac{1}{20}\sin\left(\frac{x_5}{2}\right)(2x_5 - \sin x_3) - \frac{1}{5}\cos\left(\frac{x_5}{2}\right) + \frac{1}{10}\cos\left(\frac{x_5}{2}\right)\cos(x_3)$. Therefore, for all $x \in M_1$, the rank of the matrix in (22) is equal to 2. Thus, $R_1(x)$, $\Phi_1(x)$, and $H_2(x)$ are taken as

$$R_1(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; \Phi_1(x) = \begin{pmatrix} 0 \\ x_6 \\ x_7 \end{pmatrix}$$

$$H_2(x) = \begin{pmatrix} -\frac{1}{10}(2x_5 - \sin x_3)\cos(\frac{x_5}{2}) \\ x_6 \\ x_7 \end{pmatrix}$$

Since $\operatorname{rank} dH_2(x) = 3$ on U_1, $M_2 = \left\{x \in U_1 : x_6 = x_7 = 0 \text{ and } 2x_5 = \sin x_3\right\}$.

Step 3 (end). Now, we have that $\operatorname{col}(G(x), L_g H_2(x)) = \operatorname{col}\left(G(x), L_g H_2(x), \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\right)$, which has rank equal to 3 on M_2. Thus, matrix $R_2(x) \in \mathbb{R}^{3\times 6}$ has zeros everywhere except in the entries (1,1), (2,3), and (3,5), which have a number one. As for $\Phi_2(x)$, it takes the form $\Phi_2(x) = \begin{pmatrix} 0 & x_6 & x_7 \end{pmatrix}^T$. Finally, we obtain that $H_3(x) = H_2(x)$, which implies that $Z^* = M_2$ and $H^* = H_2$. Moreover, $\operatorname{rank}\operatorname{col}(G(x), L_g H^*(x)) = 3$ for all $x \in Z^*$. Therefore, $\mu = \mu^*(x)$ satisfies the equation (14) with $k^* = 2$, for $x \in Z^*$. Thus, we obtain $\mu_1^*(x) = \frac{2x_3 - \left(\frac{1}{10}x_1\cos\left(\frac{x_5}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2}$, $\mu_2^*(x) = x_2 x_5$, and $\mu_3^*(x) = \frac{1}{10}\cos\left(\frac{x_5}{2}\right)$. Thus, $f^*(x)$ ($x \in Z^*$) has the form:

$$\begin{aligned}
f_1^* &= x_2, \quad f_6^* = f_7^* = 0 \\
f_2^* &= x_4 + x_2 \frac{2x_3 - \left(\frac{1}{10}x_1\cos\left(\frac{2\sin(x_3)}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2} \\
f_3^* &= x_1 x_4 + \frac{2x_3 - \left(\frac{1}{10}x_1\cos\left(\frac{x_5}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2} \\
&\quad + \frac{1}{10}x_1\cos\left(\frac{2\sin(x_3)}{2}\right) \\
f_4^* &= 2\sin(x_3) + x_2 \frac{2x_3 - \left(\frac{1}{10}x_1\cos\left(\frac{2\sin(x_3)}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2} \\
f_5^* &= x_3 + \frac{2x_3 - \left(\frac{1}{10}x_1\cos\left(\frac{2\sin(x_3)}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2}
\end{aligned}$$

In this case, matrix in (18) is equal to $\operatorname{col}\left(dH^*, dh, dL_{f^*}h, dL_{f^*}^2 h\right)$, which has rank 7 in a vicinity of $x = 0$. Then, the system is (locally) uniformly observable according to Theorem 2. Furthermore, as the dimension of Z^* is equal to 4, at most 3 derivatives of y are required for the estimation of the entire state x. In fact, we have that x can be expressed in terms of $(y, \dot{y}, \ddot{y})$ as follows:

$$\begin{gathered}
x_1 = y_1,\ x_2 = \dot{y}_1,\ x_3 = y_2,\ x_4 = \ddot{y}_1 + \dot{y}_1\left(y_2 - \tfrac{1}{2}\dot{y}_2\cos y_2\right) \\
x_5 = \tfrac{1}{2}\sin y_2,\ x_6 = x_7 = 0
\end{gathered} \tag{23}$$

$\dot{y}$ and $\ddot{y}$ are estimated using a SMHOD proposed in [20], i.e.,

$$\begin{aligned}
\dot{\bar{y}}_{1,0} &= -\lambda_2 3^{\frac{1}{3}} |\bar{y}_{1,0} - y_1|^{\frac{2}{3}} \operatorname{sign}(\bar{y}_{1,0} - y_1) + \bar{y}_{1,1} \\
\dot{\bar{y}}_{1,1} &= -\lambda_1 3^{\frac{1}{2}} \left|\bar{y}_{1,1} - \dot{\bar{y}}_{1,0}\right|^{\frac{1}{2}} \operatorname{sign}\left(\bar{y}_{1,1} - \dot{\bar{y}}_{1,0}\right) + \bar{y}_{1,2} \\
\dot{\bar{y}}_{1,2} &= -\lambda_0 3\left(\bar{y}_{1,2} - \dot{\bar{y}}_{1,1}\right)
\end{aligned}$$

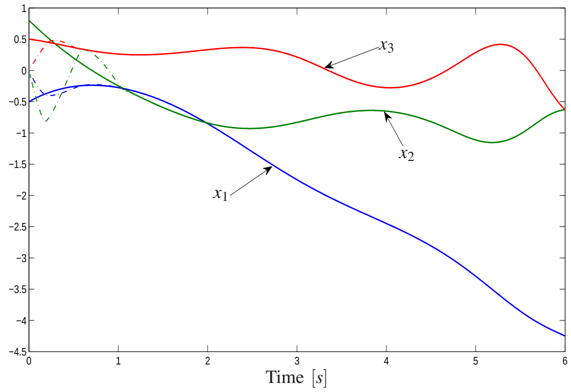

Fig. 1. Original states x_i (solid) and its estimate $\hat{x}_i$ (dash-dotted), ($i = \overline{1,3}$).

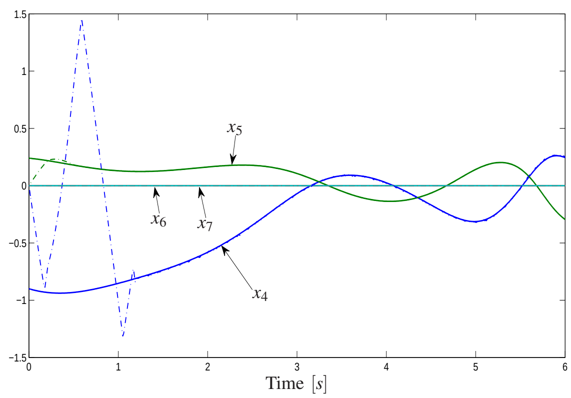

Fig. 2. Original states x_i (solid) and its estimate $\hat{x}_i$ (dash-dotted), ($i = \overline{4,7}$).

$$\begin{aligned} \dot{\bar{y}}_{2,0} &= -\lambda_1 3^{\frac{1}{2}} \left|\bar{y}_{2,0} - y_2\right|^{\frac{1}{2}} \operatorname{sign}\left(\bar{y}_{2,0} - y_2\right) + \bar{y}_{2,1} \\ \dot{\bar{y}}_{2,1} &= -\lambda_0 3 \left(\bar{y}_{2,1} - \bar{y}_{2,0}\right) \end{aligned}$$

with $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$. Hence, the estimate of $\dot{y}$ and $\ddot{y}$ is given by $\hat{y}_1 = \bar{y}_{1,0}$, $\widehat{\dot{y}}_1 = \bar{y}_{1,1}$, $\widehat{\ddot{y}}_1 = \bar{y}_{1,2}$, $\hat{y}_2 = \bar{y}_{2,0}$, and $\widehat{\dot{y}}_2 = \bar{y}_{2,1}$. Then, estimate of x is obtained with $\hat{x}$:

$$\begin{gathered} \hat{x}_1 = \bar{y}_{1,0},\ \hat{x}_2 = \bar{y}_{1,1},\ \hat{x}_3 = \bar{y}_{2,0},\ \hat{x}_6 = \hat{x}_7 = 0 \\ \hat{x}_4 = \bar{y}_{1,2} + \bar{y}_{1,1}(\bar{y}_{2,0} - \tfrac{1}{2}\bar{y}_{2,1}\cos\bar{y}_{2,0}),\ \hat{x}_5 = \tfrac{1}{2}\sin\bar{y}_{2,0} \end{gathered}$$

The states and their estimations are shown in Fig. 1 and 2.

VI. Conclusions

A new method to carry out the state estimation has been proposed. By means of a *zeroing manifold algorithm*, provided that suitable conditions are satisfied, the state space whose dynamics is governed by a sole system of differential equations is found. This has allowed to apply standard techniques for the state and unknown input reconstruction. Nevertheless, the observability conditions allowing the state estimation can be checked also in terms of the original system with DAE. For a future work, one could look for considering a class of system with states having no explicit differential equations governing their dynamics.

Acknowledgment

F.J. Bejarano acknowledges INRIA Lille-Nord Europe, where he held a Post-doctoral position with Non-A team, and the support of Proyecto SIP 20141364 from IPN.

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