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Uli Fahrenberg, Axel Legay

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ULI FAHRENBERG AND AXEL LEGAY

IRISA / Inria Rennes, Campus de Beaulieu, 35042 Rennes Cedex, France
e-mail address: ulrich.fahrenberg@inria.fr

Abstract. We introduce a new category of higher-dimensional automata in which the morphisms are functional homotopy simulations, *i.e.* functional simulations up to concurrency of independent events. For this, we use unfoldings of higher-dimensional automata into higher-dimensional trees. Using a notion of open maps in this category, we define homotopy bisimilarity. We show that homotopy bisimilarity is equivalent to a straightforward generalization of standard bisimilarity to higher dimensions, and that it is finer than split bisimilarity and incomparable with history-preserving bisimilarity.

1. Introduction

The dominant notion for behavioral equivalence of processes is *bisimulation* as introduced by Park [23] and Milner [21]. It is compelling because it enjoys good algebraic properties, admits several easy characterizations using modal logics, fixed points, or game theory, and generally has low computational complexity.

Bisimulation, or rather its underlying semantic model of *transition systems*, applies to a setting in which concurrency of actions is the same as non-deterministic interleaving; using CCS notation [21], $a|b = a.b + b.a$. For some applications however, a distinction between these two is necessary, which has led to development of so-called *non-interleaving* or *truly concurrent* models such as Petri nets [24], event structures [22], asynchronous transition systems [2, 26] and others; see [33] for a survey.

Higher-dimensional automata (or *HDA*) is another non-interleaving formalism for reasoning about behavior of concurrent systems. Introduced by Pratt [25] and van Glabbeek [28] in 1991 for the purpose of a *geometric* interpretation to the theory of concurrency, it has since been shown by van Glabbeek [29] that HDA provide a generalization (up to *history-preserving bisimilarity*) to “the main models of concurrency proposed in the literature” [29], including the ones mentioned above. Hence HDA are useful as a tool for comparing and relating different models, and also as a modeling formalism by themselves.

HDA are geometric in the sense that they are very similar to the *simplicial complexes* used in algebraic topology, and research on HDA has drawn on a lot of tools and methods

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from geometry and algebraic topology such as homotopy [7,9], homology [10,15], and model categories [11,12], see also the survey [13].

There are a number of popular notions of equivalence for HDA and other non-interleaving models, see [29,31]. *Split bisimilarity* takes interleavings of beginning and ending actions into account; *ST-bisimilarity* additionally distinguishes between different occurrences of the same action; *history-preserving bisimilarity* takes entire computing histories into account; and *hereditary history-preserving bisimilarity* additionally distinguishes different possible futures of past computations.

We have in earlier work [4] introduced a new such equivalence, higher-dimensional bisimilarity. Contrary to the previously mentioned ones, this is not a relation between *computations*, but directly at the level of states, transitions etc. Using *unfoldings* of HDA, which geometrically are similar to *universal coverings*, we show in the present paper that this notion is equivalent to another one, *homotopy bisimilarity*, which compares homotopy classes of computations. Placing homotopy bisimulation on the spectrum of non-interleaving equivalences, we show that homotopy bisimilarity is finer than split bisimilarity and incomparable with history-preserving bisimilarity.

Our results imply *decidability* of homotopy bisimilarity for finite HDA. They also put homotopy bisimilarity firmly into the open-maps framework of [18] and tighten the connections between bisimilarity and weak topological *fibrations* [1,19].

Outline. We start by reviewing the category HDA of higher-dimensional automata introduced in [14] in Section 2. This is the category used in [4] as a framework to define composition, following [33], and a notion of bisimilarity via open maps, following [18], for HDA. This latter construction, together with its notion of path category, we recall in Section 3.

Computations in HDA are modeled by cube paths, the higher-dimensional analogue of paths in transition systems. These come with a notion of homotopy which we introduce in Section 4. Based on homotopy classes of cube paths we can then define the construction at the heart of this paper, the unfolding of a HDA.

In Section 5 we introduce the category HDA_h of higher-dimensional automata up to homotopy, based on unfoldings. We also show in this section that unfolding provides a coreflection between HDA and higher-dimensional trees, and between HDA-up-to-homotopy and higher-dimensional trees. In Section 6 we define homotopy bisimilarity via open maps in HDA_h and show that this is the same as bisimilarity in HDA.

All these first sections deal with unlabeled higher-dimensional automata. In Section 7, we introduce labels using an arrow category construction and show that things can easily be transferred to the labeled setting. In Section 8 we compare homotopy bisimilarity to other equivalence notions for non-interleaving models.

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2. Higher-Dimensional Automata

As a formalism for concurrent behavior, HDA have the specific feature that they can express all higher-order dependencies between events in a concurrent system. Like for transition systems, they consist of states and transitions which are labeled with events.

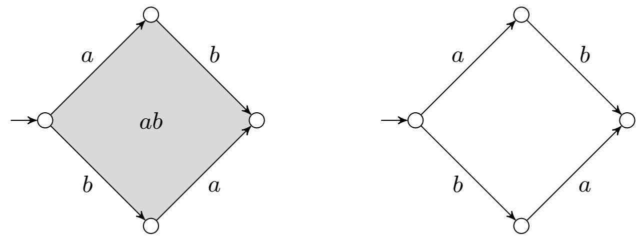

Figure 1: HDA for the CCS expressions $a|b$ (left) and $a.b + b.a$ (right). In the left HDA, the square is filled in by a two-dimensional transition labeled ab, signifying independence of events a and b. On the right, a and b are not independent.

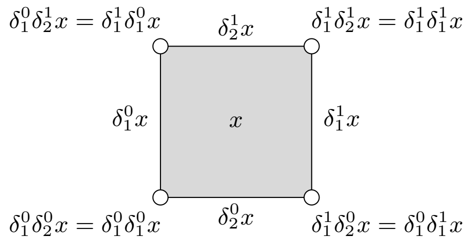

Figure 2: A 2-cube x with its four faces δ_1^0x, δ_1^1x, δ_2^0x, δ_2^1x and four corners.

Now if two transitions from a state, with labels a and b for example, are independent, then this is expressed by the existence of a *two-dimensional* transition with label ab. Fig. 1 shows two examples; on the left, transitions a and b are independent, on the right, they can merely be executed in any order. Hence for HDA, as indeed for any formalism employing the so-called *true concurrency* paradigm, the algebraic law $a|b = a.b + b.a$ does *not* hold; concurrency is not the same as interleaving.

The above considerations can equally be applied to sets of more than two events: if three events a, b, c are independent, then this is expressed using a three-dimensional transition labeled abc. Hence this is different from mutual pairwise independence (expressed by transitions ab, ac, bc), a distinction which cannot be made in formalisms such as asynchronous transition systems [2,26] or transition systems with independence [33] which only consider binary independence relations.

An unlabeled HDA is essentially a pointed precubical set as defined below. For labeled HDA, one can pass to an arrow category; this is what we shall do in Section 7. Until then, we concentrate on the unlabeled case.

A *precubical set* is a graded set $X = \{X_n\}_{n\in\mathbb{N}}$ together with mappings $\delta_k^\nu : X_n \to X_{n-1}$, $k \in \{1, \dots, n\}$, $\nu \in \{0, 1\}$, satisfying the *precubical identity*

$$\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu \qquad (k < \ell)\,. \tag{2.1}$$

The mappings δ_k^ν are called *face maps*, and elements of X_n are called *n-cubes*. As above, we shall usually omit the extra subscript (n) in the face maps. Faces $\delta_k^0 x$ of an element $x \in X$ are to be thought of as *lower faces*, $\delta_k^1 x$ as *upper faces*. The precubical identity expresses the fact that $(n-1)$-faces of an n-cube meet in common $(n-2)$-faces, see Fig. 2 for an example of a 2-cube and its faces.

We will always assume the sets X_n to be disjoint. For an n-cube $x \in X_n$, we denote by $\dim x = n$ its *dimension.*

Morphisms $f : X \to Y$ of precubical sets are graded mappings $f = \{f_n : X_n \to Y_n\}_{n \in \mathbb{N}}$ which commute with the face maps: $\delta_k^\nu \circ f_n = f_{n-1} \circ \delta_k^\nu$ for all $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$, $\nu \in \{0, 1\}$. This defines a category pCub of precubical sets and morphisms.

It can be shown [16] that the category pCub is complete and cocomplete, with pointwise limits and colimits. In elementary terms this means that, for instance, the *product* $Z = X \times Y$ of two precubical sets X, Y is given by $Z_n = X_n \times Y_n$ and face maps $\delta_k^\nu(x, y) = (\delta_k^\nu x, \delta_k^\nu y)$. Likewise, a *precubical subset* $Y \subseteq X$ of $X \in \mathsf{pCub}$ is a precubical set Y for which $Y_n \subseteq X_n$ for all n.

A *pointed* precubical set is a precubical set X with a specified 0-cube $i \in X_0$, and a pointed morphism is one which respects the point. This defines a category which is isomorphic to the comma category $* \downarrow \mathsf{pCub}$, where $* \in \mathsf{pCub}$ is the precubical set with one 0-cube and no other n-cubes. Note that $*$ is *not* terminal in pCub (instead, the terminal object is the somewhat unwieldy infinite-dimensional precubical set with one cube in every dimension).

Definition 2.1. The category of *higher-dimensional automata* is the comma category $\mathsf{HDA} = * \downarrow \mathsf{pCub}$, with objects pointed precubical sets and morphisms commutative diagrams

$$\begin{tikzcd}[column sep=small] & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} .$$

Hence a one-dimensional HDA is a transition system; indeed, the category of transition systems [33] is isomorphic to the full subcategory of HDA spanned by the one-dimensional objects. Similarly one can show [14] that the category of asynchronous transition systems is isomorphic to the full subcategory of HDA spanned by the (at most) two-dimensional objects. The category HDA as defined above was used in [4] to provide a categorical framework (in the spirit of [33]) for parallel composition of HDA. In this article we also introduced a notion of higher-dimensional bisimilarity which we will review in the next section.

3. Path Objects, Open Maps and Bisimilarity

With the purpose of introducing bisimilarity via *open maps* in the sense of [18], we identify here a subcategory of HDA consisting of path objects and path-extending morphisms. We say that a precubical set X is a *precubical path object* if there is a (necessarily unique) sequence $(x_1, \dots, x_m)$ of elements in X such that $x_i \neq x_j$ for $i \neq j$,

- for each $x \in X$ there is $j \in \{1, \dots, m\}$ for which $x = \delta_{k_1}^{\nu_1} \cdots \delta_{k_p}^{\nu_p} x_j$ for some indices $\nu_1, \dots, \nu_p$ and a *unique* sequence $k_1 < \cdots < k_p$, and
- for each $j = 1, \dots, m-1$, there is $k \in \mathbb{N}$ for which $x_j = \delta_k^0 x_{j+1}$ or $x_{j+1} = \delta_k^1 x_j$.

Note that precubical path objects are *non-selflinked* in the sense of [7]. If X and Y are precubical path objects with representations $(x_1, \dots, x_m)$, $(y_1, \dots, y_p)$, then a morphism $f : X \to Y$ is called a *cube path extension* if $x_j = y_j$ for all $j = 1, \dots, m$ (hence $m \leq p$).

Definition 3.1. The category HDP of *higher-dimensional paths* is the subcategory of HDA which as objects has pointed precubical paths, and whose morphisms are generated by pointed cube path extensions and isomorphisms.

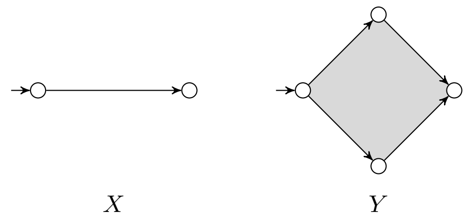

Figure 3: Two higher-dimensional paths with no HDP-morphism between them.

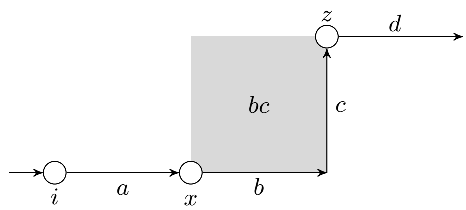

Figure 4: Graphical representation of the two-dimensional cube path $(i, a, x, b, bc, c, z, d)$. Its computational interpretation is that a is executed first, then execution of b starts, and while b is running, c starts to execute. After this, b finishes, then c, and then execution of d is started. Note that the computation is partial, as d does not finish.

Example 3.2. HDP is not a full subcategory of HDA: If X and Y are the two higher-dimensional paths depicted in Fig. 3, then none of the two mappings $X \to Y$ is a HDP-morphism.

A *cube path* in a precubical set X is a morphism $P \to X$ from a precubical path object P. In elementary terms, this is a sequence $(x_1, \dots, x_m)$ of elements of X such that for each $j = 1, \dots, m-1$, there is $k \in \mathbb{N}$ for which $x_j = \delta_k^0 x_{j+1}$ (start of a new part of a computation) or $x_{j+1} = \delta_k^1 x_j$ (end of a computation part).

Cube paths were introduced in [28], where they are simply called paths. They are intended to model (partial) computations of HDA. We show an example of a cube path in Fig. 4.

A cube path in a HDA $i : * \to X$ is *pointed* if $x_1 = i$, hence if it is a pointed morphism $P \to X$ from a higher-dimensional path P. We will say that a cube path $(x_1, \dots, x_m)$ is *from x_1 to x_m*, and that a cube $x \in X$ in a HDA X is *reachable* if there is a pointed cube path to x in X.

Cube paths can be *concatenated* if the end of one is compatible with the beginning of the other: If $\rho = (x_1, \dots, x_m)$ and $\sigma = (y_1, \dots, y_p)$ are cube paths with $y_1 = \delta_k^1 x_m$ or $x_m = \delta_k^0 y_1$ for some k, then their *concatenation* is the cube path $\rho * \sigma = (x_1, \dots, x_m, y_1, \dots, y_p)$. We say that ρ is a *prefix* of χ and write $\rho \sqsubseteq \chi$ if there is a cube path ρ for which $\chi = \rho * \sigma$.

Definition 3.3. A pointed morphism $f : X \to Y$ in HDA is an *open map* if it has the right lifting property with respect to HDP, *i.e.* if it is the case that there is a lift r in any commutative diagram as below, for morphisms $g : P \to Q \in$ HDP, $p : P \to X, q : Q \to Y \in$

HDA:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, dotted, "r"] & Y \end{tikzcd}$$

HDA X, Y are *hd-bisimilar* if there is $Z \in \mathsf{HDA}$ and a span of open maps $X \leftarrow Z \rightarrow Y$ in HDA.

It follows straight from the definition that composites of open maps again are open. By the next lemma, morphisms are open precisely when they have a zig-zag property similar to the one of [18].

Lemma 3.4. *For a morphism $f : X \to Y \in \mathsf{HDA}$, the following are equivalent:*
(1) *f is open;*
(2) *for any reachable $x_1 \in X$ and any $y_2 \in Y$ with $f(x_1) = \delta_k^0 y_2$ for some k, there is $x_2 \in X$ for which $x_1 = \delta_k^0 x_2$ and $y_2 = f(x_2)$;*
(3) *for any reachable $x_1 \in X$ and any cube path $(y_1, \dots, y_m)$ in Y with $y_1 = f(x_1)$, there is a cube path $(x_1, \dots, x_m)$ in X for which $y_j = f(x_j)$ for all $j = 1, \dots, m$.*

Proof. For the implication (1) $\implies$ (2), let $p : P \to X$ be a pointed cube path with P represented by $(p_1, \dots, p_m)$ and $p(p_m) = x_1$. Let p_{m+1} be a cube of dimension one higher than p_m, set $p_m = \delta_k^0 p_{m+1}$, and let Q be the higher-dimensional path represented by $(p_1, \dots, p_m, p_{m+1})$. Let $g : P \to Q$ be the inclusion, and define $q : Q \to Y$ by $q(p_j) = f(p(p_j))$ for $j = 1, \dots, m$ and $q(p_{m+1}) = y_2$. We have a lift $r : Q \to X$ and can set $x_2 = r(p_{m+1})$.

The implication (2) $\implies$ (3) can be easily shown by induction. The case $y_m = \delta_k^0 y_{m+1}$ follows directly from (2), and the case $y_{m+1} = \delta_k^1 y_m$ is clear by $\delta_k^1 \circ f = f \circ \delta_k^1$.

To finish the proof, we show the implication (3) $\implies$ (1). Let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd}$$

be a commutative diagram, with P represented by $(p_1, \dots, p_m)$. Up to isomorphism we can assume that Q is represented by $(p_1, \dots, p_m, p_{m+1}, \dots, p_t)$ and that g is the inclusion. The cube $p(p_m)$ is reachable in X, and $(q(p_m), \dots, q(p_t))$ is a cube path in Y which starts in $q(p_m) = f(p(p_m))$. Hence we have a cube path $(x_m, \dots, x_t)$ in X with $x_m = p(p_m)$ and $q(p_j) = f(x_j)$ for all $j = m, \dots, t$, and we can define a lift $r : Q \to X$ by $r(p_j) = p(p_j)$ for $j = 1, \dots, m$ and $r(p_j) = x_j$ for $j = m+1, \dots, t$. $\square$

Theorem 3.5. *For HDA $i : * \to X$, $j : * \to Y$, the following are equivalent:*
(1) *X and Y are hd-bisimilar;*
(2) *there exists a precubical subset $R \subseteq X \times Y$ for which $(i, j) \in R$, and such that for all $(x_1, y_1) \in R$,*
- *for any $x_2 \in X$ for which $x_1 = \delta_k^0 x_2$ for some k, there exists $y_2 \in Y$ for which $y_1 = \delta_k^0 y_2$ and $(x_2, y_2) \in R$,*
- *for any $y_2 \in Y$ for which $y_1 = \delta_k^0 y_2$ for some k, there exists $x_2 \in X$ for which $x_1 = \delta_k^0 x_2$ and $(x_2, y_2) \in R$;*

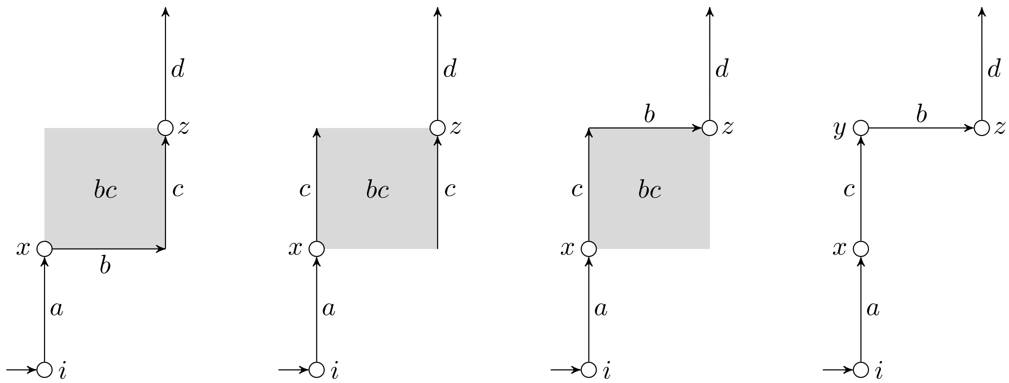

Figure 5: Graphical representation of the cube path homotopy $(i, a, x, b, bc, c, z, d) \sim (i, a, x, c, bc, c, z, d) \sim (i, a, x, c, bc, b, z, d) \sim (i, a, x, c, y, b, z, d)$.

(3) *there exists a precubical subset $R \subseteq X \times Y$ for which $(i, j) \in R$, and such that for all $(x_1, y_1) \in R$,*
- *for any cube path $(x_1, \dots, x_m)$ in X, there exists a cube path $(y_1, \dots, y_m)$ in Y with $(x_p, y_p) \in R$ for all $p = 1, \dots, m$,*
- *for any cube path $(y_1, \dots, y_m)$ in Y, there exists a cube path $(x_1, \dots, x_m)$ in X with $(x_p, y_p) \in R$ for all $p = 1, \dots, m$.*

Proof. For the implication (1) $\Longrightarrow$ (2), let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a span of open maps and define $R = \{(x, y) \in X \times Y \mid \exists z \in Z : x = f(z), y = g(z)\}$. Then $(i, j) \in R$ because f and g are pointed morphisms, and the other properties follow by Lemma 3.4. The implication (2) $\Longrightarrow$ (3) can be shown by a simple induction, and for the implication (3) $\Longrightarrow$ (1), the projections give a span $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$ and are open by Lemma 3.4. □

4. Homotopies and Unfoldings

In order to connect our notion of hd-bisimilarity with other common notions, we need to introduce in which cases different cube paths are equivalent due to independence of actions. Following [29], we model this equivalence by a combinatorial version of *homotopy* which is an extension of the equivalence defining *Mazurkiewicz traces* [20].

We say that cube paths $(x_1, \dots, x_m)$, $(y_1, \dots, y_m)$ are *adjacent* if $x_1 = y_1$, $x_m = y_m$, there is precisely one index $p \in \{1, \dots, m\}$ at which $x_p \neq y_p$, and

- $x_{p-1} = \delta^0_k x_p$, $x_p = \delta^0_\ell x_{p+1}$, $y_{p-1} = \delta^0_{\ell-1} y_p$, and $y_p = \delta^0_k y_{p+1}$ for some $k < \ell$, or vice versa,
- $x_p = \delta^1_k x_{p-1}$, $x_{p+1} = \delta^1_\ell x_p$, $y_p = \delta^1_{\ell-1} y_{p-1}$, and $y_{p+1} = \delta^1_k y_p$ for some $k < \ell$, or vice versa,
- $x_p = \delta^0_k \delta^1_\ell y_p$, $y_{p-1} = \delta^0_k y_p$, and $y_{p+1} = \delta^1_\ell y_p$ for some $k < \ell$, or vice versa, or
- $x_p = \delta^1_k \delta^0_\ell y_p$, $y_{p-1} = \delta^0_\ell y_p$, and $y_{p+1} = \delta^1_k y_p$ for some $k < \ell$, or vice versa.

Homotopy of cube paths is the reflexive, transitive closure of the adjacency relation. We denote homotopy of cube paths using the symbol $\sim$, and the homotopy class of a cube path $(x_1, \dots, x_m)$ is denoted $[x_1, \dots, x_m]$. The intuition of adjacency is rather simple, even though the combinatorics may look complicated, see Fig. 5. Note that adjacencies come in two basic "flavors": the first two above in which the dimensions of x_ℓ and y_ℓ are the same, and the last two in which they differ by 2.

The following lemma shows that, not surprisingly, cube paths entirely contained in one cube are homotopic (provided that they share endpoints).

Lemma 4.1. *Let $x \in X_n$ in a precubical set X and $(k_1, \dots, k_n)$, $(\ell_1, \dots, \ell_n)$ sequences of indices with $k_j, \ell_j \le j$ for all $j = 1, \dots, n$. Let $x_j = \delta^0_{k_j} \cdots \delta^0_{k_n} x$, $y_j = \delta^0_{\ell_j} \cdots \delta^0_{\ell_n} x$. Then the cube paths $(x_1, \dots, x_n, x) \sim (y_1, \dots, y_n, x)$.*

Proof. (*cf.* [6, Ex. 2.15]). We can represent a cube path $(x_1, \dots, x_n, x)$ as above by an element $(p_1, \dots, p_n)$ of the symmetric group S_n by setting $p_n = k_n$ and, working backwards, $p_j = (\{1, \dots, n\} \setminus \{p_{j+1}, \dots, p_n\})[k_j]$, denoting by this the k_j-largest element of the set in parentheses. This introduces a bijection between the set of cube paths from the lower left corner of x to x on the one hand, and elements of S_n on the other hand, and under this bijection adjacencies of cube paths are transpositions in S_n. These generate all of S_n, hence all such cube paths are homotopic. □

We extend concatenation and prefix to homotopy classes of cube paths by defining $[x_1, \dots, x_m] * [y_1, \dots, y_p] = [x_1, \dots, x_m, y_1, \dots, y_p]$ and saying that $\tilde x \sqsubseteq \tilde z$, for homotopy classes $\tilde x$, $\tilde z$ of cube paths, if there are $(x_1, \dots, x_m) \in \tilde x$ and $(z_1, \dots, z_q) \in \tilde z$ for which $(x_1, \dots, x_m) \sqsubseteq (z_1, \dots, z_q)$. It is easy to see that concatenation is well-defined, and that $\tilde x \sqsubseteq \tilde z$ if and only if there is a homotopy class $\tilde y$ for which $\tilde z = \tilde x * \tilde y$.

Using homotopy classes of cube paths, we can now define the *unfolding* of a HDA. Unfoldings of HDA are similar to unfoldings of transition systems [33] or Petri nets [17, 22], but also to *universal covering spaces* in algebraic topology. The intention is that the unfolding of a HDA captures all its computations, up to homotopy.

We say that a HDA X is a *higher-dimensional tree* if it holds that for any $x \in X$, there is precisely one homotopy class of pointed cube paths to x. The full subcategory of HDA spanned by the higher-dimensional trees is denoted HDT. Note that any higher-dimensional path is a higher-dimensional tree; indeed there is an inclusion $\mathsf{HDP} \hookrightarrow \mathsf{HDT}$.

Definition 4.2. The *unfolding* of a HDA $i : * \to X$ consists of a HDA $\tilde i : * \to \tilde X$ and a pointed *projection* morphism $\pi_X : \tilde X \to X$, which are defined as follows:

- $\tilde X_n = \big\{[x_1, \dots, x_m] \mid (x_1, \dots, x_m) \text{ pointed cube path in } X, x_m \in X_n\big\}$; $\tilde i = [i]$
- $\tilde\delta^0_k[x_1, \dots, x_m] = \big\{(y_1, \dots, y_p) \mid y_p = \delta^0_k x_m, (y_1, \dots, y_p, x_m) \sim (x_1, \dots, x_m)\big\}$
- $\tilde\delta^1_k[x_1, \dots, x_m] = [x_1, \dots, x_m, \delta^1_k x_m]$
- $\pi_X[x_1, \dots, x_m] = x_m$

Proposition 4.3. *The unfolding $(\tilde X, \pi_X)$ of a HDA X is well-defined, and $\tilde X$ is a higher-dimensional tree. If X itself is a higher-dimensional tree, then the projection $\pi_X : \tilde X \to X$ is an isomorphism.*

Before proving the proposition, we need an auxiliary notion of *fan-shaped* cube path together with a technical lemma. Say that a cube path $(x_1, \dots, x_m)$ in a precubical set X, with $x_m \in X_n$, is fan-shaped if

$$x_j \in \begin{cases} X_0 & \text{for } 1 \le j \le m - n \text{ odd,} \\ X_1 & \text{for } 1 \le j \le m - n \text{ even,} \\ X_{n+j-m} & \text{for } m - n < j \le m\,. \end{cases}$$

Hence a fan-shaped cube path is a one-dimensional path up to the point where it needs to build up to hit the possibly high-dimensional end cube x_m; in computational terms, it is *serialized.*

Lemma 4.4. *Any pointed cube path in a higher-dimensional automaton $i : * \to X$ is homotopic to a fan-shaped one.*

Proof. Let us first introduce some notation: For any pointed cube path $(x_1, \dots, x_m)$, let $n_j = \dim x_j$ be the j-th component's dimension, and let $T(x_1, \dots, x_m) = n_1 + \dots + n_m$. An easy induction shows that $j - n_j$ is odd for all j. Also, $T(x_1, \dots, x_m) \geq \frac{1}{2}(n_m^2 + m - 1)$, with equality if and only if $(x_1, \dots, x_m)$ is fan-shaped.

Next we show that $n_1 + \dots + n_m \equiv \frac{1}{2}(n_m^2 + m - 1) \mod 2$. By oddity of $j - n_j$ we have $\sum_{j=1}^m n_j - \sum_{j=1}^m j \equiv m \mod 2$, and also $\frac{1}{2}(n_m^2 + m - 1) - \sum_{j=1}^m j = \frac{1}{2}(n_m^2 - m^2 - 1) \equiv m \mod 2$, hence the claim follows.

We can now finish the proof by showing how to convert a cube path $(x_1, \dots, x_m)$ with $T(x_1, \dots, x_m) > \frac{1}{2}(n_m^2 + m - 1)$ into an adjacent cube path $(x'_1, \dots, x'_m)$ which has $T(x'_1, \dots, x'_m) = T(x_1, \dots, x_m) - 2$, essentially by replacing one of its cubes, called x_ℓ below, with another one of dimension $n_\ell - 2$.

If $(x_1, \dots, x_m)$ is a cube path which is not fan-shaped, then there is an index $\ell \in \{3, \dots, m-1\}$ for which $n_\ell \geq 2$, $x_{\ell-1} = \delta^0_{k_2} x_\ell$ for some k_2, and $x_{\ell+1} = \delta^1_{k_3} x_\ell$ for some k_3. Assuming ℓ to be the *least* such index, we must also have $x_{\ell-2} = \delta^0_{k_1} x_{\ell-1}$ for some k_1.

Now if $k_2 < k_3$, then $\delta^0_{k_2} x_{\ell+1} = \delta^0_{k_2} \delta^1_{k_3} x_\ell = \delta^1_{k_3-1} \delta^0_{k_2} x_\ell = \delta^1_{k_3-1} x_{\ell-1}$ by the precubical identity (2.1), hence we can let $(x'_1, \dots, x'_m)$ be the cube path with $x'_j = x_j$ for $j \neq \ell$ and $x'_\ell = \delta^0_{k_2} x_{\ell+1}$.

If $k_2 > k_3$, then similarly $\delta^1_{k_3} x_{\ell-1} = \delta^1_{k_3} \delta^0_{k_2} x_\ell = \delta^0_{k_2-1} \delta^1_{k_3} x_\ell = \delta^0_{k_2-1} x_{\ell+1}$, and we can let $x'_j = x_j$ for $j \neq \ell$ and $x'_\ell = \delta^1_{k_3} x_{\ell-1}$.

For the remaining case $k_2 = k_3$, we replace $x_{\ell-1}$ by another cube of equal dimension first: If $k_1 < k_2$, then $x_{\ell-2} = \delta^0_{k_1} \delta^0_{k_2} x_\ell = \delta^0_{k_2-1} \delta^0_{k_1} x_\ell$, hence the cube path $(x''_1, \dots, x''_m)$ with $x''_j = x_j$ for $j \neq \ell - 1$ and $x''_{\ell-1} = \delta^0_{k_1} x_\ell$ is adjacent to $(x_1, \dots, x_m)$, and $T(x''_1, \dots, x''_m) = T(x_1, \dots, x_m)$. For this new cube path, we have $x''_{\ell-2} = \delta^0_{k_2-1} x''_{\ell-1}$, $x''_{\ell-1} = \delta^0_{k_1} x''_\ell$, and $x''_{\ell+1} = \delta^1_{k_3} x''_\ell$, and as $k_1 < k_3$, we can apply to the cube path $(x''_1, \dots, x''_m)$ the argument for the case $k_2 < k_3$ above.

If $k_1 \geq k_2$, then $x_{\ell-2} = \delta^0_{k_1} \delta^0_{k_2} x_\ell = \delta^0_{k_2} \delta^0_{k_1+1} x_\ell$ by another application of the precubical identity (2.1). Hence we can let $x''_j = x_j$ for $j \neq \ell - 1$ and $x''_{\ell-1} = \delta^0_{k_1+1} x_\ell$. Then $x''_{\ell-2} = \delta^0_{k_2} x''_{\ell-1}$, $x''_{\ell-1} = \delta^0_{k_1+1} x''_\ell$, and $x''_{\ell+1} = \delta^1_{k_3} x''_\ell$, and as $k_1 + 1 > k_3$, we can apply the argument for the case $k_2 > k_3$ above. □

Proof of Theorem 4.3. It is clear that the structure maps $\tilde\delta^1_k$ are well-defined. For showing that also the mappings $\tilde\delta^0_k$ are well-defined, we note first that $\tilde\delta^0_k[x_1, \dots, x_m]$ is independent of the representative chosen for $[x_1, \dots, x_m]$: If $(x'_1, \dots, x'_m) \sim (x_1, \dots, x_m)$, then $(y_1, \dots, y_p) \in \tilde\delta^0_k[x'_1, \dots, x'_m]$ if and only if $y_p = \delta^0_k x'_m = \delta^0_k x_m$ and $(y_1, \dots, y_p, x'_m) = (y_1, \dots, y_p, x_m) \sim (x'_1, \dots, x'_m) \sim (x_1, \dots, x_m)$, if and only if $(y_1, \dots, y_p) \in \tilde\delta^0_k[x_1, \dots, x_m]$.

We are left with showing that $\tilde\delta^0_k[x_1, \dots, x_m]$ is non-empty. By Lemma 4.4 there is a fan-shaped cube path $(x'_1, \dots, x'_m) \in [x_1, \dots, x_m]$, and by Lemma 4.1 we can assume that $x'_{m-1} = \delta^0_k x'_m = \delta^0_k x_m$, hence $(x'_1, \dots, x'_{m-1}) \in \tilde\delta^0_k[x_1, \dots, x_m]$.

We need to show the precubical identity $\tilde\delta_k^\nu \tilde\delta_\ell^\mu = \tilde\delta_{\ell-1}^\mu \tilde\delta_k^\nu$ for $k < \ell$ and $\nu, \mu \in \{0,1\}$. For $\nu = \mu = 1$ this is clear, and for $\nu = \mu = 0$ one sees that $(y_1, \dots, y_p) \in \tilde\delta_k^0 \tilde\delta_\ell^0 [x_1, \dots, x_m]$ if and only if $y_p = \delta_k^0 \delta_\ell^0 x_m = \delta_{\ell-1}^0 \delta_k^0 x_m$ and $(x_1, \dots, x_m) \sim (y_1, \dots, y_p, \delta_\ell^0 x_m, x_m) \sim (y_1, \dots, y_p, \delta_k^0 x_m, x_m)$, by adjacency.

The cases $\nu = 1$, $\mu = 0$ and $\nu = 0$, $\mu = 1$ are similar to each other, so we only show the former. Let $(x_1', \dots, x_m') \in [x_1, \dots, x_m]$ be a fan-shaped cube path with $x_{m-1}' = \delta_\ell^0 x_m'$, *cf.* Lemma 4.1. Then $\tilde\delta_k^1 \tilde\delta_\ell^0 [x_1, \dots, x_m] = \tilde\delta_k^1 [x_1', \dots, x_{m-1}'] = [x_1', \dots, x_{m-1}', \delta_k^1 x_{m-1}']$. Now $\delta_k^1 x_{m-1}' = \delta_k^1 \delta_\ell^0 x_m' = \delta_{\ell-1}^0 \delta_k^1 x_m$, and by adjacency, $(x_1', \dots, x_{m-1}', \delta_k^1 x_{m-1}', \delta_k^1 x_m') \sim (x_1', \dots, x_{m-1}', x_m', \delta_k^1 x_m')$, so that we have $(x_1', \dots, x_{m-1}', \delta_k^1 x_{m-1}') \in \tilde\delta_{\ell-1}^0 [x_1', \dots, x_m', \delta_k^1 x_m'] = \tilde\delta_{\ell-1}^0 \tilde\delta_k^1 [x_1', \dots, x_m']$.

For showing that the projection $\pi_X : \tilde X \to X$ is a precubical morphism, we note first that $\pi_X \tilde\delta_k^1 [x_1, \dots, x_m] = \pi_X [x_1, \dots, x_m, \delta_k^1 x_m] = \delta_k^1 x_m = \delta_k^1 \pi_X [x_1, \dots, x_m]$ as required. For $\tilde\delta_k^0$, let again $(x_1', \dots, x_m') \in [x_1, \dots, x_m]$ be a fan-shaped cube path with $x_{m-1}' = \delta_k^0 x_m'$. Then $\pi_X \tilde\delta_k^0 [x_1, \dots, x_m] = \pi_X [x_1', \dots, x_{m-1}'] = x_{m-1}' = \delta_k^0 x_m' = \delta_k^0 x_m = \delta_k^0 \pi_X [x_1, \dots, x_m]$.

The proof that $* \to \tilde X$ is a higher-dimensional tree follows from Lemma 4.5 below: Let $(\tilde x_1, \dots, \tilde x_m)$, $(\tilde y_1, \dots, \tilde y_m)$ be pointed cube paths in $\tilde X$ with $\tilde x_m = \tilde y_m$, then we need to prove that $(\tilde x_1, \dots, \tilde x_m) \sim (\tilde y_1, \dots, \tilde y_m)$. Let $x_j = \pi_X \tilde x_j$, $y_j = \pi_X \tilde y_j$ for $j = 1, \dots, m$ be the projections, then $(x_1, \dots, x_m)$, $(y_1, \dots, y_m)$ are pointed cube paths in X. By Lemma 4.5, $(x_1, \dots, x_j) \in \tilde x_j$ and $(y_1, \dots, y_j) \in \tilde y_j$ for all $j = 1, \dots, m$.

By $\tilde x_m = \tilde y_m$, we know that $(x_1, \dots, x_m) \sim (y_1, \dots, y_m)$. Let $(x_1, \dots, x_m) = (z_1^1, \dots, z_m^1) \sim \dots \sim (z_1^p, \dots, z_m^p) = (y_1, \dots, y_m)$ be a sequence of adjacencies, and let $\tilde z_j^\ell = [z_1^\ell, \dots, z_j^\ell]$. This defines pointed cube paths $(\tilde z_1^\ell, \dots, \tilde z_m^\ell)$ in $\tilde X$; we show that $(\tilde x_1, \dots, \tilde x_m) = (\tilde z_1^1, \dots, \tilde z_m^1) \sim \dots \sim (\tilde z_1^p, \dots, \tilde z_m^p) = (\tilde y_1, \dots, \tilde y_m)$ is a sequence of adjacencies:

Let $\ell \in \{1, \dots, p-1\}$, and let $\alpha \in \{1, \dots, m-1\}$ be the index such that $z_\alpha^\ell \ne z_\alpha^{\ell+1}$ and $z_j^\ell = z_j^{\ell+1}$ for all $j \ne \alpha$. Then $(z_1^\ell, \dots, z_j^\ell) = (z_1^{\ell+1}, \dots, z_j^{\ell+1})$ for $j < \alpha$ and $(z_1^\ell, \dots, z_j^\ell) \sim (z_1^{\ell+1}, \dots, z_j^{\ell+1})$ for $j > \alpha$, hence there is an adjacency $(\tilde z_1^\ell, \dots, \tilde z_m^\ell) \sim (\tilde z_1^{\ell+1}, \dots, \tilde z_m^{\ell+1})$.

For the last claim of the proposition, if X itself is a higher-dimensional tree, then an inverse to π_X is given by mapping $x \in X$ to the unique equivalence class $[x_1, \dots, x_m] \in \tilde X$ of any pointed cube path $(x_1, \dots, x_m)$ in X with $x_m = x$. $\square$

Lemma 4.5. *If X is a higher-dimensional automaton and $(\tilde x_1, \dots, \tilde x_m)$ is a pointed cube path in $\tilde X$, then $(\pi_X \tilde x_1, \dots, \pi_X \tilde x_j) \in \tilde x_j$ for all $j = 1, \dots, m$.*

Proof. Let $x_j = \pi_X \tilde x_j$, for $j = 1, \dots, m$, then $(x_1, \dots, x_m)$ is a pointed cube path in X. We show the claim by induction: We have $\tilde x_1 = \tilde i = [i] = [x_1]$, so assume that $(x_1, \dots, x_j) \in \tilde x_j$ for some $j \in \{1, \dots, m-1\}$. If $\tilde x_{j+1} = \tilde\delta_k^1 \tilde x_j$ for some k, then $x_{j+1} = \delta_k^1 x_j$, and $(x_1, \dots, x_{j+1}) \in \tilde x_{j+1}$ by definition of $\tilde\delta_k^1$. Similarly, if $\tilde x_j = \tilde\delta_k^0 \tilde x_{j+1}$ for some k, then $x_j = \delta_k^0 x_{j+1}$, and $(x_1, \dots, x_{j+1}) \in \tilde x_{j+1}$ by definition of $\tilde\delta_k^0$. $\square$

Lemma 4.6. *For any HDA X there is a unique lift r in any commutative diagram as below, for morphisms $g : P \to Q \in \mathsf{HDP}$, $p : P \to \tilde X, q : Q \to X \in \mathsf{HDA}$:*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & \tilde X \arrow[d, "\pi_X"] \\ Q \arrow[r, "q"'] \arrow[ur, dotted, "r"] & X \end{tikzcd}$$

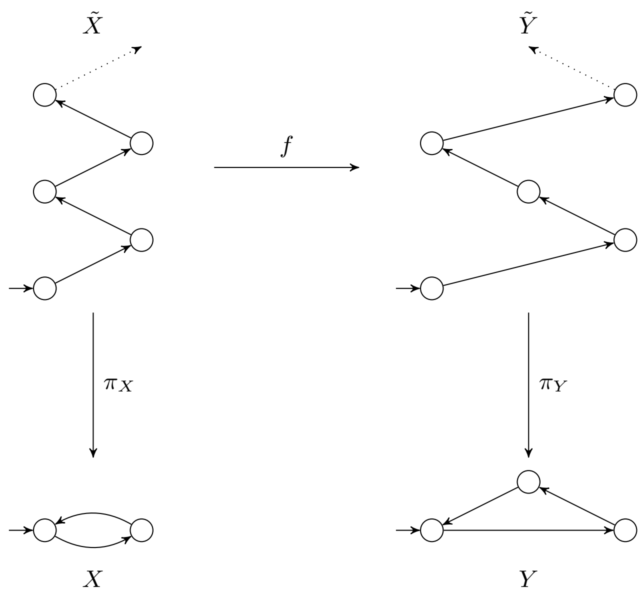

Figure 6: Two simple one-dimensional HDA as objects of HDA and $\mathsf{HDA_h}$. In HDA there is no morphism $X \to Y$, in $\mathsf{HDA_h}$ there is precisely one morphism $f : X \to Y$.

Proof. Let $(\tilde{x}_1, \dots, \tilde{x}_m)$ be a pointed cube path in $\tilde{X}$, and write $x_j = \pi_X \tilde{x}_j$ for $j = 1, \dots, m$. Let $(x_1, \dots, x_m, y_1, \dots, y_p)$ be an extension in X and define $\tilde{y}_j = [x_1, \dots, x_m, y_1, \dots, y_j]$ for $j = 1, \dots, p$. Then $(\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_p)$ is the required extension in $\tilde{X}$, which is unique as $\tilde{X}$ is a higher-dimensional tree. □

Corollary 4.7. *Projections are open, and any HDA is hd-bisimilar to its unfolding.* □

5. Higher-dimensional Automata up to Homotopy

Definition 5.1. The category of *higher-dimensional automata up to homotopy* $\mathsf{HDA_h}$ has as objects HDA and as morphisms pointed precubical morphisms $f : \tilde{X} \to \tilde{Y}$ of unfoldings.

Hence any morphism $X \to Y$ in HDA gives, by the unfolding functor, rise to a morphism $X \to Y$ in $\mathsf{HDA_h}$. The simple example in Fig. 6 shows that the converse is not the case. By restriction to higher-dimensional trees, we get a full subcategory $\mathsf{HDT_h} \hookrightarrow \mathsf{HDA_h}$.

Lemma 5.2. *The natural projection isomorphisms $\pi_X : \tilde{X} \to X$ for $X \in \mathsf{HDT}$ extend to an isomorphism of categories $\mathsf{HDT_h} \cong \mathsf{HDT}$.*

Proof. Using the projection isomorphisms, any morphism $f : X \to Y$ in $\mathsf{HDT_h}$ can be "pulled down" to a morphism $\pi_Y \circ f \circ \pi_X^{-1} : X \to Y$ of HDT. □

Restricting the above isomorphism to the subcategory HDP of HDT allows us to identify a subcategory $\mathsf{HDP_h}$ of $\mathsf{HDT_h}$ isomorphic to HDP.

Analogously to the coreflection between transition systems and synchronization trees in [33], we have a coreflection between higher-dimensional automata and higher-dimensional trees:

Proposition 5.3. *The functor $U : \mathsf{HDA} \to \mathsf{HDT}$ given on objects by mapping $X \in \mathsf{HDA}$ to its unfolding $\tilde X$ and on morphisms by mapping $f : X \to Y$ to $\tilde f : \tilde X \to \tilde Y$ given by $\tilde f[x_1, \dots, x_m] = [f(x_1), \dots, f(x_m)]$ is right adjoint to the forgetful functor $\mathsf{HDT} \hookrightarrow \mathsf{HDA}$. The counit morphisms are the projections $\pi_X : \tilde X \to X$.*

Proof. First, U is indeed functorial, as f maps adjacent cube paths $(x_1, \dots, x_m) \sim (y_1, \dots, y_m)$ to cube paths $(fx_1, \dots, fx_m)$, $(fy_1, \dots, fy_m)$ which are identical or adjacent, hence $\tilde f : \tilde X \to \tilde Y$ is well-defined.

To show adjointness, we need to see that any pointed morphism $f : T \to Y \in \mathsf{pCub}$ from a higher-dimensional tree $* \to T$ to a higher-dimensional automaton $* \to Y$ factors uniquely as $f = \pi_Y \circ g : T \to \tilde Y \to Y$. This amounts to filling-in the dotted arrow in the diagram

$$\begin{tikzcd} \tilde Y \arrow[r, "\pi_Y"] & Y \\ \tilde T \arrow[u, "\tilde f"] \arrow[r, "\pi_T"'] & T \arrow[u, "f"'] \arrow[ul, dotted, "g"] \end{tikzcd}.$$

By Proposition 4.3, π_T has an inverse ψ_T, hence $g = \tilde f \circ \psi_T$ is the unique filler. □

Note that by Proposition 4.3, the unit morphisms are isomorphisms, hence the above adjunction is indeed a coreflection.

The following is the analogue of Proposition 5.3 for the homotopy categories, with a similar proof. Note however that here, $U_{\mathsf h}$ is an *isomorphism* on morphisms.

Proposition 5.4. *The forgetful functor $\mathsf{HDT_h} \hookrightarrow \mathsf{HDA_h}$ has a right adjoint $U_{\mathsf h}$ given on objects by mapping $X \in \mathsf{HDA_h}$ to its unfolding $\tilde X$ and on morphisms by mapping $f : X \to Y$ to $\tilde f : \tilde X \to \tilde Y$. The counit morphisms are the projections $\pi_X : \tilde X \to X$.* □

The unit morphisms are again isomorphisms, hence the adjunction is a coreflection.

Combining the functors of Propositions 5.3 and 5.4 with the isomorphism of Lemma 5.2, we have the following diagram of categories and coreflections. Note that the adjunctions do not compose.

$$\begin{tikzcd} \mathsf{HDA_h} \arrow[r, shift left, "U_{\mathsf h}"] \arrow[r, phantom, "\top"] & \mathsf{HDT_h} \arrow[l, shift left, hook, "J_{\mathsf h}"] \arrow[r, shift left, "\cong"] & \mathsf{HDT} \arrow[l, shift left] \arrow[r, shift left, hook, "J"] \arrow[r, phantom, "\perp"] & \mathsf{HDA} \arrow[l, shift left, "U"] \end{tikzcd}$$

The endofunctor $J \circ U$ on HDA, which maps objects and morphisms to their unfoldings, splits into an adjunction between HDA and $\mathsf{HDA_h}$. Its left part is the "inclusion" $\mathsf{HDA} \hookrightarrow \mathsf{HDA_h}$ which we already saw above.

Proposition 5.5. *There is a coreflection $U_1 : \mathsf{HDA_h} \leftrightarrows \mathsf{HDA} : U_2$, with U_1 left and U_2 right adjoint given by $U_1(X) = \tilde X$ on objects, $U_1(f) = f$ on morphisms, $U_2(X) = X$ on objects, and $U_2(f) = \tilde f$ on morphisms. The counit morphisms are the projections $\pi_X : \tilde X \to X$.*

Proof. We need to see that any precubical morphism $f : \tilde X \to Y$ factors uniquely as $f = \pi_Y \circ g : \tilde X \to \tilde Y \to Y$, but as $\tilde X$ is a higher-dimensional tree, this is clear by the isomorphism $\pi_{\tilde X} : \tilde{\tilde X} \to \tilde X$ in the diagram

$$\begin{tikzcd} \tilde Y \arrow[r, "\pi_Y"] & Y \\ \tilde{\tilde X} \arrow[u, "\tilde f"'] \arrow[r, "\pi_{\tilde X}"'] & \tilde X . \arrow[u, "f"'] \arrow[ul, dotted, "g"] \end{tikzcd}$$

∎

6. Homotopy Bisimilarity

Definition 6.1. A pointed morphism $f : X \to Y$ in $\mathsf{HDA_h}$ is *open* if it has the right lifting property with respect to $\mathsf{HDP_h}$, *i.e.* if it is the case that there is a lift r in any commutative diagram as below, for all morphism $g : P \to Q \in \mathsf{HDP_h}$, $p : P \to X, q : Q \to Y \in \mathsf{HDA_h}$:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, dotted, "r"] & Y \end{tikzcd}$$

HDA X, Y are *homotopy bisimilar* if there is $Z \in \mathsf{HDA_h}$ and a span of open maps $X \leftarrow Z \to Y$ in $\mathsf{HDA_h}$.

The connections between open maps in $\mathsf{HDA_h}$ and open maps in HDA are as follows.

Lemma 6.2. *A morphism $f : X \to Y$ in $\mathsf{HDA_h}$ is open if and only if $f : \tilde X \to \tilde Y$ is open as a morphism of HDA. If $g : X \to Y$ is open in HDA, then so is $\tilde g : \tilde X \to \tilde Y$.*

Proof. For the forward implication of the first claim, let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & \tilde X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & \tilde Y \end{tikzcd} \tag{6.1}$$

be a diagram in HDA with $g : P \to Q \in \mathsf{HDP}$; we need to find a lift $Q \to \tilde X$.

Using the isomorphisms $\pi_P : \tilde P \to P$, $\pi_Q : \tilde Q \to Q$, we can extend this diagram to the left; note that $\tilde g : \tilde P \to \tilde Q$ is a morphism of HDP:

$$\begin{tikzcd} \tilde P \arrow[r, "\cong"'] \arrow[d, "\tilde g"'] \arrow[rr, bend left, "p'"] & P \arrow[r, "p"'] \arrow[d, "g"'] & \tilde X \arrow[d, "f"] \\ \tilde Q \arrow[r, "\cong"] \arrow[rr, bend right, "q'"'] & Q \arrow[r, "q"] & \tilde Y \end{tikzcd} \tag{6.2}$$

Hence we have a diagram

$$\begin{tikzcd} P \arrow[r, "p'"] \arrow[d, "\tilde g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q'"'] & Y \end{tikzcd}$$

in $\mathsf{HDA_h}$, and as $\tilde g : P \to Q$ is a morphism of $\mathsf{HDP_h}$, we have a lift $r : Q \to X$ in $\mathsf{HDA_h}$. This gives a morphism $r : \tilde Q \to \tilde X \in \mathsf{HDA}$ in Diagram (6.2), and by composition with the inverse of the isomorphism $\pi_Q : \tilde Q \to Q$, a lift $r' : Q \to \tilde X \in \mathsf{HDA}$ in Diagram (6.1).

For the back implication in the first claim, assume $f : \tilde X \to \tilde Y \in \mathsf{HDA}$ open and let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd}$$

be a diagram in $\mathsf{HDA_h}$ with $g : P \to Q \in \mathsf{HDP_h}$; we need to find a lift $Q \to X$. Transferring this diagram to the category HDA, we have

$$\begin{tikzcd} \tilde P \arrow[r, "p"] \arrow[d, "g"'] & \tilde X \arrow[d, "f"] \\ \tilde Q \arrow[r, "q"'] & \tilde Y \end{tikzcd}$$

and as $g : \tilde P \to \tilde Q$ is a morphism of HDP, we get the required lift.

To prove the second claim, let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "h"'] & \tilde X \arrow[d, "\tilde g"] \\ Q \arrow[r, "q"'] & \tilde Y \end{tikzcd}$$

be a diagram in HDA with $h : P \to Q \in \mathsf{HDP}$. We can extend it using the projection morphisms:

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "h"'] & \tilde X \arrow[r, "\pi_X"] \arrow[d, "\tilde g"] & X \arrow[d, "g"] \\ Q \arrow[r, "q"'] & \tilde Y \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Because g is open in HDA, we hence have a lift

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "h"'] & \tilde X \arrow[r, "\pi_X"] \arrow[d, "\tilde g"] & X \arrow[d, "g"] \\ Q \arrow[r, "q"'] \arrow[urr, dotted, "r"'] & \tilde Y \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

and Lemma 4.6 then gives the required lift r' in the diagram

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "\pi_X"] \\ Q \arrow[r, "r"'] \arrow[ur, dotted, "r'"] & X \end{tikzcd}$$

□

Example 6.3. The morphism $f : X \to Y$ in Fig. 6 is open in $\mathsf{HDA_h}$, showing that X and Y are, as expected, homotopy bisimilar.

We also need a lemma on prefixes in unfoldings.

Lemma 6.4. *Let X be a HDA and $\tilde{x}, \tilde{z} \in \tilde{X}$. Then there is a cube path from $\tilde{x}$ to $\tilde{z}$ in $\tilde{X}$ if and only if $\tilde{x} \sqsubseteq \tilde{z}$.*

Proof. For the forward implication, let $(\tilde{x}, \tilde{y}_1, \dots, \tilde{y}_p)$ be a cube path in $\tilde{X}$ with $\tilde{y}_p = \tilde{z}$, let $(x_1, \dots, x_m) \in \tilde{x}$, and write $y_j = \pi_X \tilde{y}_j$ for all j. By Lemma 4.5, $(x_1, \dots, x_m, y_1, \dots, y_p) \in \tilde{z}$.

For the other direction, let $(x_1, \dots, x_m, y_1, \dots, y_p) \in \tilde{z}$ such that $(x_1, \dots, x_m) \in \tilde{x}$, and define $\tilde{y}_j = [x_1, \dots, x_m, y_1, \dots, y_j]$ for all j. Then $(\tilde{x}, \tilde{y}_1, \dots, \tilde{y}_p)$ is the required cube path from $\tilde{x}$ to $\tilde{z}$ in $\tilde{X}$. □

Proposition 6.5. *For HDA $i : * \to X$, $j : * \to Y$, the following are equivalent:*

(1) *X and Y are homotopy bisimilar;*
(2) *there exists a precubical subset $\tilde{R} \subseteq \tilde{X} \times \tilde{Y}$ with $(\tilde{i}, \tilde{j}) \in \tilde{R}$, and such that for all $(\tilde{x}_1, \tilde{y}_1) \in \tilde{R}$,*
   - *for any $\tilde{x}_2 \in \tilde{X}$ for which $\tilde{x}_1 = \delta_k^0 \tilde{x}_2$ for some k, there exists $\tilde{y}_2 \in \tilde{Y}$ for which $\tilde{y}_1 = \delta_k^0 \tilde{y}_2$ and $(\tilde{x}_2, \tilde{y}_2) \in \tilde{R}$,*
   - *for any $\tilde{y}_2 \in \tilde{Y}$ for which $\tilde{y}_1 = \delta_k^0 \tilde{y}_2$ for some k, there exists $\tilde{x}_2 \in \tilde{X}$ for which $\tilde{x}_1 = \delta_k^0 \tilde{x}_2$ and $(\tilde{x}_2, \tilde{y}_2) \in \tilde{R}$;*
(3) *there exists a precubical subset $\tilde{R} \subseteq \tilde{X} \times \tilde{Y}$ with $(\tilde{i}, \tilde{j}) \in \tilde{R}$, and such that for all $(\tilde{x}_1, \tilde{y}_1) \in \tilde{R}$,*
   - *for any cube path $(\tilde{x}_1, \dots, \tilde{x}_n)$ in $\tilde{X}$, there exists a cube path $(\tilde{y}_1, \dots, \tilde{y}_n)$ in $\tilde{Y}$ with $(\tilde{x}_p, \tilde{y}_p) \in \tilde{R}$ for all $p = 1, \dots, n$,*
   - *for any cube path $(\tilde{y}_1, \dots, \tilde{y}_n)$ in $\tilde{Y}$, there exists a cube path $(\tilde{x}_1, \dots, \tilde{x}_n)$ in $\tilde{X}$ with $(\tilde{x}_p, \tilde{y}_p) \in \tilde{R}$ for all $p = 1, \dots, n$;*
(4) *there exists a precubical subset $\tilde{R} \subseteq \tilde{X} \times \tilde{Y}$ with $(\tilde{i}, \tilde{j}) \in \tilde{R}$, and such that for all $(\tilde{x}_1, \tilde{y}_1) \in \tilde{R}$,*
   - *for any $\tilde{x}_2 \sqsupseteq \tilde{x}_1$ in $\tilde{X}$, there exists $\tilde{y}_2 \sqsupseteq \tilde{y}_1$ in $\tilde{Y}$ for which $(\tilde{x}_2, \tilde{y}_2) \in \tilde{R}$,*
   - *for any $\tilde{y}_2 \sqsupseteq \tilde{y}_1$ in $\tilde{Y}$, there exists $\tilde{x}_2 \sqsupseteq \tilde{x}_1$ in $\tilde{X}$ for which $(\tilde{x}_2, \tilde{y}_2) \in \tilde{R}$.*

Again, the requirement that $\tilde{R}$ be a precubical subset is equivalent to saying that whenever $(\tilde{x}, \tilde{y}) \in \tilde{R}$, then also $(\delta_k^\nu \tilde{x}, \delta_k^\nu \tilde{y}) \in \tilde{R}$ for any k and $\nu \in \{0, 1\}$.

Proof. The implication (1) $\implies$ (2) follows directly from Theorem 3.5, and (3) can be proven from (2) by induction. Equivalence of (3) and (4) is immediate from Lemma 6.4.

For the implication (3) $\implies$ (1), we can use Theorem 3.5 to get a span $\tilde{X} \xleftarrow{f} R \xrightarrow{g} \tilde{Y}$ of open maps in HDA. Connecting these with the projection $\pi_R : \tilde{R} \to R$ gives a span

$\tilde X \xleftarrow{f\circ\pi_R} \tilde R \xrightarrow{g\circ\pi_R} \tilde Y$. By Corollary 4.7, the maps in the span are open in HDA, hence by Lemma 6.2, $X \xleftarrow{f\circ\pi_R} R \xrightarrow{g\circ\pi_R} Y$ is a span of open maps in $\mathsf{HDA_h}$. □

Theorem 6.6. *HDA X, Y are homotopy bisimilar if and only if they are hd-bisimilar.*

Proof. A span of open maps $X \xleftarrow{f} Z \xrightarrow{g} Y$ in HDA lifts to a span $X \xleftarrow{\tilde f} Z \xrightarrow{\tilde g} Y$ in $\mathsf{HDA_h}$, and $\tilde f$ and $\tilde g$ are open by Lemma 6.2. Hence hd-bisimilarity implies homotopy bisimilarity.

For the other direction, let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a span of open maps in $\mathsf{HDA_h}$. In HDA, this is a span $\tilde X \xleftarrow{f} \tilde Z \xrightarrow{g} \tilde Y$, and composing with the projections yields $X \xleftarrow{\pi_X\circ f} \tilde Z \xrightarrow{\pi_Y\circ g} Y$. By Lemma 6.2 and Corollary 4.7, both $\pi_x \circ f$ and $\pi_Y \circ g$ are open in HDA. □

Corollary 6.7. *Homotopy bisimilarity is decidable for finite HDA.*

Proof. The condition in Thm. 3.5(2) immediately gives rise to a fixed-point algorithm similar to the one used to decide standard bisimilarity, *cf.* [21]. □

In order to be able to relate our notion of bisimilarity to other common notions in Section 8 below, we translate it to a relation between pointed cube paths, *i.e.* executions:

Theorem 6.8. *HDA $i : * \to X$, $j : * \to Y$ are homotopy bisimilar if and only if there exists a relation R between pointed cube paths in X and pointed cube paths in Y for which $((i),(j)) \in R$, and such that for all $(\rho,\sigma) \in R$ with $\rho = (x_1,\dots,x_m)$ and $\sigma = (y_1,\dots,y_p)$,*

- $\dim x_m = \dim y_p$,
- *for all $k = 1,\dots,\dim x_m$, $(\rho * \delta^1_k x_m, \sigma * \delta^1_k y_p) \in R$,*
- *for all $k = 1,\dots,\dim x_m$, there exist $\rho' \in \tilde\delta^0_k[\rho]$ and $\sigma' \in \tilde\delta^0_k[\sigma]$ with $(\rho',\sigma') \in R$,*
- *for all $\rho' \sim \rho$, there exists $\sigma' \sim \sigma$ with $(\rho',\sigma') \in R$,*
- *for all $\sigma' \sim \sigma$, there exists $\rho' \sim \rho$ with $(\rho',\sigma') \in R$,*
- *for all $\rho' \sqsupseteq \rho$, there exists $\sigma' \sqsupseteq \sigma$ with $(\rho',\sigma') \in R$,*
- *for all $\sigma' \sqsupseteq \sigma$, there exists $\rho' \sqsupseteq \rho$ with $(\rho',\sigma') \in R$.*

Note how the last four conditions are reminiscent of the ones for *history-preserving bisimilarity* [29].

Proof. For the "if" part of the theorem, assume that we have a relation R as in the theorem and define $\tilde R \subseteq \tilde X \times \tilde Y$ by $\tilde R = \{([\rho],[\sigma]) \mid (\rho,\sigma) \in R\}$. Then $(\tilde i, \tilde j) \in \tilde R$, and the first three conditions ensure that $\tilde R$ is a precubical subset of $\tilde X \times \tilde Y$: By $\dim x_m = \dim y_p$, $\tilde R_n \subseteq \tilde X_n \times \tilde Y_n$ for all n, the second condition implies that for all $([\rho],[\sigma]) \in \tilde R$ and all k, also $\tilde\delta^1_k([\rho],[\sigma]) = ([\rho * \delta^1_k x_m],[\sigma * \delta^1_k y_m]) \in \tilde R$, and using the third condition, also $\tilde\delta^0_k([\rho],[\sigma]) = ([\rho'],[\sigma']) \in \tilde R$.

Now let $(\tilde x_1, \tilde y_1) \in \tilde R$ and $\tilde x_2 \sqsupseteq \tilde x_1$. We have $\rho_1 \in \tilde x_1$ and $\sigma_1 \in \tilde y_1$ for which $(\rho_1,\sigma_1) \in R$. Let $\rho_1' \in \tilde x_1$ and $\rho_2 \in \tilde x_2$ such that $\rho_2 \sqsupseteq \rho_1'$, then $\rho_1' \sim \rho_1$, hence we have $\sigma_1' \sim \sigma_1$ for which $(\rho_1',\sigma_1') \in R$. By $\rho_2 \sqsupseteq \rho_1'$ we also have $\sigma_2 \sqsupseteq \sigma_1'$ for which $(\rho_2,\sigma_2) \in R$, hence $(\tilde x_2 = [\rho_2],[\sigma_2]) \in \tilde R$ as was to be shown. The symmetric condition in Theorem 6.5(4) can be shown analogously.

For the other implication, let $\tilde R \subseteq \tilde X \times \tilde Y$ be a precubical subset as in Theorem 6.5(4) and define a relation of pointed cube paths by $R = \{(\rho,\sigma) \mid ([\rho],[\sigma]) \in \tilde R\}$. Then $((i),(j)) \in R$. Let $(\rho = (x_1,\dots,x_m), \sigma = (y_1,\dots,y_p)) \in R$, then $\dim x_m = \dim y_p$ by $\tilde R_n \subseteq \tilde X_n \times \tilde Y_n$. Let $k \in \{1,\dots,\dim x_m\}$, then $\tilde\delta^1_k([\rho],[\sigma]) \in \tilde R$ and hence $(\rho * \delta^1_k x_m, \sigma * \delta^1_k y_p) \in R$. Using $\tilde\delta^0_k([\rho],[\sigma]) \in \tilde R$, we see that there must exist $\rho' \in \tilde\delta^0_k[\rho]$ and $\sigma' \in \tilde\delta^0_k[\sigma]$ with $(\rho',\sigma') \in R$.

Now let $(\rho, \sigma) \in R$, then also $(\rho', \sigma') \in R$ for any $\rho' \sim \rho$, $\sigma' \sim \sigma$, showing the fourth and fifth conditions of the theorem. For the sixth one, let $\rho' \sqsupseteq \rho$, then $[\rho'] \sqsupseteq [\rho]$, hence we have $\tilde{y}_2 \sqsupseteq [\sigma]$ for which $([\rho'], \tilde{y}_2) \in \tilde{R}$. By definition of R we have $(\rho', \sigma') \in R$ for any $\sigma' \in \tilde{y}_2$, and by $\tilde{y}_2 \sqsupseteq [\sigma]$, there is $\sigma' \in \tilde{y}_2$ for which $\sigma' \sqsupseteq \sigma$, showing the sixth condition. The seventh condition is proved analogously. □

7. Labels

For labeling HDA, we need a subcategory of pCub isomorphic to the category of sets and functions. Given a finite or countably infinite set $S = \{a_1, a_2, \dots\}$, we construct a precubical set $!S = \{!S_n\}$ by letting

$$!S_n = \big\{(a_{i_1}, \dots, a_{i_n}) \mid i_k \le i_{k+1} \text{ for all } k = 1, \dots, n-1\big\}$$

with face maps defined by $\delta_k^\nu(a_{i_1}, \dots, a_{i_n}) = (a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}, \dots, a_{i_n})$.

Definition 7.1. The category of *higher-dimensional tori* HDO is the full subcategory of pCub generated by the objects $!S$.

As any object in HDO has precisely one 0-cube, the pointed category $* \downarrow \mathsf{HDO}$ is isomorphic to HDO. Note that the objects in HDO indeed are tori: by definition, lower and upper boundaries of any n-cube agree, hence all n-cubes are loops.

Lemma 7.2. HDO *is isomorphic to the category of sets and functions.*

Proof. A function $f : S \to T$ is lifted to $!f : !S \to !T$ by $f(a_1, \dots, a_n) = \langle f(a_1), \dots, f(a_n)\rangle$, where the elements on the right-hand side are re-sorted. This is easily seen to be a precubical mapping. The inverse direction follows from the fact that the objects in HDO are *coskeletal* on their 1-cubes, *cf.* [3, 4]. □

Definition 7.3. The category of *labeled higher-dimensional automata* is the pointed arrow category $\mathsf{LHDA} = * \downarrow \mathsf{pCub} \to \mathsf{HDO}$, with objects $* \to X \to !S$ labeled pointed precubical sets and morphisms commutative diagrams

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] \arrow[d] & & Y \arrow[d] \\ !S \arrow[rr, "\sigma"'] & & !T \end{tikzcd}$$

Remark 7.4. If morphisms of labeled higher-dimensional automata are to model (functional) *simulations*, then one needs *partial* labeling morphisms σ. This can be achieved by introducing *degeneracies* for precubical sets, passing to the category Cub of *cubical* sets. One can then show that the full subcategory of Cub spanned by free cubical sets on higher-dimensional tori is isomorphic to the category of finite sets and partial functions and define LHDA accordingly. This is indeed the approach taken in [4, 14]. As we are only concerned with bisimilarity here, we do not need partial labeling morphisms.

We now fix a labeling set Σ; we will work in the category with morphisms

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] \arrow[d] & & Y \arrow[d] \\ !\Sigma \arrow[rr, "\mathrm{id}"'] & & !\Sigma \end{tikzcd}$$

Definition 7.5. A morphism $(f, \mathrm{id}) : (* \to X \to !\Sigma) \to (* \to Y \to !\Sigma)$ in LHDA is *open* if its component f is open in HDA. Labeled HDA $* \to X \to !\Sigma$, $* \to Y \to !\Sigma$ are *hd-bisimilar* if there is $* \to Z \to !\Sigma \in \mathsf{LHDA}$ and a span of open maps $X \leftarrow Z \to Y$ in LHDA.

The definitions of open maps and bisimilarity in $\mathsf{HDA_h}$ can now easily be extended to the labeled case. Again, we will only need label-preserving morphisms.

Definition 7.6. The category of *labeled higher-dimensional automata up to homotopy* $\mathsf{LHDA_h}$ has as objects labeled HDA $* \to X \to !S$ and as morphisms pairs of precubical morphisms $(f, \sigma) : (* \to \tilde X \to !\tilde S) \to (* \to \tilde Y \to !\tilde T)$ of unfoldings.

Definition 7.7. A morphism $(f, \mathrm{id}) : (* \to X \to !\Sigma) \to (* \to Y \to !\Sigma)$ in $\mathsf{LHDA_h}$ is *open* if its component f is open in $\mathsf{HDA_h}$. Labeled HDA $* \to X \to !\Sigma$, $* \to Y \to !\Sigma$ are *homotopy bisimilar* if there is a labeled HDA $* \to Z \to !\Sigma$ and a span of open maps $X \leftarrow Z \to Y$ in $\mathsf{LHDA_h}$.

As a corollary, we see that $* \to X \xrightarrow{\lambda} !\Sigma$, $* \to Y \xrightarrow{\mu} !\Sigma$ are homotopy bisimilar if and only if there exists a precubical subset $\tilde R \subseteq \tilde X \times \tilde Y$ like in Theorem 6.5 which *respects homotopy classes of labels*, *i.e.* for which $\tilde\lambda(\tilde x) = \tilde\mu(\tilde y)$ for each $(\tilde x, \tilde y) \in \tilde R$.

The proof of the next theorem is exactly the same as the one for Theorem 6.6.

Theorem 7.8. *Labeled HDA X, Y are homotopy bisimilar if and only if they are hd-bisimilar.* □

8. Relation to Other Equivalences

It remains to be seen how our homotopy bisimilarity relates to other notions of equivalence for concurrent systems.

For a labeled HDA $* \to X \xrightarrow{\lambda} !\Sigma$, we extend λ to cube paths in X by $\lambda(x_1, \dots, x_m) = (\lambda(x_1), \dots, \lambda(x_m))$.

The following is a labeled version of Theorem 6.8.

Theorem 8.1. *Labeled HDA $* \xrightarrow{i} X \xrightarrow{\lambda} !\Sigma$, $* \xrightarrow{j} Y \xrightarrow{\mu} !\Sigma$ are homotopy bisimilar if and only if there exists a relation R between pointed cube paths in X and pointed cube paths in Y for which $((i), (j)) \in R$, and such that for all $(\rho, \sigma) \in R$ with $\rho = (x_1, \dots, x_m)$ and $\sigma = (y_1, \dots, y_p)$,*

- *for all $k = 1, \dots, \dim x_m$, $(\rho * \delta_k^1 x_m, \sigma * \delta_k^1 y_p) \in R$,*
- *for all $k = 1, \dots, \dim x_m$, there exist $\rho' \in \tilde\delta_k^0[\rho]$ and $\sigma' \in \tilde\delta_k^0[\sigma]$ with $(\rho', \sigma') \in R$,*
- *$\lambda(\rho) \sim \mu(\sigma)$,*
- *for all $\rho' \sim \rho$, there exists $\sigma' \sim \sigma$ with $(\rho', \sigma') \in R$,*
- *for all $\sigma' \sim \sigma$, there exists $\rho' \sim \rho$ with $(\rho', \sigma') \in R$,*
- *for all $\rho' \sqsupseteq \rho$, there exists $\sigma' \sqsupseteq \sigma$ with $(\rho', \sigma') \in R$,*

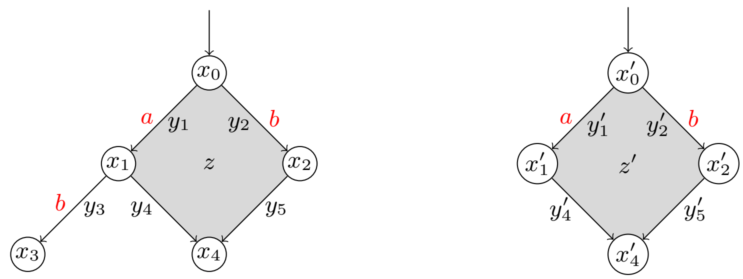

Figure 7: Two HDA pertaining to Example 8.3.

- *for all $\sigma' \sqsupseteq \sigma$, there exists $\rho' \sqsupseteq \rho$ with $(\rho', \sigma') \in R$.*

Proof. For the "if" part of the theorem, assume that we have a relation R as in the theorem and define $\tilde{R} \subseteq \tilde{X} \times \tilde{Y}$ by $\tilde{R} = \{([\rho], [\sigma]) \mid (\rho, \sigma) \in R\}$, as in the proof of Theorem 6.8. Let $(\rho, \sigma) \in R$ and write $\rho = (x_1, \dots, x_m)$ and $\sigma = (y_1, \dots, y_p)$. By $\lambda(\rho) \sim \mu(\sigma)$, also $\lambda(x_m) = \mu(y_p)$, which, as λ and μ are precubical mappings, implies that $\dim x_m = \dim y_p$.

Thus R satisfies the conditions of Theorem 6.8, so we can infer that $\tilde{R} \subseteq \tilde{X} \times \tilde{Y}$ is a precubical subset for which the conditions in Theorem 6.5(4) hold. Let $([\rho], [\sigma]) \in \tilde{R}$, then $\lambda(\rho) \sim \mu(\sigma)$ entails $\tilde{\lambda}[\rho] = \tilde{\mu}[\sigma]$.

For the other direction, let $\tilde{R} \subseteq \tilde{X} \times \tilde{Y}$ be a precubical subset as in Theorem 6.5(4) which respects labels. Define a relation of pointed cube paths by $R = \{(\rho, \sigma) \mid ([\rho], [\sigma]) \in \tilde{R}\}$, then R satisfies the conditions of Theorem 6.8. Let $(\rho, \sigma) \in R$, then $([\rho], [\sigma]) \in \tilde{R}$ implies $\tilde{\lambda}[\rho] = \tilde{\mu}[\sigma]$, hence $\lambda(\rho) \sim \mu(\sigma)$. □

Theorem 8.2. *Homotopy bisimilarity is not implied by ST-bisimilarity and incomparable with history-preserving bisimilarity.*

Proof. This will follow from the examples below. □

We finish this section by exposing several examples. The first two serve to position homotopy bisimilarity with regard to history-preserving bisimilarity, and the last shows a case in which homotopy bisimilarity distinguishes auto-concurrency in a way similar to ST-bisimilarity. Whether homotopy bisimilarity implies ST-bisimilarity, and whether it is implied by hereditary history-preserving (hhp) bisimilarity, is open.

Example 8.3. The two HDA in Fig. 7 are hd-bisimilar, as witnessed by the following precubical subset $R \subseteq X \times X'$:

$$R_0 = \{(x_0, x'_0), (x_1, x'_1), (x_2, x'_2), (x_3, x'_4), (x_4, x'_4)\}$$
$$R_1 = \{(y_1, y'_1), (y_2, y'_2), (y_3, y'_4), (y_4, y'_4), (y_5, y'_5)\}$$
$$R_2 = \{(z, z')\}$$

In [32, Example 5.2.2] it is shown that the Petri-net translations of these HDA are ST-bisimilar, but not history-preserving bisimilar.

Example 8.4. We show by a bisimulation-game [27] type argument that the HDA in Fig. 8 are not hd-bisimilar. Note that in [31] it is shown that these systems are history-preserving bisimilar but not hhp-bisimilar.

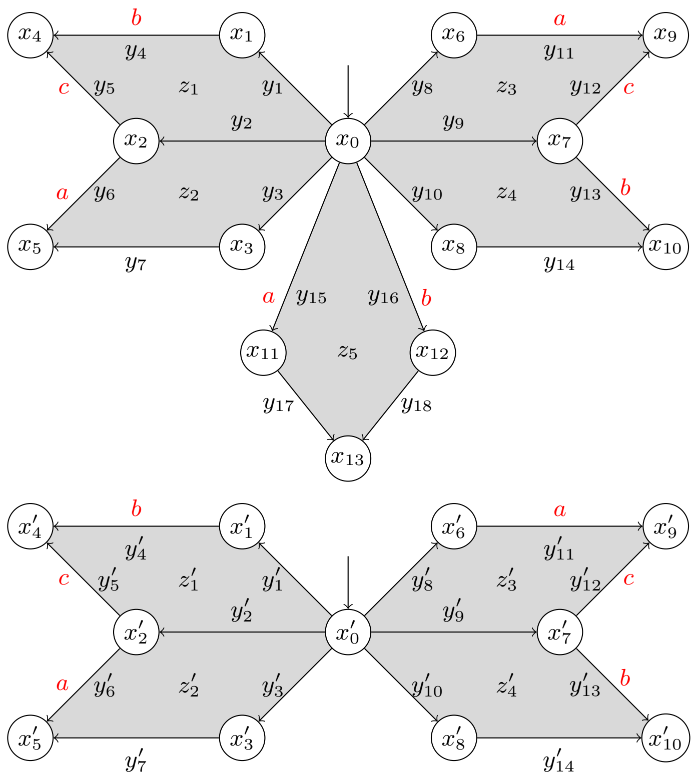

Figure 8: Two HDA pertaining to Example 8.4.

The starting configuration is (x_0, x'_0), in which Player 1 (the spoiler) plays the x_0-extension y_{16}. Player 2 (the duplicator) must answer with either y'_2 or y'_{10}. Playing y'_2 is losing, as Player 1 then can play the y'_2-extension z'_1, with label bc, which Player 2 cannot duplicate. Hence Player 2 must play y'_{10}. Then Player 1 attacks by extending y_{16} with z_5, to which Player 2 can only answer z'_4. Player 1 now retreats to the other lower boundary of z_5, y_{15}, to which Player 2 must answer y'_9. But then Player 1 plays the y'_9-extension z'_3, with label ac, which Player 2 cannot duplicate. Hence the game is decided in favor of the spoiler.

Example 8.5. Again using a hd-bisimulation game, we show that the HDA in Fig. 9 are not hd-bisimilar. Note that according to [31], they are split bisimilar, but not ST-bisimilar.

From the initial configuration (x_0, x'_0) of the game, the spoiler plays y_1 and then z_1, leading to the configuration (z_1, z'_1). Playing y_4 and then z_2, the spoiler forces the configuration (z_2, z'_2) and, playing y_8 and then z_4, leads the game to the cc-labeled configuration (z_4, z'_4). Here the spoiler plays y_{12}, which the duplicator has to answer by the z'_4-boundary *in the same direction*, hence y'_{12}. But then the spoiler can play the cd-labeled z_5, to which the duplicator has no answer.

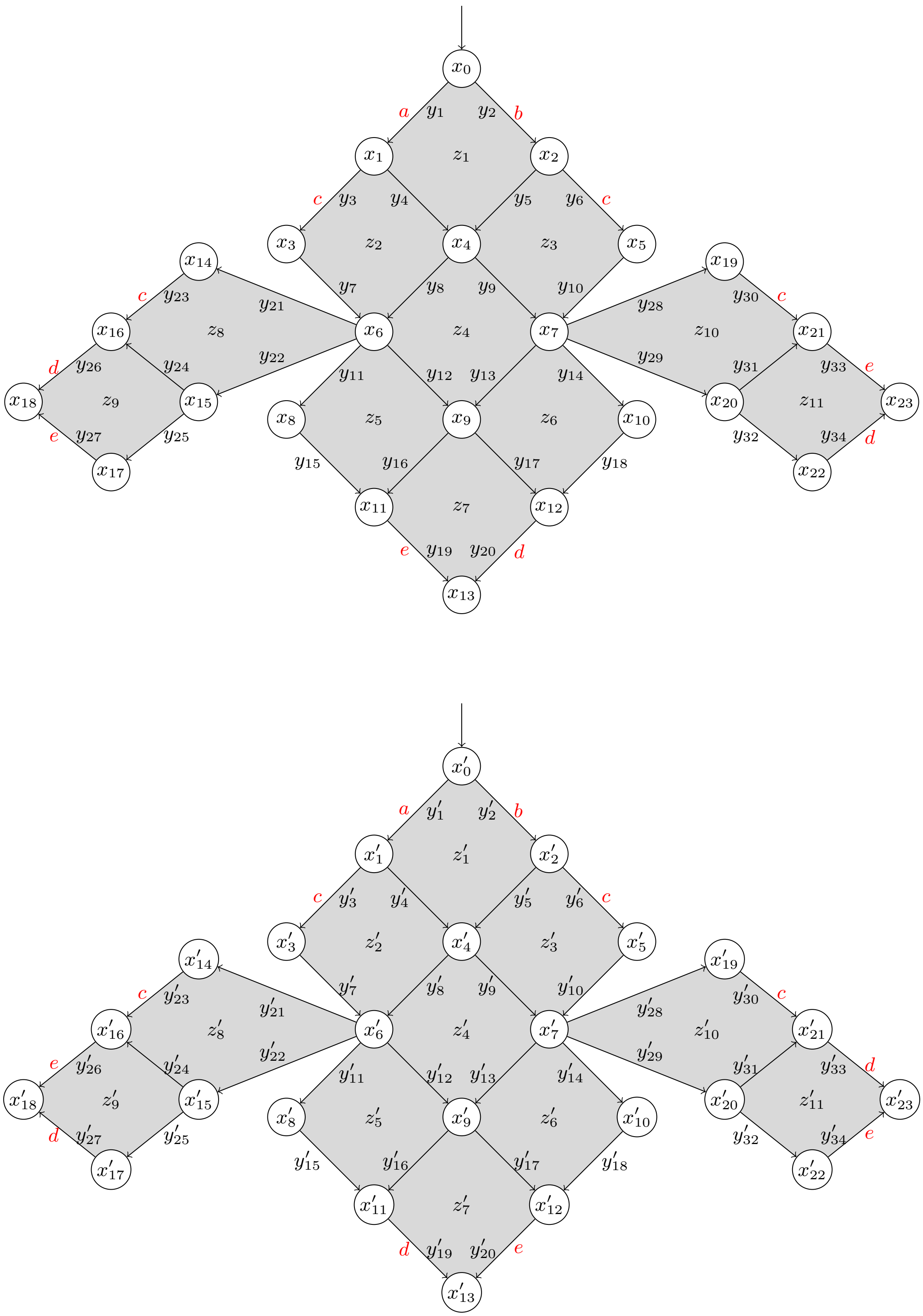

Figure 9: Two HDA pertaining to Example 8.5.

9. Conclusion

We have introduced a notion of homotopy bisimilarity for HDA which can be characterized as an equivalence relation between homotopy classes of computations, or equivalently by a zig-zag relation between cubes in all dimensions. Aside from implying decidability of homotopy bisimilarity for finite HDA, and together with the results of [29], this confirms that HDA is a useful formalism for concurrency: not only does it generalize the main models for concurrency which people have been working with, but it also is remarkably simple and natural.

One major question which remains is how precisely homotopy bisimilarity fits into the spectrum of equivalence notions for non-interleaving models. We have shown that it is finer than split bisimilarity and incomparable with history-preserving bisimilarity, but we miss to see whether homotopy bisimilarity implies ST-bisimilarity and whether it is implied by hhp-bisimilarity.

With regard to the geometric interpretation of HDA as directed topological spaces, there are two open questions related to the work laid out in the paper: In [4] we show that morphisms in HDA are open if and only if their geometric realizations lift pointed directed paths. This shows that there are some connections to weak factorization systems [1] here which should be explored; see [19] for a related approach.

In [5] we relate homotopy of cube paths to directed homotopy of directed paths in the geometric realization. Based on this, one should be able to prove that the geometric realization of the unfolding of a HDA is the same as the universal directed covering [8] of its geometric realization.

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