The quantitative linear-time–branching-time spectrum

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Abstract

We present a distance-agnostic approach to quantitative verification. Taking as input an unspecified distance on system traces, or executions, we develop a game-based framework which allows us to define a spectrum of different interesting system distances corresponding to the given trace distance. Thus we extend the classic linear-time–branching-time spectrum to a quantitative setting, parametrized by trace distance. We also prove a general transfer principle which allows us to transfer counterexamples from the qualitative to the quantitative setting, showing that all system distances are mutually topologically inequivalent.

Keywords: Quantitative verification, system distance, distance hierarchy, linear time, branching time

1. Introduction

For rigorous design and verification of embedded systems, both qualitative and quantitative information and constraints have to be taken into account [23, 26, 30]. This applies to the *models* considered, to the *properties* one wishes to be satisfied, and to the *verification* itself. Hence the question asked in quantitative verification is not "Does the system satisfy the requirements?", but rather "*To which extent* does the system satisfy the requirements?" Standard qualitative verification techniques are inherently *fragile*: either the requirements are satisfied, or they are not, regardless of how close the actual system might come to the specification. To overcome this lack of robustness, notions of *distance* between systems are essential.

As pointed out in [23], qualitative and quantitative aspects of verification are best treated orthogonally in any theory of quantitative verification. In practical applications these aspects may indeed interfere with each other, but for the purpose of theory, they are best treated separately. The formalism we propose in this paper addresses this separation by modeling qualitative aspects using standard *labeled transition systems* and expressing the quantitative aspects using *trace distances*, or distances on system executions. Based on these ingredients, we develop a comprehensive theory of *system distances* which

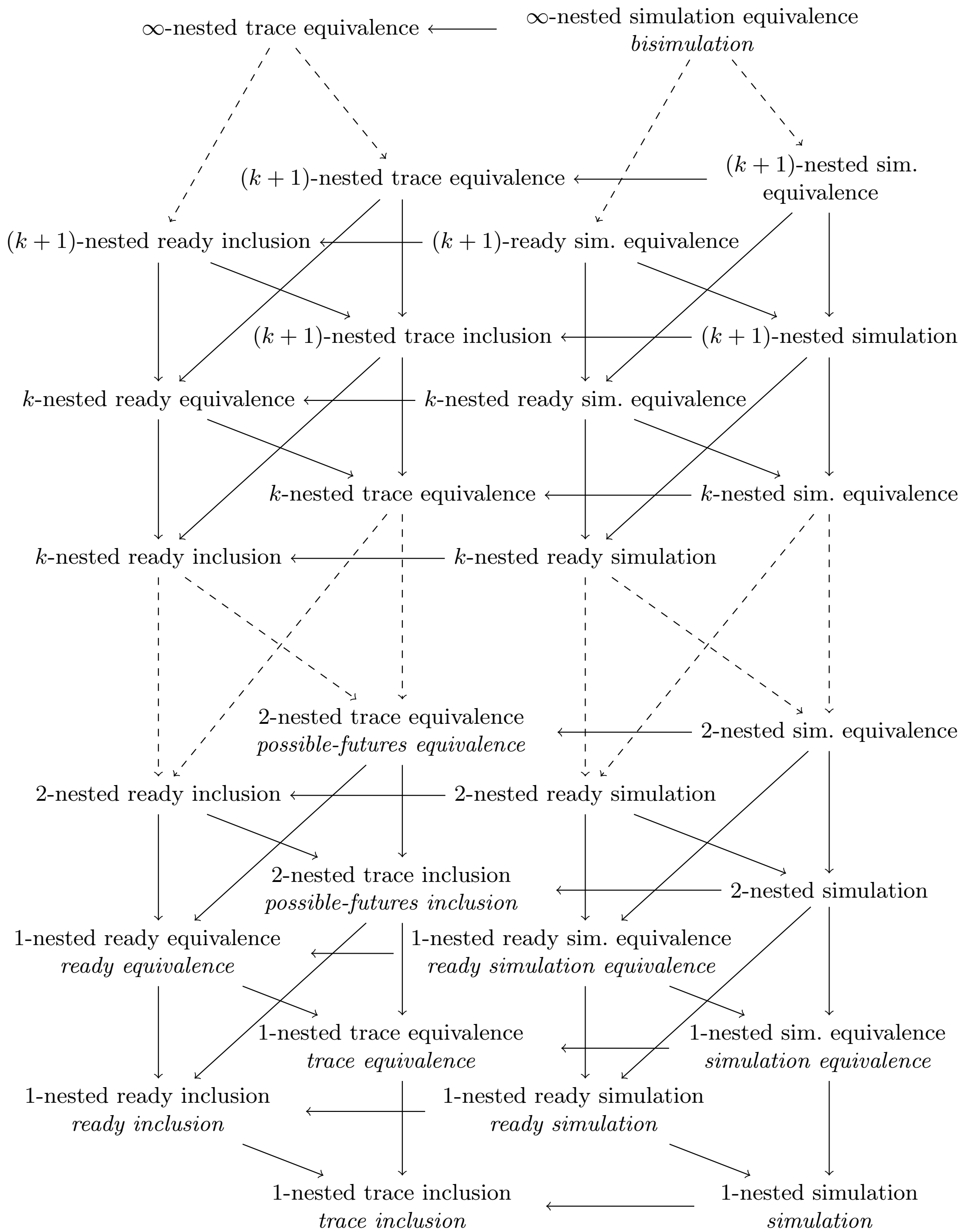

Figure 1: The quantitative linear-time–branching-time spectrum. The nodes are the different system distances introduced in this paper, and an edge $d_1 \longrightarrow d_2$ or $d_1 \dashrightarrow d_2$ indicates that $d_1(s,t) \geq d_2(s,t)$ for all states s, t, and that d_1 and d_2 in general are topologically inequivalent.

generalizes the standard linear-time–branching-time spectrum [17, 18, 36] to a quantitative setting, see Figure 1. Similarly to [4], our theory relies on Ehrenfeucht-Fraïssé games and allows for a more refined analysis of systems. More precisely, our parametrized framework forms a hierarchy of games, for each trace distance used in its instantiation. In the quantitative setting, using games with quantitative objectives as opposed to discrete games, effectively allows us to obtain a continuous verdict on the relationship between systems, and hence to detect the difference between minor and major discrepancies between systems. We refer to [15] for a good introduction to the theory of quantitative games.

Indeed the view of this paper is that in a theory of quantitative verification, the quantitative aspects should be treated just as much as an input to a verification problem as the qualitative aspects are. Hence it is of limited use to develop a theory pertaining only to some *specific* quantitative measures like the ones in [2, 3, 11, 24, 33] and other papers which all treat only a few specific ways of measuring distances; any theory of quantitative verification should work just as well regardless of the way the engineers decide to measure differences between system executions. We note that the framework we lay out here may be equally instantiated with labels (or propositions) in states rather than on transitions, hence it also generalizes the formalisms of [5, 35].

We take as input a labeled transition system and a trace distance; both are unspecified except for some general characteristic properties. Based on this information and using the theory of *quantitative games*, we lift most of the linear-time–branching-time spectrum of van Glabbeek [36] to the quantitative setting, while the rest may be obtained in a similar way using minor additional conditions as described in [4]. We show that all the distinct equivalences in van Glabbeek's spectrum correspond to topologically inequivalent distances in the quantitative setting.

As our framework is independent of the chosen trace distance, we are essentially adding a second, quantitative, dimension to the linear-time–branching-time spectrum. In this terminology, the first dimension is the qualitative one which concerns the different linear and branching ways of specifying qualitative constraints, and the second dimension bridges the gap between the trivial van-Glabbeek spectrum in which everything is equivalent, and the discrete spectrum in which everything is fragile.

We start the paper by recalling some preliminaries and fixing notation in Section 2. Section 3 then shows that our general framework of trace distances is applicable to a large number of system distances found in the literature [2, 3, 5–8, 10–12, 20, 22, 24, 32, 33, 35, 37]; indeed we show in Section 8 that it generalizes all of them.

Before this, we devote Section 4 to introducing a game with quantitative objectives on which all the subsequent developments build, and Section 5 to show some general properties of this game. In Section 6 we reap the fruits of our labor and develop all of the quantitative linear-time–branching-time spectrum of Figure 1. Section 7 then prepares for Section 8 by treating an important special case where the trace distance in question has a certain recursive characterization. Here we show that in this case, all distances in the spectrum can be expressed as fixed points of certain functionals.

Seeing in Section 8 that all known applications of our framework are instantiations of the special case of recursively defined trace distances, it is most probable that the fixed-point characterizations of Section 7 will be the most important part of this work, and that the game-based framework introduced and used in the preceding sections should be seen only as a justification of Section 7.

This paper is an extended and revised version of papers which have appeared as [13, 14]. Compared to these papers, some more background and motivation has been added to Sections 4 and 6, Section 7 has been greatly extended, and all proofs are included in the paper.

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2. Traces, Trace Distances, and Transition Systems

In this paper, the set $\mathbb{N}$ of natural numbers includes 0; the set of positive natural numbers is denoted by $\mathbb{N}_+$. For a finite non-empty sequence $a = (a_0, \dots, a_n)$, we write $\text{last}(a) = a_n$ and $\mathsf{len}(a) = n+1$ for the length of a; for an infinite sequence a we let $\mathsf{len}(a) = \infty$. Concatenation of finite sequences a and b is denoted $a \cdot b$. We denote by $a^k = (a_k, a_{k+1}, \dots)$ the k-shift, and by a_i the $(i+1)$st element, of a (finite or infinite) sequence, and by $\boldsymbol{\epsilon}$ the empty sequence.

We need to recall some terminology and constructions regarding distances. A *hemimetric* on a set X is a function $d : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies $d(x, x) = 0$ and $d(x, y) + d(y, z) \geq d(x, z)$ (the *triangle inequality*) for all $x, y, z \in X$. The hemimetric is said to be *symmetric* if also $d(x, y) = d(y, x)$ for all $x, y \in X$; it is said to be *separating* if $d(x, y) = 0$ implies $x = y$. The terms "pseudometric" for a symmetric hemimetric, "quasimetric" for a separating hemimetric, and "metric" for a hemimetric which is both symmetric and separating are also in use, but we will not use them here. The tuple (X, d) is called a *hemimetric space*.

Note that our hemimetrics are *extended* in that they can take the value ∞. This is convenient for several reasons, *cf.* [25], one of them being that it allows for a disjoint union, or coproduct, of hemimetric spaces: the disjoint union of (X_1, d_1) and (X_2, d_2) is the hemimetric space $(X_1, d_1) \uplus (X_1, d_2) = (X_1 \uplus X_2, d)$ where points from different components are infinitely far away from each other, *i.e.* with d defined by

$$d(x, y) = \begin{cases} d_1(x, y) & \text{if } x, y \in X_1, \\ d_2(x, y) & \text{if } x, y \in X_2, \\ \infty & \text{otherwise.} \end{cases}$$

The *product* of two hemimetric spaces (X_1, d_1) and (X_2, d_2) is the hemimetric space $(X_1, d_1) \times (X_2, d_2) = (X_1 \times X_2, d)$ with d given by $d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$.

The *symmetrization* of a hemimetric d on X is the symmetric hemimetric $\bar{d} : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by $\bar{d}(x, y) = \max(d(x, y), d(y, x))$; this is the smallest of all symmetric hemimetrics d' on X for which $d \leq d'$.

The *topology* generated by a hemimetric d on X is the same as the one generated by its symmetrization $\bar{d}$; it has as open sets all unions of open balls $B(x; r) = \{y \in X \mid \bar{d}(x, y) < r\}$, for $x \in X$ and $r > 0$. Two hemimetrics d_1 and d_2 on X are said to be *topologically equivalent* if the topologies on X generated by d_1 and d_2 coincide. Topological equivalence hence preserves topological notions such as convergence of sequences: If a sequence (x_j) of points in X converges in one hemimetric, then it also converges in the other. As a consequence, topological equivalence of d_1 and d_2 implies that for all $x, y \in X$, $d_1(x, y) = 0$ if, and only if, $d_2(x, y) = 0$.

Topological equivalence is the weakest of the common notions of equivalence for metrics; it does not preserve geometric properties such as distances or angles. We are hence mainly interested in topological equivalence as a tool for showing *negative* properties; we will later prove a number of results on topological *inequivalence* of hemimetrics which imply that any other reasonable metric equivalence, such as uniform or Lipschitz equivalence, also fails for these cases.

Throughout this paper we fix a set $\mathbb{K}$ of labels, and we let $\mathbb{K}^\infty = \mathbb{K}^* \cup \mathbb{K}^\omega$ denote the set of finite and infinite traces (*i.e.* sequences) in $\mathbb{K}$. A hemimetric $d^T : \mathbb{K}^\infty \times \mathbb{K}^\infty \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is called a *trace distance* if $\mathsf{len}(\sigma) \neq \mathsf{len}(\tau)$ implies $d^T(\sigma, \tau) = \infty$.

A *labeled transition system* (LTS) is a pair (S, T) consisting of states S and transitions $T \subseteq S \times \mathbb{K} \times S$. We often write $s \xrightarrow{x} t$ to signify that $(s, x, t) \in T$. Given $e = (s, x, t) \in T$, we write $src(e) = s$, $tgt(e) = t$ for the source and target of e. A *path* in (S, T) is a finite or infinite sequence $\pi = ((s_0, x_0, t_0), (s_1, x_1, t_1), \dots)$ of transitions $(s_j, x_j, t_j) \in T$ which satisfy $t_j = s_{j+1}$ for all j. We denote by $\mathrm{tr}(\pi) = (x_0, x_1, \dots)$ the trace induced by such a path π. For $s \in S$ we denote by $\mathrm{Pa}(s)$ the set of (finite or infinite) paths from s and by $\mathrm{Tr}(s) = \{\mathrm{tr}(\pi) \mid \pi \in \mathrm{Pa}(s)\}$ the set of traces from s.

3. Examples of Trace Distances

We show here a number of trace distances with which our quantitative framework can be instantiated. Note that each such distance gives rise to its own linear-time–branching-time spectrum in the quantitative dimension.

Most of the trace distances one finds in the literature are defined by giving a hemimetric d on $\mathbb{K}$ and a method to combine the so-defined distances on individual symbols to a distance on traces. Three general methods are used for this combination (recall that σ_j denotes the $(j+1)$-th symbol in $\sigma = (\sigma_0, \sigma_1, \dots)$):

- The *point-wise* trace distance: $\mathsf{PW}_\lambda(d)(\sigma, \tau) = \sup_j \lambda^j d(\sigma_j, \tau_j)$;
- the *accumulating* trace distance: $\mathsf{ACC}_\lambda(d)(\sigma, \tau) = \sum_j \lambda^j d(\sigma_j, \tau_j)$;
- The *limit-average* trace distance: $\mathsf{AVG}(d)(\sigma, \tau) = \liminf_j \frac{1}{j+1} \sum_{i=0}^{j} d(\sigma_i, \tau_i)$.

Note that the trace distances are parametrized by the label distance $d : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. Also, λ is a *discounting* factor with $0 < \lambda \leq 1$, and we assume that the involved traces have equal length; otherwise any trace distance has value ∞. The point-wise distance thus measures the (discounted) greatest individual symbol distance in the traces, whereas accumulating and limit-average distance accumulate these individual distances along the traces.

If the distance on $\mathbb{K}$ is the *discrete* distance given by $d_{\text{disc}}(x, x) = 0$ and $d_{\text{disc}}(x, y) = \infty$ for $x \neq y$, then all trace distances above agree, for any λ. This defines the *discrete trace distance* $d^T_{\text{disc}} = \mathsf{PW}_\lambda(d_{\text{disc}}) = \mathsf{ACC}_\lambda(d_{\text{disc}}) = \mathsf{AVG}(d_{\text{disc}})$ given by $d^T_{\text{disc}}(\sigma, \tau) = 0$ if $\sigma = \tau$ and ∞ otherwise. We will show below that for the discrete trace distance, our quantitative linear-time–branching-time spectrum specializes to the qualitative one of [36].

If one lets $d(x, x) = 0$ and $d(x, y) = 1$ for $x \neq y$ instead, then $\mathsf{ACC}_1(d)$ is *Hamming distance* [20] for finite traces, and $\mathsf{ACC}_\lambda(d)$ with $\lambda < 1$ and $\mathsf{AVG}(d)$ are two sensible ways to define Hamming distance also for infinite traces. $\mathsf{PW}_1(d)$ is topologically equivalent to the discrete distance; indeed, $\mathsf{PW}_1(d)(\sigma, \tau) = 1$ if, and only if, $d^T_{\text{disc}}(\sigma, \tau) = \infty$.

A generalization of the above distances may be obtained by equipping $\mathbb{K}$ with a preorder $\sqsubseteq \subseteq \mathbb{K} \times \mathbb{K}$ indicating that a label $x \in \mathbb{K}$ may be replaced by any $y \in \mathbb{K}$ with $x \sqsubseteq y$, as *e.g.* in [32]. If we define $d(x, y) = 0$ if $x \sqsubseteq y$ and $d(x, y) = \infty$ otherwise (note that this is a hemimetric which is not necessarily symmetric), then again $\mathsf{PW}_\lambda(d) = \mathsf{ACC}_\lambda(d) = \mathsf{AVG}(d)$ for any λ.

Point-wise and accumulating distances have been studied in a number of papers [2, 3, 5, 11, 24, 33, 35]. $\mathsf{PW}_1(d)$ is the point-wise distance from [5, 7, 11, 24, 33], and $\mathsf{PW}_\lambda(d)$ for $\lambda < 1$ is the discounted distance from [5, 6]. Accumulating distance $\mathsf{ACC}_\lambda(d)$ has been studied in [11, 24, 33], and $\mathsf{AVG}(d)$ *e.g.* in [2, 3]. Both $\mathsf{ACC}_\lambda(d)$ and $\mathsf{AVG}(d)$ are well-known from the theory of discounted and mean-payoff games [10, 37].

All distances above were obtained from distances on individual symbols in $\mathbb{K}$. A trace distance for which this is *not* the case is the *maximum-lead* distance from [22, 33] defined for $\mathbb{K} \subseteq \Sigma \times \mathbb{R}$, where Σ is a finite alphabet. Writing $x \in \mathbb{K}$ as $x = (x^\ell, x^w)$, it is given by

$$d^T_\pm(\sigma, \tau) = \begin{cases} \sup_j \left| \sum_{i=0}^j \sigma_i^w - \sum_{i=0}^j \tau_i^w \right| & \text{if } \sigma_j^\ell = \tau_j^\ell \text{ for all } j, \\ \infty & \text{otherwise.} \end{cases}$$

As this measures differences of accumulated labels along runs, it is especially useful for real-time systems, *cf.* [12, 22].

As a last example of a trace distance we mention the *Cantor* distance given by $d^T_{\text{C}}(\sigma, \tau) = (1 + \inf\{j \mid \sigma_j \neq \tau_j\})^{-1}$. Cantor distance hence measures the (inverse of the) length of the common prefix of the sequences and has been used for verification *e.g.* in [8]. Both Hamming and Cantor distance have applications in information theory and pattern matching.

We will return to our example trace distances in Section 8 to show how our framework may be applied to yield concrete formulations of distances in the linear-time–branching-time spectrum relative to these.

4. Quantitative Ehrenfeucht-Fraïssé Games

To lift the linear-time–branching-time spectrum to the quantitative setting, we define below a quantitative Ehrenfeucht-Fraïssé game [9, 16] on a given LTS (S, T) which is similar to the game hierarchy in [4] and the well-known bisimulation game of [31].

The intuition of the game is as follows: The two players, with Player 1 starting the game, alternate to choose transitions, or *moves*, in T, starting with transitions from given start states s and t and continuing their choices from the targets of the transitions chosen in the previous step. At each of his turns, Player 1 also makes a choice whether to choose a transition from the target of his own previous choice, or from the target of his opponent's previous choice (to "switch paths"). We use a *switch counter* to keep track of how often Player 1 has chosen to switch paths. Player 2 has then to respond with a transition from the remaining target. This game is played for an infinite number of rounds, or until one player runs out of choices, thus building two finite or infinite paths. The value of the game is then the trace distance of the traces of these two paths.

We proceed to formalize the above intuition. A Player-1 *configuration* of the game is a tuple $(\pi, \rho, m) \in T^n \times T^n \times \mathbb{N}$, for $n \in \mathbb{N}$, such that for all $i \in \{0, \dots, n-2\}$, either $src(\pi_{i+1}) = tgt(\pi_i)$ and $src(\rho_{i+1}) = tgt(\rho_i)$, or $src(\pi_{i+1}) = tgt(\rho_i)$ and $src(\rho_{i+1}) = tgt(\pi_i)$. Similarly, a Player-2 configuration is a tuple $(\pi, \rho, m) \in T^{n+1} \times T^n \times \mathbb{N}$ such that for all $i \in \{0, \dots, n-2\}$, either $src(\pi_{i+1}) = tgt(\pi_i)$ and $src(\rho_{i+1}) = tgt(\rho_i)$, or $src(\pi_{i+1}) = tgt(\rho_i)$ and $src(\rho_{i+1}) = tgt(\pi_i)$; and $src(\pi_n) = tgt(\pi_{n-1})$ or $src(\pi_n) = tgt(\rho_{n-1})$. The set of all Player-$i$ configurations is denoted Conf_i.

Intuitively, the configuration (π, ρ, m) keeps track of the history of the game; π stores the choices of Player 1, ρ the choices of Player 2, and m is the switch counter. Hence π and ρ are sequences of transitions in T which can be arranged by suitable swapping to form two paths $(\bar{\pi}, \bar{\rho})$. How exactly these sequences are constructed is determined by a pair of *strategies* which specify for each player which edge to play from any configuration.

A Player-1 strategy is hence a partial mapping $\theta_1 : \mathsf{Conf}_1 \to T \times \mathbb{N}$ such that for all $(\pi, \rho, m) \in \mathsf{Conf}_1$ for which $\theta_1(\pi, \rho, m) = (e', m')$ is defined,

- $src(e') = tgt(\mathrm{last}(\pi))$ and $m' = m$ or $m' = m + 1$, or
- $src(e') = tgt(\mathrm{last}(\rho))$ and $m' = m + 1$.

A Player-2 strategy is a partial mapping $\theta_2 : \mathsf{Conf}_2 \to T \times \mathbb{N}$ such that for all $(\pi \cdot e, \rho, m) \in \mathsf{Conf}_2$ for which $\theta_2(\pi \cdot e, \rho, m) = (e', m')$ is defined, $m' = m$, and $src(e') = tgt(\mathrm{last}(\rho))$ if $src(e) = tgt(\mathrm{last}(\pi))$, $src(e') = tgt(\mathrm{last}(\pi))$ if $src(e) = tgt(\mathrm{last}(\rho))$. The sets of Player-1 and Player-2 strategies are denoted Θ_1 and Θ_2.

Note that if Player 1 chooses a transition from the end of the previous choice of Player 2 (case $src(e') = tgt(\mathrm{last}(\rho))$ above), then the switch counter is increased; but Player 1 may also choose to increase the switch counter without switching paths. Player 2 does not touch the switch counter.

We can now define what it means to *update* a configuration according to a strategy: For $\theta_1 \in \Theta_1$ and $(\pi, \rho, m) \in \mathsf{Conf}_1$, $\mathsf{upd}_{\theta_1}(\pi, \rho, m)$ is defined if $\theta_1(\pi, \rho, m) = (e', m')$ is defined, and then $\mathsf{upd}_{\theta_1}(\pi, \rho, m) = (\pi \cdot e', \rho, m')$. Similarly, for $\theta_2 \in \Theta_2$ and $(\pi \cdot e, \rho, m) \in \mathsf{Conf}_2$, $\mathsf{upd}_{\theta_2}(\pi \cdot e, \rho, m)$ is defined if $\theta_2(\pi \cdot e, \rho, m) = (e', m')$ is defined, and then $\mathsf{upd}_{\theta_2}(\pi \cdot e, \rho, m) = (\pi \cdot e, \rho \cdot e', m')$.

For any pair of states $(s, t) \in S \times S$, a pair of strategies $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$ inductively determines a sequence (π^j, ρ^j, m^j) of configurations, by

$$(\pi^0, \rho^0, m^0) = (s, t, 0);$$

$$(\pi^{2j+1}, \rho^{2j+1}, m^{2j+1}) = \begin{cases} \text{undefined} & \text{if } \mathsf{upd}_{\theta_1}(\pi^{2j}, \rho^{2j}, m^{2j}) \text{ is undefined,} \\ \mathsf{upd}_{\theta_1}(\pi^{2j}, \rho^{2j}, m^{2j}) & \text{otherwise;} \end{cases}$$

$$(\pi^{2j}, \rho^{2j}, m^{2j}) = \begin{cases} \text{undefined} & \text{if } \mathsf{upd}_{\theta_2}(\pi^{2j-1}, \rho^{2j-1}, m^{2j-1}) \\ & \qquad \text{is undefined,} \\ \mathsf{upd}_{\theta_2}(\pi^{2j-1}, \rho^{2j-1}, m^{2j-1}) & \text{otherwise.} \end{cases}$$

Note that indeed, we are updating configurations by alternating between the two strategies θ_1, θ_2.

The configurations in this sequence satisfy $\pi^j \sqsubseteq \pi^{j+1}$, $\rho^j \sqsubseteq \rho^{j+1}$ for all j, where $\sqsubseteq$ denotes prefix ordering, hence the limits $\pi = \varinjlim \pi^j$, $\rho = \varinjlim \rho^j$ exist (as infinite paths). By our conditions on configurations, the pair (π, ρ) in turn determines a pair $(\bar{\pi}, \bar{\rho})$ of *paths* in S, as follows:

$$(\bar{\pi}_1, \bar{\rho}_1) = \begin{cases} (\pi_1, \rho_1) & \text{if } \mathit{src}(\pi_1) = s \\ (\rho_1, \pi_1) & \text{if } \mathit{src}(\pi_1) = t \end{cases}$$

$$(\bar{\pi}_j, \bar{\rho}_j) = \begin{cases} (\pi_j, \rho_j) & \text{if } \mathit{src}(\pi_j) = \mathit{tgt}(\bar{\pi}_{j-1}) \\ (\rho_j, \pi_j) & \text{if } \mathit{src}(\pi_j) = \mathit{tgt}(\bar{\rho}_{j-1}) \end{cases}$$

The *outcome* of the game when played from (s, t) according to a strategy pair (θ_1, θ_2) is defined to be $\mathsf{out}(\theta_1, \theta_2)(s, t) = (\bar{\pi}, \bar{\rho})$, and its *utility* is defined by $\mathsf{util}(\theta_1, \theta_2)(s, t) = d^T(\mathrm{tr}(\mathsf{out}(\theta_1, \theta_2)(s, t))) = d^T(\mathrm{tr}(\bar{\pi}), \mathrm{tr}(\bar{\rho}))$.

Recall that d^T is given as a parameter to the game; if we want to make explicit the parametrization on the trace distance d^T which utility depends on, we write $\mathsf{util}_{d^T}(\theta_1, \theta_2)(s, t)$.

Note that $\mathsf{util}(\theta_1, \theta_2)(s, t)$ is defined both in case the paths $\bar{\pi}$ and $\bar{\rho}$ are finite and in case they are infinite (the case where one is finite and the other is infinite cannot occur). Also, if the paths are finite because $\theta_1(\pi^j, \rho^j, m^j)$ was undefined for some configuration (π^j, ρ^j, m^j) in the sequence, then $\bar{\pi}$ and $\bar{\rho}$ have the same length; if on the other hand the reason is that $\theta_2(\pi^j, \rho^j, m^j)$ was undefined, then the paths have different length, and $\mathsf{util}(\theta_1, \theta_2)(s, t) = \infty$. Hence if the game reaches a configuration in which Player 2 has no moves available, the utility is ∞.

The objective of Player 1 in the game is to maximize utility, whereas Player 2 wants to minimize it. Hence we define the *value* of the game from (s, t) to be

$$v(s, t) = \sup_{\theta_1 \in \Theta_1} \inf_{\theta_2 \in \Theta_2} \mathsf{util}(\theta_1, \theta_2)(s, t)\,.$$

For a given subset $\Theta'_1 \subseteq \Theta_1$ we will write

$$v(\Theta'_1)(s,t) = \sup_{\theta_1 \in \Theta'_1} \inf_{\theta_2 \in \Theta_2} \mathsf{util}(\theta_1, \theta_2)(s,t)\,,$$

and if we need to emphasize dependency of the value on the given trace distance, we write $v(d^T, \Theta'_1)$. The following lemma states the immediate fact that if Player 1 has fewer strategies available, the game value decreases.

Lemma 1. *For all $\Theta'_1 \subseteq \Theta''_1 \subseteq \Theta_1$ and all $s,t \in S$, $v(\Theta'_1)(s,t) \leq v(\Theta''_1)(s,t)$.*

We introduce two technical conditions on strategies and on trace distances. We say that a strategy $\theta_1 \in \Theta_1$ is *uniform* if it holds for all configurations $(\pi, \rho, m), (\pi, \rho', m), (\pi', \rho, m) \in \mathsf{Conf}_1$ that whenever $\theta_1(\pi, \rho, m) = (e', m')$ is defined,

- if $src(e') = tgt(\pi)$, then also $\theta_1(\pi, \rho', m)$ is defined, and
- if $src(e') = tgt(\rho)$, then also $\theta_1(\pi', \rho, m)$ is defined.

Uniformity of strategies is used to combine paths built from different starting states in the proof of Proposition 3 below. A subset $\Theta'_1 \subseteq \Theta_1$ is uniform if all strategies in Θ'_1 are uniform; the concrete strategy subsets we will consider in later sections will all be uniform.

We say that a pair (Θ'_1, d^T) of a strategy subset $\Theta'_1 \subseteq \Theta_1$ and a trace distance is *well-behaved* if

$$\sup_{\theta_1 \in \Theta'_1} \inf_{\theta_2 \in \Theta_2} \mathsf{util}(\theta_1, \theta_2)(s,t) = \inf_{\theta_2 \in \Theta_2} \sup_{\theta_1 \in \Theta'_1} \mathsf{util}(\theta_1, \theta_2)(s,t)$$

for all $s, t \in S$. This assumption is related to determinacy of the quantitative path-building game, asserting that each pair of states *has* a value. We will need well-behavedness in our proof of Proposition 3 below; whether a pair (Θ'_1, d^T) is well-behaved depends on both Θ'_1 and d^T, and to consider any necessary or sufficient conditions for well-behavedness for concrete strategy subsets or concrete trace distances is beyond the scope of this paper.

However, we will later see that for the important special case where the trace distance d^T is given by a recursive characterization, *cf.* Section 7, another, simpler proof may be given for Proposition 3 which does not need well-behavedness.

Remark 1. The type of games we have defined here are *Blackwell games* as introduced in [1]; note however that we do not need randomized strategies, as our games are turn-based. In [28], it is shown that all *Borel* Blackwell games are determined, and that also for other aspects, the situation is similar to the one for infinite games with qualitative objectives [27]. In particular, determinacy for any interesting bigger class than Borel Blackwell games is not provable in Zermelo-Fränkel set theory including the axiom of choice.

5. General Properties

We show here that under certain conditions, the game value is indeed a distance, and that results concerning inequalities in the qualitative dimension can be transfered to topological inequivalences in the quantitative setting. Say that a Player-1 strategy $\theta_1 \in \Theta_1$ is *non-switching* if it holds for all (π, ρ, m) for which $\theta_1(\pi, \rho, m) = (e', m')$ is defined that $m = m'$, and let Θ_1^0 be the set of non-switching Player-1 strategies. We first show a lemma which shows that any pair of traces can be generated by a non-switching strategy:

Lemma 2. *For all $s, t \in S$ and all $\sigma \in \mathrm{Tr}(s)$, $\tau \in \mathrm{Tr}(t)$ there exist $\theta_1 \in \Theta_1^0$ and $\theta_2 \in \Theta_2$ for which $\mathsf{util}(\theta_1, \theta_2)(s, t) = d^T(\sigma, \tau)$.*

Proof. Let $(\pi, \rho, 0) \in \mathsf{Conf}_1$ for finite paths π, ρ with $\mathsf{len}(\pi) = \mathsf{len}(\rho) = k \geq 0$ and $\mathrm{tr}(\pi) = \sigma_0 \dots \sigma_{k-1}$, $\mathrm{tr}(\rho) = \tau_0 \dots \tau_{k-1}$. If $\mathsf{len}(\sigma) \geq k$, then there is $e = (\mathrm{last}(\pi), \sigma_k, s') \in T$, and we define $\theta_1(\pi, \rho, 0) = (e, 0)$. If also $\mathsf{len}(\tau) \geq k$, then there is $e' = (\mathrm{last}(\rho), \tau_k, t') \in T$, and we let $\theta_2(\pi \cdot e, \rho, 0) = (e', 0)$.

Let $(\bar\pi, \bar\rho) = \mathsf{out}(\theta_1, \theta_2)(s, t)$. If both σ and τ are infinite paths, then $\mathrm{tr}(\bar\pi) = \sigma$ and $\mathrm{tr}(\bar\rho) = \tau$; otherwise, $\mathrm{tr}(\bar\pi)$ and $\mathrm{tr}(\bar\rho)$ will be finite prefixes of σ and τ for which $d^T(\mathrm{tr}(\bar\pi), \mathrm{tr}(\bar\rho)) = d^T(\sigma, \tau)$. □

The following proposition shows under which conditions we can expect the distance defined by our quantitative game to be a hemimetric. Well-behavednes is used in the proof of the triangle inequality.

Proposition 3. *If $\Theta_1' \subseteq \Theta_1$ is uniform and $\Theta_1^0 \subseteq \Theta_1'$, and if (Θ_1', d^T) is well-behaved, then $v(\Theta_1')$ is a hemimetric on S.*

Proof. We write $v = v(\Theta_1')$ during this proof. It is clear that $v(s, s) = 0$ for all $s \in S$: if the players are making their choices from the same state, Player 2 can always answer by choosing exactly the same transition as Player 1. For proving the triangle inequality $v(s, u) \leq v(s, t) + v(t, u)$, let $\varepsilon > 0$ and use well-behavedness of d^T to choose Player-2 strategies $\theta_2^{s,t}, \theta_2^{t,u} \in \Theta_2$ for which

$$\begin{aligned} \sup_{\theta_1 \in \Theta_1'} \mathsf{util}(\theta_1, \theta_2^{s,t})(s, t) < v(s, t) + \tfrac{\varepsilon}{2}\,, \\ \sup_{\theta_1 \in \Theta_1'} \mathsf{util}(\theta_1, \theta_2^{t,u})(t, u) < v(t, u) + \tfrac{\varepsilon}{2}\,. \end{aligned} \tag{1}$$

We define a strategy $\theta_2^{s,u} \in \Theta_2$ which uses three paths and two configurations in S as extra memory. This is only for convenience, as these can be reconstructed by Player 2 at any time; hence we do not extend the capabilities of Player 2:

$$\theta_2^{s,u}(\pi\cdot e,\chi,m;\bar\pi,\bar\rho',\bar\chi,\pi',\rho_1',\rho_2',\chi') =$$
$$\begin{cases}
\Big(\theta_2^{t,u}\big(\rho_2'\cdot\theta_{2,1}^{s,t}(\pi'\cdot e,\rho_1',m),\chi',m\big); \\
\bar\pi\cdot e, \\
\bar\rho'\cdot\theta_{2,1}^{s,t}(\pi'\cdot e,\rho_1',m), \\
\bar\chi\cdot\theta_{2,1}^{t,u}\big(\rho_2'\cdot\theta_{2,1}^{s,t}(\pi'\cdot e,\rho_1',m)\big), \\
\pi'\cdot e, \\
\rho_1'\cdot\theta_{2,1}^{s,t}(\pi'\cdot e,\rho_1',m), \\
\rho_2'\cdot\theta_{2,1}^{s,t}(\pi'\cdot e,\rho_1',m), \\
\chi'\cdot\theta_{2,1}^{t,u}\big(\rho_2'\cdot\theta_{2,1}^{s,t}(\pi'\cdot e,\rho_1',m)\big)\Big) & \text{if } \mathit{src}(e)=\mathit{tgt}(\mathrm{last}(\bar\pi)), \\
\Big(\theta_2^{s,t}\big(\pi'\cdot\theta_{2,1}^{t,u}(\rho_2'\cdot e,\chi',m),\rho_1',m\big); \\
\bar\pi\cdot\theta_{2,1}^{s,t}\big(\pi'\cdot\theta_{2,1}^{t,u}(\rho_2'\cdot e,\chi',m)\big), \\
\bar\rho'\cdot\theta_{2,1}^{t,u}(\rho_2'\cdot e,\chi',m), \\
\bar\chi\cdot e, \\
\pi'\cdot\theta_{2,1}^{t,u}(\rho_2'\cdot e,\chi',m), \\
\rho_1'\cdot\theta_{2,1}^{s,t}\big(\pi'\cdot\theta_{2,1}^{t,u}(\rho_2'\cdot e,\chi',m)\big) \\
\rho_2'\cdot e, \\
\chi'\cdot\theta_{2,1}^{t,u}(\rho_2'\cdot e,\chi',m)\Big) & \text{if } \mathit{src}(e)=\mathit{tgt}(\mathrm{last}(\bar\chi)).
\end{cases}$$

In the beginning of the game, all memory paths are initialized to be empty.

In the expression above, the strategy $\theta_2^{s,u}$ is constructed from the strategies $\theta_2^{s,t}$ and $\theta_2^{t,u}$ by using the answer to the move of Player 1 in one of the games as an emulated Player-1 move in the other. The paths $\bar\pi$, $\bar\chi$ are constructed from the configuration (π,χ) of the (s,u)-game and are only kept in memory so that we can see whether Player 1 is playing an edge prolonging $\bar\pi$ or $\bar\chi$. The pair (π',ρ_1') is the configuration in the (s,t)-game we are emulating, and (ρ_2',χ') is the (t,u)-configuration. The path $\bar\rho'=\bar\rho_1'=\bar\rho_2'$ is common for the paths $(\bar\pi',\bar\rho_1')$, $(\bar\rho_2',\bar\chi')$ constructed from (π',ρ_1') and (ρ_2',χ').

If Player 1 has played an edge e prolonging $\bar\pi$ (first case above), we compute an answer move $(e',m)=\theta_2^{s,t}(\pi'\cdot e,\rho_1',m)$ to this in the (s,t)-game. This answer is then used to emulate a Player-1 move in the (t,u)-game, and the answer $\theta_2^{t,u}(\rho_2'\cdot e',\chi',m)$ to this is what Player 2 plays in the (s,u)-game. The memory is updated accordingly. If on the other hand, Player 1 has played an edge e prolonging $\bar\chi$, we play in the (t,u)-game first and use the answer $(e',m)=\theta_2^{t,u}(\rho_2'\cdot e,\chi',m)$ in the (s,t)-game to compute $\theta_2^{s,t}(\pi'\cdot e',\rho_1',m)$. Figure 2 gives an illustration of how the configurations are updated during the game; note that uniformity of Θ_1' is necessary for being able to emulate Player-1 moves from one game in another.

Take now any $\theta_1^{s,u}\in\Theta_1'$, let $(\bar\pi,\bar\chi)=\mathsf{out}(\theta_1^{s,u},\theta_2^{s,u})(s,u)$, and let $\bar\rho'$ be the corresponding memory path. By Lemma 2 there exist $\theta_1^{s,t},\theta_1^{t,u}\in\Theta_1'$ for which $d^T(\mathrm{tr}(\bar\pi),\mathrm{tr}(\bar\rho'))=\mathsf{util}(\theta_1^{s,t},\theta_2^{s,t})(s,t)$ and $d^T(\mathrm{tr}(\bar\rho'),\mathrm{tr}(\bar\chi))=\mathsf{util}(\theta_1^{t,u},\theta_2^{t,u})(t,u)$.

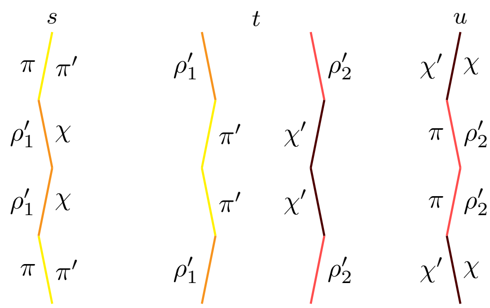

Figure 2: Configuration update in the game used for showing the triangle inequality

Using Equation (1) we have

$$\begin{aligned}\mathsf{util}(\theta_1^{s,u},\theta_2^{s,u})(s,u) &= d^T(\mathrm{tr}(\bar\pi),\mathrm{tr}(\bar\chi))\\ &\le d^T(\mathrm{tr}(\bar\pi),\mathrm{tr}(\bar\rho)) + d^T(\mathrm{tr}(\bar\rho),\mathrm{tr}(\bar\chi))\\ &< v(s,t)+v(t,u)+\varepsilon\end{aligned}$$

and hence also $\inf_{\theta_2\in\Theta_2}\mathsf{util}(\theta_1^{s,u},\theta_2)(s,u) < v(s,t)+v(t,u)+\varepsilon$. As the choice of $\theta_1^{s,u}$ was arbitrary, this implies

$$\sup_{\theta_1\in\Theta_1'}\inf_{\theta_2\in\Theta_2}\mathsf{util}(\theta_1,\theta_2)(s,u) \le v(s,t)+v(t,u)+\varepsilon\,,$$

and as also ε was chosen arbitrarily, we have $v(s,u)\le v(s,t)+v(t,u)$. □

Next we show a *transfer principle* which allows us to generalize counterexamples regarding the equivalences in the qualitative linear-time–branching-time spectrum [36] to the qualitative setting. We will make use of this principle later to show that all distances we introduce are topologically inequivalent.

Lemma 4. *Let $\Theta_1', \Theta_1'' \subseteq \Theta_1$, and assume (Θ_1', d^T) and (Θ_1'', d^T) to be well-behaved and d^T to be separating. If there exist states $s,t\in S$ for which $v(d^T_{\mathrm{disc}},\Theta_1')(s,t)=0$ and $v(d^T_{\mathrm{disc}},\Theta_1'')(s,t)=\infty$, then $v(d^T,\Theta_1')$ and $v(d^T,\Theta_1'')$ are topologically inequivalent.*

Proof. By $v(d^T_{\mathrm{disc}},\Theta_1')(s,t)=0$, we know that for any $\theta_1\in\Theta_1'$ there exists $\theta_2\in\Theta_2$ for which $(\bar\pi,\bar\rho)=\mathsf{out}(\theta_1,\theta_2)(s,t)$ satisfy $\mathrm{tr}(\bar\pi)=\mathrm{tr}(\bar\rho)$, hence also $v(d^T,\Theta_1')(s,t)=0$. Conversely, and as d^T is separating, $v(d^T,\Theta_1'')(s,t)=0$ would imply that also $v(d^T_{\mathrm{disc}},\Theta_1'')(s,t)=0$, hence we must have $v(d^T,\Theta_1'')(s,t)\ne 0$, entailing topological inequivalence. □

6. The Distance Spectrum

In this section we introduce the distances depicted in Figure 1 and show their relationship. Note again that the results obtained here are independent of the particular trace distance considered; in the terminology of the introduction we are developing a linear-time–branching-time spectrum at every point of the quantitative dimension. In order to capture the remaining relations in the original spectrum, we may easily adopt the approach from [4] which imposes one of three extra conditions which Player 1 may choose to invoke and thereby terminate the game.

Throughout this section, we fix a LTS $(S, T \subseteq S \times \mathbb{K} \times S)$ and a trace distance $d^T : \mathbb{K}^\infty \times \mathbb{K}^\infty \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

6.1. Branching Distances

If the switching counter in the game introduced in Section 4 is unbounded, Player 1 can choose at any move whether to prolong the previous choice or to switch paths, hence this resembles the bisimulation game [31].

Definition 5. The *bisimulation distance* between s and t is $d^{\text{bisim}}(s,t) = v(s,t)$.

Theorem 6. *For $d^T = d^T_{\text{disc}}$ the discrete trace distance, $d^{\text{bisim}}_{\text{disc}}(s,t) = 0$ if, and only if, s and t are bisimilar.*

Proof. By discreteness of d^T_{disc}, we have $d^{\text{bisim}}_{\text{disc}}(s,t) = 0$ if, and only if, it holds that for all $\theta_1 \in \Theta_1$ there exists $\theta_2 \in \Theta_2$ for which $\mathsf{util}(\theta_1, \theta_2)(s,t) = 0$. Hence for each reachable Player-1 configuration (π, ρ, m) with $\theta_1(\pi, \rho, m) = (e', m')$, we have $\theta_2(\pi \cdot e', \rho, m') = (e'', m')$ with $\text{tr}(e') = \text{tr}(e'')$, *i.e.* Player 2 matches the labels chosen by Player 1 precisely, implying that s and t are bisimilar. The proof of the other direction is trivial. $\square$

We can restrict the strategies available to Player 1 by allowing only a predefined finite number of switches:

$$\Theta_1^{k\text{-sim}} = \{\theta_1 \in \Theta_1 \mid \text{if } \theta_1(\pi, \rho, m) = (e', m') \text{ is defined, then } m' \leq k-1\}$$

In the so-defined k-nested simulation game, Player 1 is only allowed to switch paths $k-1$ times during the game. Note that $\Theta_1^{1\text{-sim}} = \Theta_1^0$ is the set of non-switching strategies.

Definition 7. The *k-nested simulation distance* from s to t, for $k \in \mathbb{N}_+$, is $d^{k\text{-sim}}(s,t) = v(\Theta_1^{k\text{-sim}})(s,t)$. The *$k$-nested simulation equivalence distance* between s and t is $d^{k\text{-sim-eq}}(s,t) = \max(v(\Theta_1^{k\text{-sim}})(s,t), v(\Theta_1^{k\text{-sim}})(t,s))$.

Theorem 8. *For $d^T = d^T_{\text{disc}}$ the discrete trace distance,*

- *$d^{k\text{-sim}}_{\text{disc}}(s,t) = 0$ if, and only if, there is a k-nested simulation from s to t,*
- *$d^{k\text{-sim-eq}}_{\text{disc}}(s,t) = 0$ if, and only if, there is a k-nested simulation equivalence between s and t.*

Especially, $d^{1\text{-sim}}_{\text{disc}}$ corresponds to the usual *simulation* preorder, and $d^{2\text{-sim}}_{\text{disc}}$ to *nested simulation*. Similarly, $d^{1\text{-sim-eq}}_{\text{disc}}$ is *similarity*, and $d^{2\text{-sim-eq}}_{\text{disc}}$ is *nested similarity*. We refer to [19, 21] for definitions and discussion of nested and k-nested simulations.

Proof. This is similar to the proof of Theorem 6: If $d_{\text{disc}}^{k\text{-sim}}(s,t)=0$, then any $\theta_1\in\Theta_1^{k\text{-sim}}$ has a counter-strategy $\theta_2\in\Theta_2$ which matches the labels chosen by Player 1 precisely, implying k-nested simulation from s to t. The other direction is again trivial. □

Theorem 9. *For all $k,\ell\in\mathbb{N}_+$ with $k<\ell$ and all $s,t\in S$,*

$$d^{k\text{-sim-eq}}(s,t)\le d^{\ell\text{-sim}}(s,t)\le d^{\ell\text{-sim-eq}}(s,t)\le d^{\text{bisim}}(s,t).$$

If the trace distance d^T is separating and, together with the involved strategy subsets, well-behaved, then all distances above are topologically inequivalent.

Proof. The first part of the theorem follows from $\Theta_1^{k\text{-sim-eq}}\subseteq\Theta_1^{\ell\text{-sim}}\subseteq\Theta_1^{\ell\text{-sim-eq}}\subseteq\Theta_1$ and Lemma 1. Topological inequivalence follows from Lemma 4 and the fact that for the discrete relations corresponding to the distances above (obtained by letting $d^T=d_{\text{disc}}^T$), the inequalities are strict [36]. □

As a variation of k-nested simulation, we can consider strategies which allow Player 1 to switch paths k times during the game, but at the last switch, he may only pose *one* transition as a challenge, to which Player 2 must answer, and then the game finishes:

$$\Theta_1^{k\text{-rsim}}=\{\theta_1\in\Theta_1\mid \text{if } \theta_1(\pi,\rho,m) \text{ is defined, then } m\le k-1\}$$

Hence after his kth switch, Player 1 has no more moves available, and the game finishes after the answer move of Player 2. Again, we allow Player 1 to increase the switch counter without actually switching paths.

Definition 10. The *k-nested ready simulation distance* from s to t, for $k\in\mathbb{N}_+$, is $d^{k\text{-rsim}}(s,t)=v(\Theta_1^{k\text{-rsim}})(s,t)$. The *$k$-nested ready simulation equivalence distance* between s and t is $d^{k\text{-rsim-eq}}(s,t)=\max(v(\Theta_1^{k\text{-rsim}})(s,t),v(\Theta_1^{k\text{-rsim}})(t,s))$.

For the discrete case, it seems only the case $k=1$ has been considered; the proof is similar to the one of Theorem 6.

Theorem 11. *For $d^T=d_{\text{disc}}^T$ the discrete trace distance,*

- $d_{\text{disc}}^{1\text{-rsim}}(s,t)=0$ *if, and only if, there is a ready simulation from s to t,*
- $d_{\text{disc}}^{1\text{-rsim-eq}}(s,t)=0$ *if, and only if, s and t are ready simulation equivalent.*

The next theorem finishes our work on the right half of Figure 1.

Theorem 12. *For all $k,\ell\in\mathbb{N}_+$ with $k<\ell$ and all $s,t\in S$,*

$$d^{k\text{-sim}}(s,t)\le d^{k\text{-rsim}}(s,t)\le d^{\ell\text{-sim}}(s,t)\,,$$
$$d^{k\text{-sim-eq}}(s,t)\le d^{k\text{-rsim-eq}}(s,t)\le d^{\ell\text{-sim-eq}}(s,t)\,.$$

Additionally, $d^{k\text{-rsim}}$ and $d^{k\text{-sim-eq}}$ are incomparable, and also $d^{k\text{-rsim-eq}}$ and $d^{(k+1)\text{-sim}}$ are incomparable. If the trace distance d^T is separating and, together with the involved strategy subsets, well-behaved, then all distances above are topologically inequivalent.

Proof. Like in the proof of Theorem 9, the inequalities follow from strategy set inclusions and topological inequivalence from Lemma 4. The incomparability results follow from the corresponding results for d^T_{disc} and Lemma 4. $\square$

6.2. Linear Distances

Above we have introduced the distances in the right half of the quantitative linear-time–branching-time spectrum in Figure 1 and shown the relations claimed in the diagram. To develop the left half, we need the notion of *blind* strategies. For any subset $\Theta'_1 \subseteq \Theta_1$ we define the set of blind Θ'_1-strategies by

$$\begin{aligned}\tilde{\Theta}'_1 = \{\theta_1 \in \Theta'_1 \mid \forall \pi, \rho, \rho', m : \theta_1(\pi, \rho, m) = \theta_1(\pi, \rho', m), \\ \text{or } \theta_1(\pi, \rho, m) = (e, m+1) \text{ and } \mathit{tgt}(\text{last}(\rho)) \neq \mathit{tgt}(\text{last}(\rho'))\}.\end{aligned}$$

Hence in such a blind strategy, either the edge chosen by Player 1 does not depend on the choices of Player 2, or the switch counter is increased, in which case the Player-1 choice only depends on the target of the last choice of Player 2 (note that this dependency is necessary if Player 1 wants to switch paths).

Now we can define, for $s, t \in S$ and $k \in \mathbb{N}_+$,

- the ∞*-nested trace equivalence distance*: $d^{\infty\text{-trace-eq}}(s,t) = v(\tilde{\Theta}_1)(s,t)$,
- the *k-nested trace distance*: $d^{k\text{-trace}}(s,t) = v(\tilde{\Theta}_1^{k\text{-sim}})(s,t)$,
- the *k-nested trace equivalence distance*:
$$d^{k\text{-trace-eq}}(s,t) = \max(v(\tilde{\Theta}_1^{k\text{-sim}})(s,t), v(\tilde{\Theta}_1^{k\text{-sim}})(t,s)),$$
- the *k-nested ready distance*: $d^{k\text{-ready}}(s,t) = v(\tilde{\Theta}_1^{k\text{-rsim}})(s,t)$, and
- the *k-nested ready equivalence distance*:
$$d^{k\text{-ready-eq}}(s,t) = \max(v(\tilde{\Theta}_1^{k\text{-rsim}})(s,t), v(\tilde{\Theta}_1^{k\text{-rsim}})(t,s)).$$

Our approach is justified by the following lemma which shows that the (1-nested) trace distance from s to t is precisely the Hausdorff distance between the sets of traces available from s and t, respectively.

Lemma 13. *For $s, t \in S$, $d^{1\text{-trace}}(s,t) = \sup_{\sigma \in \text{Tr}(s)} \inf_{\tau \in \text{Tr}(t)} d^T(\sigma, \tau)$.*

Proof. We have $d^{1\text{-trace}}(s,t) = v(\tilde{\Theta}_1^0)(s,t)$, with $\tilde{\Theta}_1^0 = \{\theta_1 \in \Theta_1^0 \mid \forall \pi, \rho, \rho', m : \theta_1(\pi, \rho, m) = \theta_1(\pi, \rho', m)\}$. Hence, and as strategies in Θ_1^0 are non-switching, every strategy $\theta_1 \in \tilde{\Theta}_1^0$ gives rise to precisely one trace $\sigma = \sigma(\theta_1) \in \text{Tr}(s)$ independently of Player-2 strategy $\theta_2 \in \Theta_2$. Conversely, by Lemma 2 (noticing that indeed, we have constructed a blind Player-1 strategy in the proof of that lemma), every trace $\sigma \in \text{Tr}(s)$ is generated by a strategy $\theta_1 \in \tilde{\Theta}_1^0$ with $\sigma = \sigma(\theta_1)$.

We can finish the proof by showing that for all $\theta_1 \in \tilde{\Theta}_1^0$,

$$\inf_{\theta_2 \in \Theta_2} d^T(\sigma(\theta_1), \text{tr}(\bar{\rho}(\theta_1, \theta_2))) = \inf_{\tau \in \text{Tr}(t)} d^T(\sigma(\theta_1), \tau)\,.$$

But again using Lemma 2, we see that any $\tau \in \text{Tr}(t)$ is generated by a strategy $\theta_2 \in \Theta_2$, hence this is clear. $\square$

Using the discrete trace distance, we recover the following standard relations [36]. The theorem follows by Lemma 13 and arguments similar to the ones used in the proofs of the corresponding theorems in the preceding section. We refer to [21, 29] for definitions and discussion of possible-futures inclusion and equivalence.

Theorem 14. *For $d^T = d^T_{\text{disc}}$ the discrete trace distance and $s, t \in S$ we have*

- $d^{\text{1-trace}}_{\text{disc}}(s,t) = 0$ *if, and only if, there is a trace inclusion from s to t,*
- $d^{\text{1-trace-eq}}_{\text{disc}}(s,t) = 0$ *if, and only if, s and t are trace equivalent,*
- $d^{\text{2-trace}}_{\text{disc}}(s,t) = 0$ *if, and only if, there is a possible-futures inclusion from s to t,*
- $d^{\text{2-trace-eq}}_{\text{disc}}(s,t) = 0$ *if, and only if, s and t are possible-futures equivalent,*
- $d^{\text{1-ready}}_{\text{disc}}(s,t) = 0$ *if, and only if, there is a readiness inclusion from s to t,*
- $d^{\text{1-ready-eq}}_{\text{disc}}(s,t) = 0$ *if, and only if, s and t are ready equivalent.*

The following theorem entails all relations in the left side of Figure 1; the right-to-left arrows follow from the strategy set inclusions $\tilde{\Theta}'_1 \subseteq \Theta'_1$ for any $\Theta'_1 \subseteq \Theta_1$ and Lemma 1. As with Theorems 9 and 12, the theorem follows by strategy set inclusion, Lemma 4, and corresponding results for the discrete relations.

Theorem 15. *For all $k, \ell \in \mathbb{N}_+$ with $k < \ell$ and $s, t \in S$,*

$$\begin{aligned}
&d^{k\text{-trace-eq}}(s,t) \le d^{\ell\text{-trace}}(s,t) \le d^{\ell\text{-trace-eq}}(s,t) \le d^{\infty\text{-trace-eq}}(s,t),\\
&d^{k\text{-trace}}(s,t) \le d^{k\text{-ready}}(s,t) \le d^{\ell\text{-trace}}(s,t),\\
&d^{k\text{-trace-eq}}(s,t) \le d^{k\text{-ready-eq}}(s,t) \le d^{\ell\text{-trace-eq}}(s,t).
\end{aligned}$$

Additionally, $d^{k\text{-ready}}$ and $d^{k\text{-trace-eq}}$ are incomparable, and also $d^{k\text{-ready-eq}}$ and $d^{(k+1)\text{-trace}}$ are incomparable. If the trace distance d^T is separating and, together with the involved strategy subsets, well-behaved, then all distances above are topologically inequivalent.

7. Recursive Characterizations

We now turn our attention to an important special case in which the given trace distance has a specific recursive characterization; we show that in this case, all distances in the spectrum can be characterized as least fixed points. We will see in Section 8 that this can be applied to all examples of trace distances mentioned in Section 3.

Note that all theorems require the LTS in question to be finitely branching; this is a standard assumption which goes back to [31]. In most cases it may be relaxed to *compact branching* in the sense of [34], but to keep things simple, we do not do this here.

Let L be a complete lattice with order $\sqsubseteq$ and bottom and top elements $\bot$, $\top$. Let $f : \mathbb{K}^\infty \times \mathbb{K}^\infty \to L$, $g : L \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, $F : \mathbb{K} \times \mathbb{K} \times L \to L$ such that $d^T = g \circ f$, g is monotone, $F(x, y, \cdot) : L \to L$ is monotone for all $x, y \in \mathbb{K}$, and

$$f(\sigma,\tau) = \begin{cases} F(\sigma_0, \tau_0, f(\sigma^1, \tau^1)) & \text{if } \sigma, \tau \neq \boldsymbol{\epsilon}, \\ \top & \text{if } \sigma = \boldsymbol{\epsilon}, \tau \neq \boldsymbol{\epsilon} \text{ or } \sigma \neq \boldsymbol{\epsilon}, \tau = \boldsymbol{\epsilon}, \\ \bot & \text{if } \sigma = \tau = \boldsymbol{\epsilon} \end{cases} \tag{2}$$

for all $\sigma, \tau \in \mathbb{K}^\infty$.

We hence assume that d^T has a recursive characterization (using F) on top of an arbitrary lattice L which we introduce between $\mathbb{K}^\infty$ and $\mathbb{R}_{\geq 0} \cup \{\infty\}$ to serve as a *memory*. Below we will work with different endofunctions I on the set of mappings $(\mathbb{N}_+ \cup \{\infty\}) \times \{1, 2\} \to L^{S \times S}$ which are parametrized by the number m of switches in $\mathbb{N}_+ \cup \{\infty\}$ which Player 1 has left, and a value $p \in \{1, 2\}$ which keeps track of whether Player 1 currently is building the left or the right path.

Theorem 16. *The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1, 2\} \to L^{S \times S}$ defined by*

$$I(h_{m,p})(s,t) = \begin{cases} \max \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x, y, h_{m,1}(s', t')) \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x, y, h_{m-1,2}(s', t')) \end{cases} & \textit{if } m \geq 2, p = 1 \\ \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x, y, h_{m,1}(s', t')) & \textit{if } m = 1, p = 1 \\ \max \begin{cases} \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x, y, h_{m,2}(s', t')) \\ \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x, y, h_{m-1,1}(s', t')) \end{cases} & \textit{if } m \geq 2, p = 2 \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{st \xrightarrow{x} s'} F(x, y, h_{m,2}(s', t')) & \textit{if } m = 1, p = 2 \end{cases}$$

has a least fixed point $h^* : (\mathbb{N}_+ \cup \{\infty\}) \times \{1, 2\} \to L^{S \times S}$, and if the LTS (S, T) is finitely branching, then $d^{k\text{-sim}} = g \circ h^*_{k,1}$, $d^{k\text{-sim-eq}} = g \circ \max(h^*_{k,1}, h^*_{k,2})$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$.

Hence I iterates the function h over the branching structure of (S, T), computing all nested branching distances at the same time. Note the specialization of this to simulation and bisimulation distance, where we have the following fixed-point equations, using $h^*_{1,1} = h^{1\text{-sim}}$ and $h^*_{\infty,1} = h^{\text{bisim}}$:

$$h^{1\text{-sim}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x, y, h^{1\text{-sim}}(s', t'))$$

$$h^{\text{bisim}}(s,t) = \max \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x, y, h^{\text{bisim}}(s', t')) \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x, y, h^{\text{bisim}}(s', t')) \end{cases}$$

Proof. The lattice of mappings $(\mathbb{N}_+\cup\{\infty\})\times\{1,2\}\to L^{S\times S}$ with the point-wise partial order is complete, and I is monotone because F is, so by Tarski's fixed-point theorem, I has indeed a least fixed point h^*. To show that $d^{k\text{-sim}} = g\circ h^*_{k,1}$ for all k, we pull back $d^{k\text{-sim}}$ along g: Define $w : (\mathbb{N}_+\cup\{\infty\})\times\{1,2\}\to L^{S\times S}$ by

$$\begin{aligned}
w_{k,1}(s,t) &= \sup_{\theta_1\in\Theta_1^{k\text{-sim}}} \inf_{\theta_2\in\Theta_2} f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(s,t)))\\
w_{k,2}(s,t) &= \sup_{\theta_1\in\Theta_1^{k\text{-sim}}} \inf_{\theta_2\in\Theta_2} f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(t,s)))
\end{aligned}$$

then $d^{k\text{-sim}} = g\circ f(k,1)$ for all k by monotonicity of g. We will be done once we can show that $w = h^*$.

We first show that w is a fixed point for I. Let $s,t\in S$, then (assuming $k\geq 2$)

$$\begin{aligned}
I(w_{k,1})(s,t) &= \max\begin{cases}\sup_{s\xrightarrow{x}s'}\inf_{t\xrightarrow{y}t'} F(x,y,w_{k,1}(s',t'))\\ \sup_{t\xrightarrow{y}t'}\inf_{s\xrightarrow{x}s'} F(x,y,w_{k-1,2}(s',t'))\end{cases}\\
&= \max\begin{cases}\sup_{s\xrightarrow{x}s'}\inf_{t\xrightarrow{y}t'} F(x,y,\sup_{\theta_1\in\Theta_1^{k\text{-sim}}}\inf_{\theta_2\in\Theta_2} f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(s',t'))))\\ \sup_{t\xrightarrow{y}t'}\inf_{s\xrightarrow{x}s'} F(x,y,\sup_{\theta_1\in\Theta_1^{(k-1)\text{-sim}}}\inf_{\theta_2\in\Theta_2} f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(t',s'))))\end{cases}\\
&= \max\begin{cases}\sup_{s\xrightarrow{x}s'}\inf_{t\xrightarrow{y}t'}\sup_{\theta_1\in\Theta_1^{k\text{-sim}}}\inf_{\theta_2\in\Theta_2} F(x,y,f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(s',t'))))\\ \sup_{t\xrightarrow{y}t'}\inf_{s\xrightarrow{x}s'}\sup_{\theta_1\in\Theta_1^{(k-1)\text{-sim}}}\inf_{\theta_2\in\Theta_2} F(x,y,f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(t',s'))))\end{cases}\\
&= \max\begin{cases}\sup_{s\xrightarrow{x}s'}\inf_{t\xrightarrow{y}t'}\sup_{\theta_1\in\Theta_1^{k\text{-sim}}}\inf_{\theta_2\in\Theta_2}\\ \qquad f(x\cdot\mathrm{tr}(\mathsf{out}_1(\theta_1,\theta_2)(s',t')),y\cdot\mathrm{tr}(\mathsf{out}_2(\theta_1,\theta_2)(s',t')))\\ \sup_{t\xrightarrow{y}t'}\inf_{s\xrightarrow{x}s'}\sup_{\theta_1\in\Theta_1^{(k-1)\text{-sim}}}\inf_{\theta_2\in\Theta_2}\\ \qquad f(x\cdot\mathrm{tr}(\mathsf{out}_1(\theta_1,\theta_2)(t',s')),y\cdot\mathrm{tr}(\mathsf{out}_2(\theta_1,\theta_2)(t',s')))\,,\end{cases}
\end{aligned}$$

the next-to-last step by monotonicity of F. Now the choices of $t\xrightarrow{y}t'$ and $\theta_1\in\Theta_1^{k\text{-sim}}$ do not depend on each other, so the corresponding inf and sup can be exchanged, whence

$$I(w_{k,1})(s,t) = \max \begin{cases} \sup_{s \xrightarrow{x} s'} \sup_{\theta_1 \in \Theta_1^{k\text{-sim}}} \inf_{t \xrightarrow{y} t'} \inf_{\theta_2 \in \Theta_2} \\ \qquad f(x \cdot \mathrm{tr}(\mathsf{out}_1(\theta_1,\theta_2)(s',t')), y \cdot \mathrm{tr}(\mathsf{out}_2(\theta_1,\theta_2)(s',t'))) \\ \sup_{t \xrightarrow{y} t'} \sup_{\theta_1 \in \Theta_1^{(k-1)\text{-sim}}} \inf_{s \xrightarrow{x} s'} \inf_{\theta_2 \in \Theta_2} \\ \qquad f(x \cdot \mathrm{tr}(\mathsf{out}_1(\theta_1,\theta_2)(t',s')), y \cdot \mathrm{tr}(\mathsf{out}_2(\theta_1,\theta_2)(t',s'))) \end{cases}$$

$$= \max \begin{cases} \sup_{\theta_1 \in \Theta_{1,\mathrm{ns}}^{k\text{-sim}}} \inf_{\theta_2 \in \Theta_2} f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(s,t))) \\ \sup_{\theta_1 \in \Theta_{1,\mathrm{s}}^{k\text{-sim}}} \inf_{\theta_2 \in \Theta_2} f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(s,t))) \end{cases}$$

$$= w_{k,1}(s,t).$$

In the last max expression, $\Theta_{1,\mathrm{ns}}^{k\text{-sim}} \subseteq \Theta_1^{k\text{-sim}}$ is the subset of Player-1 strategies θ_1 which do not switch from the configuration $(s,t,0)$, *i.e.* for which $src(\theta_{1,1}(s,t,0)) = s$, and $\Theta_{1,\mathrm{s}}^{k\text{-sim}} = \Theta_1^{k\text{-sim}} \setminus \Theta_{1,\mathrm{ns}}^{k\text{-sim}}$ consists of the strategies which do switch from $(s,t,0)$. The other cases in the definition of I — $I(w_{1,1})$, $I(w_{1,2})$, and $I(w_{k,2})$ for $k \geq 2$ — can be shown similarly, and we can conclude that $I(w_{k,p}) = w_{k,p}$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$, $p \in \{1,2\}$.

To show that w is the least fixed point for I, let $\bar{h} : (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S \times S}$ be such that $I(\bar{h}) = \bar{h}$. We prove that $w \leq \bar{h}$, and again we show only the case $w_{k,1} \leq \bar{h}_{k,1}$ for $k \geq 2$. Note first that as the LTS (S,T) is finitely branching, we can use the equation for $I(\bar{h}_{k,1})(s,t)$ to conclude that for all $s,t \in S$,

$$\text{for any } s \xrightarrow{x} s' \text{ there is } t \xrightarrow{y} t' \text{ such that } F(x,y,\bar{h}_{k,1}(s',t')) \leq I(\bar{h}_{k,1})(s,t), \tag{3}$$

$$\text{for any } t \xrightarrow{y} t' \text{ there is } s \xrightarrow{x} s' \text{ such that } F(x,y,\bar{h}_{k-1,2}(s',t')) \leq I(\bar{h}_{k,1})(s,t). \tag{4}$$

Now let $\theta_1 \in \Theta_1^{k\text{-sim}}$; the proof will be finished once we can find $\theta_2 \in \Theta_2$ for which $f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(s,t))) \leq \bar{h}_{k,1}(s,t)$. Let $(\pi \cdot e, \rho, m) \in \mathsf{Conf}_2$ and write $s = tgt(\mathrm{last}(\pi))$, $t = tgt(\mathrm{last}(\rho))$. Assume first that $e = (s,x,s')$, let $t = tgt(\mathrm{last}(\rho))$ and $e = (t,y,t')$ an edge which satisfies the inequality of Equation (3), and define $\theta_2(\pi \cdot e, \rho, m) = (e', m)$. For the so-defined Player-2 strategy θ_2 we have $f(\mathrm{tr}(\mathsf{out}(\theta_1,\theta_2)(s,t))) \leq \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x,y,\bar{h}_{k,1}(s',t')) \leq I(\bar{h}_{k,1})(s,t) = \bar{h}_{k,1}(s,t)$ for all $s,t \in S$. The case $e = (t,y,t')$ is shown similarly, using Equation (4) instead. $\square$

The fixed-point characterization for the ready simulation distances is similar (and so is its proof, which we hence omit):

Theorem 17. *The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S\times S}$ defined by*

$$I(h_{m,p})(s,t) = \begin{cases} \max \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x,y,h_{m,1}(s',t')) \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x,y,h_{m-1,2}(s',t')) \end{cases} & \textit{if } m \geq 2, p = 1 \\ \max \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x,y,h_{m,1}(s',t')) \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} f(x,y) \end{cases} & \textit{if } m = 1, p = 1 \\ \max \begin{cases} \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} F(x,y,h_{m,2}(s',t')) \\ \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} F(x,y,h_{m-1,1}(s',t')) \end{cases} & \textit{if } m \geq 2, p = 2 \\ \max \begin{cases} \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{st \xrightarrow{x} s'} F(x,y,h_{m,2}(s',t')) \\ \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} f(x,y) \end{cases} & \textit{if } m = 1, p = 2 \end{cases}$$

has a least fixed point $h^* : (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S\times S}$, and if the LTS (S,T) is finitely branching, then $d^{k\text{-rsim}} = g \circ h^*_{k,1}$, $d^{k\text{-rsim-eq}} = g \circ \max(h^*_{k,1}, h^*_{k,2})$ for all $k \in \mathbb{N}_+ \cup \{\infty\}$.

For the *linear* distances, we extend F to a function $\mathbb{K}^n \times \mathbb{K}^n \times L \to L$, for $n \in \mathbb{N}$, by

$$F(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}, \alpha) = \alpha, \qquad F(x \cdot \sigma, y \cdot \tau, \alpha) = F(x, y, F(\sigma, \tau, \alpha)).$$

We also extend the $\xrightarrow{x}$ relation to finite traces so we can write $s \xrightarrow{\sigma} s'$ below, by letting $s \xrightarrow{\boldsymbol{\epsilon}} s$ for all $s \in S$ and $s \xrightarrow{x\cdot\sigma} s'$ if, and only if, $s \xrightarrow{x} s'' \xrightarrow{\sigma} s'$ for some $s'' \in S$. We write $s \xrightarrow{\sigma}$ if there is a (finite or infinite) trace σ from s. The proofs of the below theorems are similar to the one of Theorem 16.

Theorem 18. *The endofunction I on $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S\times S}$ defined by*

$$I(h_{m,p})(s,t) = \begin{cases} \max \begin{cases} \sup\limits_{s \xrightarrow{\sigma}} \inf\limits_{t \xrightarrow{\tau}} f(\sigma,\tau) \\ \sup\limits_{s \xrightarrow{\sigma} s'} \inf\limits_{t \xrightarrow{\tau} t'} F(\sigma,\tau,h_{m-1,1}(s',t')) \\ \sup\limits_{s \xrightarrow{\sigma} s'} \inf\limits_{t \xrightarrow{\tau} t'} F(\sigma,\tau,h_{m-1,2}(s',t')) \end{cases} & \textit{if } m \geq 2, p = 1 \\ \sup\limits_{s \xrightarrow{\sigma}} \inf\limits_{t \xrightarrow{\tau}} f(\sigma,\tau) & \textit{if } m = 1, p = 1 \\ \max \begin{cases} \sup\limits_{t \xrightarrow{\tau}} \inf\limits_{s \xrightarrow{\sigma}} f(\sigma,\tau) \\ \sup\limits_{t \xrightarrow{\tau} t'} \inf\limits_{s \xrightarrow{\sigma} s'} F(\sigma,\tau,h_{m-1,2}(s',t')) \\ \sup\limits_{t \xrightarrow{\tau} t'} \inf\limits_{s \xrightarrow{\sigma} s'} F(\sigma,\tau,h_{m-1,1}(s',t')) \end{cases} & \textit{if } m \geq 2, p = 2 \\ \sup\limits_{t \xrightarrow{\tau}} \inf\limits_{s \xrightarrow{\sigma}} f(\sigma,\tau) & \textit{if } m = 1, p = 2 \end{cases}$$

has a least fixed point $h^* : (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S\times S}$*, and if the LTS* (S,T) *is finitely branching, then* $d^{k\text{-trace}} = g \circ h^*_{k,1}$*,* $d^{k\text{-trace-eq}} = g \circ \max(h^*_{k,1}, h^*_{k,2})$ *for all* $k \in \mathbb{N}_+ \cup \{\infty\}$*.*

Theorem 19. *The endofunction* I *on* $(\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S\times S}$ *defined by*

$$I(h_{m,p})(s,t) = \begin{cases} \max\begin{cases} \sup\limits_{s\xrightarrow{\sigma}} \inf\limits_{t\xrightarrow{\tau}} f(\sigma,\tau) \\ \sup\limits_{s\xrightarrow{\sigma}s'} \inf\limits_{t\xrightarrow{\tau}t'} F(\sigma,\tau,h_{m-1,1}(s',t')) \\ \sup\limits_{s\xrightarrow{\sigma}s'} \inf\limits_{t\xrightarrow{\tau}t'} F(\sigma,\tau,h_{m-1,2}(s',t')) \end{cases} & \text{if } m \geq 2, p = 1 \\ \max\begin{cases} \sup\limits_{s\xrightarrow{\sigma}} \inf\limits_{t\xrightarrow{\tau}} f(\sigma,\tau) \\ \sup\limits_{s\xrightarrow{\sigma}s'} \inf\limits_{t\xrightarrow{\tau}t'} \sup\limits_{s'\xrightarrow{x}s''} \inf\limits_{t'\xrightarrow{y}t''} f(\sigma\cdot x,\tau\cdot y) \\ \sup\limits_{s\xrightarrow{\sigma}s'} \inf\limits_{t\xrightarrow{\tau}t'} \sup\limits_{t'\xrightarrow{y}t''} \inf\limits_{s'\xrightarrow{x}s''} f(\sigma\cdot x,\tau\cdot y) \end{cases} & \text{if } m = 1, p = 1 \\ \max\begin{cases} \sup\limits_{t\xrightarrow{\tau}} \inf\limits_{s\xrightarrow{\sigma}} f(\sigma,\tau) \\ \sup\limits_{t\xrightarrow{\tau}t'} \inf\limits_{s\xrightarrow{\sigma}s'} F(\sigma,\tau,h_{m-1,2}(s',t')) \\ \sup\limits_{t\xrightarrow{\tau}t'} \inf\limits_{s\xrightarrow{\sigma}s'} F(\sigma,\tau,h_{m-1,1}(s',t')) \end{cases} & \text{if } m \geq 2, p = 2 \\ \max\begin{cases} \sup\limits_{t\xrightarrow{\tau}} \inf\limits_{s\xrightarrow{\sigma}} f(\sigma,\tau) \\ \sup\limits_{t\xrightarrow{\tau}t'} \inf\limits_{s\xrightarrow{\sigma}s'} \sup\limits_{t'\xrightarrow{y}t''} \inf\limits_{s'\xrightarrow{x}s''} f(\sigma\cdot x,\tau\cdot y) \\ \sup\limits_{t\xrightarrow{\tau}t'} \inf\limits_{s\xrightarrow{\sigma}s'} \sup\limits_{s'\xrightarrow{x}s''} \inf\limits_{t'\xrightarrow{y}t''} f(\sigma\cdot x,\tau\cdot y) \end{cases} & \text{if } m = 1, p = 2 \end{cases}$$

has a least fixed point $h^* : (\mathbb{N}_+ \cup \{\infty\}) \times \{1,2\} \to L^{S\times S}$*, and if the LTS* (S,T) *is finitely branching, then* $d^{k\text{-ready}} = g \circ h^*_{k,1}$*,* $d^{k\text{-ready-eq}} = g \circ \max(h^*_{k,1}, h^*_{k,2})$ *for all* $k \in \mathbb{N}_+ \cup \{\infty\}$*.*

The fixed-point characterizations above have two important consequences. For the first, Proposition 3 easily follows by induction for distances with a fixed-point characterization, which means that the condition of well-behavedness is not needed to show the triangle inequality.

For the second, the fixed-point characterization immediately lead to iterative *semi-algorithms* for computing the respective distances: to compute *e.g.* simulation distance, we can initialize $h^{1\text{-sim}}(s,t) = 0$ for all states $s, t \in S$ and then iteratively apply the above equality. This assumes the LTS (S,T) to be finitely branching and uses Kleene's fixed-point theorem and continuity of F. However, this computation is only guaranteed to converge to simulation distance in finitely many steps in case the lattice $L^{S\times S}$ is *finite*; otherwise, the procedure might not terminate.

7.2. Relation Families

Below we show that both simulation and bisimulation distance admit a relational characterization akin to the one of the standard Boolean notions. Using switching counters like we did in the previous section, this can easily be generalized to give relational characterizations to all distances in this paper.

Theorem 20. *If the LTS (S, T) is finitely branching, then $d^{\text{1-sim}}(s,t) \le \varepsilon$ if, and only if, there exists a relation family $R = \{R_\alpha \subseteq S \times S \mid \alpha \in L\}$ for which $(s,t) \in R_\beta \in R$ for some β with $g(\beta) \le \varepsilon$, and such that for any $\alpha \in L$ and for all $(s', t') \in R_\alpha \in R$,*

- *for all $s' \xrightarrow{x} s''$, there exists $t' \xrightarrow{y} t''$ such that $(s'', t'') \in R_{\alpha'} \in R$ for some $\alpha' \in L$ with $F(x, y, \alpha') \sqsubseteq \alpha$.*

Similarly, $d^{\text{bisim}}(s,t) \le \varepsilon$ if, and only if, there exists a relation family $R = \{R_\alpha \subseteq S \times S \mid \alpha \in L\}$ for which $(s,t) \in R_\beta \in R$ for some β with $g(\beta) \le \varepsilon$, and such that for any $\alpha \in L$ and for all $(s', t') \in R_\alpha \in R$,

- *for all $s' \xrightarrow{x} s''$, there exists $t' \xrightarrow{y} t''$ such that $(s'', t'') \in R_{\alpha'} \in R$ for some $\alpha' \in L$ with $F(x, y, \alpha') \sqsubseteq \alpha$;*
- *for all $t' \xrightarrow{y} t''$, there exists $s' \xrightarrow{x} s''$ such that $(s'', t'') \in R_{\alpha'} \in R$ for some $\alpha' \in L$ with $F(x, y, \alpha') \sqsubseteq \alpha$.*

Proof. We only show the proof for simulation distance; for bisimulation distance it is analogous. Assume first that $d^{\text{1-sim}}(s,t) \le \varepsilon$, then we have $h : S \times S \to L$ for which $g(h(s,t)) \le \varepsilon$ and $h(s', t') = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x, y, h(s'', t''))$ for all $s', t' \in S$. Let $\beta = h(s,t)$, and define a relation family $R = \{R_\alpha \mid \alpha \in L\}$ by $R_\alpha = \{(s', t') \mid h(s', t') \sqsubseteq \alpha\}$. Let $\alpha \in L$ and $(s', t') \in R_\alpha$, then $\sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} F(x, y, h(s'', t'')) = h(s', t') \sqsubseteq \alpha$, and as (S, T) is finitely branching, this implies that for all $s' \xrightarrow{x} s''$ there is $t' \xrightarrow{y} t''$ and $\alpha' = h(s'', t'')$ such that $(s'', t'') \in R_{\alpha'}$ and $F(x, y, \alpha') \sqsubseteq \alpha$.

For the other direction, assume a relation family as in the theorem and define $h : S \times S \to L$ by $h(s', t') = \inf\{\alpha \mid (s', t' \in R_\alpha\}$. Then $(s,t) \in R_\beta$ implies that $h(s,t) \sqsubseteq \beta$ and hence $g(h(s,t)) \le \varepsilon$. Let $s', t' \in S$, then $(s', t') \in R_{h(s',t')}$, hence for all $s' \xrightarrow{x} s''$ there is $t' \xrightarrow{y} t''$ and $\alpha' \in L$ for which $F(x, y, \alpha') \sqsubseteq h(s', t')$ and $(s'', t'') \in R_{\alpha'}$, implying $h(s'', t'') \sqsubseteq \alpha'$ and hence $F(x, y, h(s'', t'')) \sqsubseteq h(s', t')$. Collecting the pieces, we get $I(h)(s', t') = \sup_{s' \xrightarrow{x} s''} \inf_{t' \xrightarrow{y} t''} F(x, y, h(s'', t'')) \sqsubseteq h(s', t')$, hence h is a pre-fixed point for I. But then $h^* \sqsubseteq h$, hence $d^{\text{1-sim}}(s,t) = g(h^*(s,t)) \le g(h(s,t)) \le \varepsilon$. □

8. Recursive Characterizations for Example Distances

We show that the considerations in Section 7 apply to all the example distances we have introduced in Section 3. We apply Theorem 16 to derive fixed-point formulas for corresponding simulation distances, but of course all other distances in the quantitative linear-time–branching-time spectrum have similar characterizations.

Let d be a hemimetric on $\mathbb{K}$, then for all $\sigma, \tau \in \mathbb{K}^\infty$ and $0 < \lambda \le 1$,

$$\mathsf{PW}_\lambda(d)(\sigma,\tau) = \begin{cases} \max(d(\sigma_0,\tau_0), \lambda\, \mathsf{PW}_\lambda(d)(\sigma^1,\tau^1)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \sigma,\tau \ne \boldsymbol{\epsilon}, \\ \infty \qquad\qquad\qquad\qquad\qquad\quad \text{if } \sigma = \boldsymbol{\epsilon}, \tau \ne \boldsymbol{\epsilon} \text{ or } \sigma \ne \boldsymbol{\epsilon}, \tau = \boldsymbol{\epsilon}, \\ 0 \qquad\qquad\qquad\qquad\qquad\quad \text{if } \sigma = \tau = \boldsymbol{\epsilon}, \end{cases}$$

$$\mathsf{ACC}_\lambda(d)(\sigma,\tau) = \begin{cases} d(\sigma_0,\tau_0) + \lambda\, \mathsf{ACC}_\lambda(d)(\sigma^1,\tau^1)) \\ \qquad\qquad\qquad\qquad\qquad\quad \text{if } \sigma,\tau \ne \boldsymbol{\epsilon}, \\ \infty \qquad\qquad\qquad\qquad\quad \text{if } \sigma = \boldsymbol{\epsilon}, \tau \ne \boldsymbol{\epsilon} \text{ or } \sigma \ne \boldsymbol{\epsilon}, \tau = \boldsymbol{\epsilon}, \\ 0 \qquad\qquad\qquad\qquad\quad \text{if } \sigma = \tau = \boldsymbol{\epsilon}, \end{cases}$$

hence we can apply the iteration theorems with lattice $L = \mathbb{R}_{\ge 0} \cup \{\infty\}$, $g = \mathrm{id}$ the identity function, and the recursion function F given like the formulas above. Using Theorem 16 we can *e.g.* derive the following fixed-point expressions for simulation distance:

$$\mathsf{PW}_\lambda(d)^{\text{1-sim}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} \max(d(x,y), \lambda\, \mathsf{PW}_\lambda(d)^{\text{1-sim}}(s',t'))$$
$$\mathsf{ACC}_\lambda(d)^{\text{1-sim}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} (d(x,y) + \lambda\, \mathsf{ACC}_\lambda(d)^{\text{1-sim}}(s',t'))$$

Incidentally, these are exactly the expressions introduced in [5, 11, 33].

Also note that if S is finite with $|S| = n$, then undiscounted point-wise distance $\mathsf{PW}_1(d)$ can only take on the finitely many values $\{d(x,y) \mid (s,x,s'), (t,y,t') \in T\}$, hence the fixed-point algorithm given by Kleene's theorem converges in at most n^2 steps. This algorithm is used in [5, 7, 24]. For undiscounted accumulating distance $\mathsf{ACC}_1(d)$, it can be shown [24] that with $D = \max\{d(x,y) \mid (s,x,s'), (t,y,t') \in T\}$, distance is either infinite or bounded above by $2n^2 D$, hence the $\mathsf{ACC}_1(d)$ algorithm either converges in at most $2n^2 D$ steps or diverges.

For the limit-average distance $\mathsf{AVG}(d)$, we let $L = (\mathbb{R}_{\ge 0} \cup \{\infty\})^{\mathbb{N}}$, $g(h) = \liminf_j h(j)$, and $f(\sigma,\tau)(j) = \frac{1}{j+1} \sum_{i=0}^{j} d(\sigma_i,\tau_i)$ the j-th average. The intuition is that L is used for "remembering" how long in the traces we have progressed with the computation. With F given by $F(x,y,h)(n) = \frac{1}{n+1} d(x,y) + \frac{n}{n+1} h(n-1)$ it can be shown that (2) holds, giving the following fixed-point expression for limit-average simulation distance (which to the best of our knowledge is new):

$$h_n^{\text{1-sim}}(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} \left(\tfrac{1}{n+1} d(x,y) + \tfrac{n}{n+1} h_{n-1}^{\text{1-sim}}(s',t') \right)$$

For the maximum-lead distance, we let $L = (\mathbb{R}_{\ge 0} \cup \{\infty\})^{\mathbb{R}}$, the lattice of mappings from leads to maximum leads. Using the notation from Section 3, we let $g(h) = h(0)$ and $f(\sigma,\tau)(\delta) = \max(|\delta|, \sup_j |\delta + \sum_{i=0}^{j} \sigma_i^w - \sum_{i=0}^{j} \tau_j^w|)$ the maximum-lead distance between σ and τ assuming that σ already has a lead of

δ over τ. With $F(x, y, h)(\delta) = \max(|\delta + x - y|, h(\delta + x - y))$ it can be shown that (2) holds, and then the fixed-point expression for maximum-lead simulation distance becomes the one given in [22]:

$$h^{\text{1-sim}}(\delta)(s,t) = \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{y} t'} \max(|\delta + x - y|, h^{\text{1-sim}}(s',t')(\delta + x - y))$$

Again it can be shown [22] that for S finite with $|S| = n$ and $D = \max\{d(x,y) \mid (s,x,s'), (t,y,t') \in T\}$, the iterative algorithm for computing maximum-lead distance either converges in at most $2n^2D$ steps or diverges.

Regarding Cantor distance, a useful recursive formulation is

$$f(\sigma,\tau)(n) = \begin{cases} f(\sigma^1,\tau^1)(n+1) & \text{if } \sigma_0 = \tau_0, \\ n & \text{otherwise,} \end{cases}$$

which iteratively counts the number of matching symbols in σ and τ. Here we use $L = (\mathbb{R}_{\geq 0} \cup \{\infty\})^{\mathbb{N}}$, and $g(h) = \frac{1}{h(0)}$; note that the order on L has to be reversed for g to be monotone. The fixed-point expression for Cantor simulation distance becomes

$$h_n^{\text{1-sim}}(s,t) = \max(n, \sup_{s \xrightarrow{x} s'} \inf_{t \xrightarrow{x} t'} h_{n+1}^{\text{1-sim}}(s',t'))$$

but as the order on L is reversed, the sup now means that Player 1 is trying to *minimize* this expression, and Player 2 tries to maximize it. Hence Player 2 tries to find maximal matching *subtrees*; the corresponding Cantor simulation equivalence distance between s and t hence is the inverse of the maximum depth of matching subtrees under s and t. The Cantor bisimulation distance in turn is the same as the inverse of *bisimulation depth*.

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