Metrics for weighted transition systems: Axiomatization and complexity

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Abstract

Simulation distances are essentially an approximation of simulation which provide a measure of the extent by which behaviors in systems are inequivalent. In this paper, we consider the general quantitative model of weighted transition systems, where transitions are labeled with elements of a finite metric space. We study the so-called point-wise and accumulating simulation distances which provide extensions to the well-know Boolean notion of simulation on labeled transition systems.

We introduce weighted process algebras for finite and regular behavior and offer sound and (approximate) complete inference systems for the proposed simulation distances. We also settle the algorithmic complexity of computing the simulation distances.

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1. Introduction

The need for an extension of the state-of-the art modeling and verification techniques to encompass systems with quantitative information has long been recognized; see [10] for a recent position paper on this subject. Classical modeling formalisms for concurrent and reactive systems have focused on describing *qualitative* aspects of systems with a range of behavioral equivalences and preorders used for the so-called *implementation verification*, see *e.g.* the survey provided in [17]. This approach requires a model of the systems and specifications, as well as a procedure for checking whether the two are related with respect to the given equivalence or preorder.

During more than a decade, classical modeling formalisms have been extended with *quantitative* aspects such as real time, probabilistic or continuous (so-called hybrid) information. Despite successful generalization of several behavioral preorders and equivalences they largely remain qualitative, *e.g.* two (quantitative) system models either are, or are not, equivalent. To properly take account of robustness, it is advocated in [10] that in the quantitative setting, equivalences and preorders are replaced by real-valued

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distances: *i.e.* from deciding on the Boolean truth of equivalence $P \sim Q$ between two models P and Q, the problem becomes that of computing their distance $|P, Q| = \epsilon \in \mathbb{R}_{\geq 0}$. It is argued that exact behavioral equivalence for quantitative models is unrealistic – as it typically requires exact matching of all quantitative aspects – whereas in practical application matching up to some error margin given by the distance ϵ suffices.

During the last years, substantial progress has been made towards defining suitable metrics or distances for various types of quantitative models including probabilistic models [6], real-time systems [9] and metrics for linear and branching systems in general [2, 4, 5, 7, 8, 11, 13, 15, 16].

In this paper, which is the third in a series of papers on general system distances [7, 15], we consider the general quantitative model of weighted transition systems where transitions are labeled with elements from a finite metric space $\mathbb{K}$. We consider two different distances on states of such transition systems, *point-wise* and *accumulating* simulation distance, and provide sound and complete axiomatizations for these distances on weighted process algebras, akin to the aximoatization of bisimulation for finite and regular process algebra in Milner's seminal paper [14]. Note that the maximum-lead distance from [9, 15] is not treated here; we leave this for future work.

We also consider the algorithmic complexity of computing point-wise and accumulating simulation distance for finite-state weighted transition systems. Whereas point-wise simulation distance is shown decidable in polynomial time – similar to that of ordinary simulation and bisimulation as shown by Smolka and Kanellakis [12] – we show that the problem of accumulating simulation distance is polynomial-time equivalent to that of computing payoff for discounted games and hence in $\mathtt{NP} \cap \mathtt{coNP}$.

2. Weighted Transition Systems

We need to fix some terminology and notation which we will use heavily: A mapping $d : X \times X \to [0, \infty] = \mathbb{R}_{\geq 0} \cup \{\infty\}$ from a set X to the non-negative reals together with positive infinity is called a *hemimetric* if $d(x, x) = 0$ for all $x \in X$ and $d(x, y) + d(y, z) \geq d(x, z)$ (the *triangle inequality*) for all $x, y, z \in X$; it is called a *metric* if additionally, $d(x, y) > 0$ for all $x \neq y \in X$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

A sequence (x_j) in a metric space X is a *Cauchy sequence* if it holds that for all $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $n, m \geq N$. X is said to be *complete* if every Cauchy sequence in X converges in X.

A continuous function $f : X \to X$ is called a *contraction* if there exists $0 \leq \alpha < 1$ (its *Lipschitz constant*) such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. Finally, we recall the Banach fixed-point theorem: Any contraction on a complete metric space has a unique fixed point.

Throughout this article we fix a *finite* metric space $\mathbb{K}$ of *weights* with a metric $d_{\mathbb{K}} : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$. We also fix a *discounting factor* λ with $0 \le \lambda < 1$, which will be used in the definition of accumulating distance below.

Definition 1. *A* weighted transition system *is a tuple* (S, T)*, where* S *is a finite set of* states *and* $T \subseteq S \times \mathbb{K} \times S$ *is a set of (weighted) transitions.*

Note that all transition systems in this paper are indeed assumed finite, hence requiring finiteness of the metric space $\mathbb{K}$ does not add extra restrictions.

3. Simulation distances

In this section we fix a weighted transition system (S, T) and introduce simulation distance between states in (S, T). We concentrate on two types here, accumulating and point-wise distance, but other kinds may indeed be defined.

3.1. Accumulating distance

Definition 2. *For states* $s, t \in S$*,* accumulating simulation distance *from* s *to* t *is defined to be the least fixed point to the set of equations*

$$\wr s, t \int = \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \big(d_{\mathbb{K}}(n, m) + \lambda \wr s', t' \int \big) \tag{1}$$

To justify this definition, we need to show that the equations (1) indeed have a least solution. To this end, write $S = \{s_1, \dots, s_p\}$ and assume for the moment that the transition system (S, T) is *non-blocking* such that every $s_i \in S$ has an outgoing transition $s_i \xrightarrow{n} s_k$ for some $s_k \in S$. Define a function $F : \mathbb{R}_{\ge 0}^{p \times p} \to \mathbb{R}_{\ge 0}^{p \times p}$ by

$$F(x)_{i,j} = \max_{s_i \xrightarrow{n} s_k} \min_{s_j \xrightarrow{m} s_\ell} \big(d_{\mathbb{K}}(n, m) + \lambda x_{k,\ell} \big)$$

Here we are using the standard linear-algebra notation $\mathbb{R}_{\ge 0}^{p \times p}$ for $p \times p$-matrices with entries in $\mathbb{R}_{\ge 0}$ and $x_{k,\ell}$ for the entry in their k'th row and ℓ'th column.

Lemma 3. *With metric on* $\mathbb{R}_{\ge 0}^{p \times p}$ *defined by* $d(x, y) = \max_{i,j=1}^{p} |x_{i,j} - y_{i,j}|$*,* F *is a contraction with Lipschitz constant* λ*.*

Proof. (*Cf.* also the proof of Theorem 5.1 in [18].) We can partition $\mathbb{R}_{\ge 0}^{p \times p}$ into finitely many (indeed at most $2^{p^2 q^2}$ with $q = |\mathbb{K}|$) closed polyhedral regions $R_{i,j}$ (some of which may be unbounded) such that for $x, y \in R_{i,j}$ in a common region, the p^2 max-min equations get resolved to the same transitions. In more precise terms, there are mappings $n, m, k, \ell : \{1, \dots, p\} \times \{1, \dots, p\} \to$

$\{1, \ldots, p\}$ such that $F(x)_{i,j} = d_{\mathbb{K}}(n(i,j), m(i,j)) + \lambda x_{k(i,j),\ell(i,j)}$ for all $x \in R_{i,j}$.

Now if $x, y \in R_{i,j}$ are in a common region, then

$$\begin{aligned} d(F(x), F(y)) &\le \lambda \max_{i,j} |x_{k(i,j),\ell(i,j)} - y_{k(i,j),\ell(i,j)}| \\ &\le \lambda \max_{i,j} |x_{i,j} - y_{i,j}| = \lambda d(x,y) \end{aligned}$$

If $x \in R_{i_1,j_1}, y \in R_{i_2,j_2}$ are in different regions, a bit more work is needed. The straight line segment between x and y admits finitely many intersection points with the regions $R_{i,j}$; denote these $x = z_0, \ldots, z_q = y$. We have

$$\begin{aligned} d(F(x), F(y)) &\le d(F(z_0), F(z_1)) + \cdots + d(F(z_{q-1}, z_q)) \\ &\le \lambda\big(d(z_0, z_1) + \cdots + d(z_{q-1}, z_q)\big) = \lambda d(x,y) \end{aligned}$$

Note that the last equality only holds because all z_i are on a straight line.□

Using the Banach fixed-point theorem and completeness of $\mathbb{R}_{\ge 0}^{p\times p}$ we can hence conclude that F has a unique fixed point. In the general case, where (S,T) may not be non-blocking, F is a function $[0,\infty] \to [0,\infty]$ with (extra) fixed point $[\infty, \ldots, \infty]$. Hence as a function $[0,\infty] \to [0,\infty]$, F has at most *two* fixed points. Now we can write the equation set from the definition as

$$\begin{bmatrix} \wr s_1, s_1 \int & \wr s_1, s_2 \int & \cdots & \wr s_1, s_p \int \\ \wr s_2, s_1 \int & \wr s_2, s_2 \int & \cdots & \wr s_2, s_p \int \\ \vdots & \vdots & \ddots & \vdots \\ \wr s_p, s_1 \int & \wr s_p, s_2 \int & \cdots & \wr s_p, s_p \int \end{bmatrix} = F \begin{bmatrix} \wr s_1, s_1 \int & \wr s_1, s_2 \int & \cdots & \wr s_1, s_p \int \\ \wr s_2, s_1 \int & \wr s_2, s_2 \int & \cdots & \wr s_2, s_p \int \\ \vdots & \vdots & \ddots & \vdots \\ \wr s_p, s_1 \int & \wr s_p, s_2 \int & \cdots & \wr s_p, s_p \int \end{bmatrix}$$

hence (1) has indeed a unique least fixed point.

3.2. Point-wise distance

For point-wise simulation distance we follow a lattice-theoretic rather than a contraction approach.

Definition 4. *For states $s, t \in S$,* point-wise simulation distance *from s to t is defined to be the least fixed point to the set of equations*

$$\wr s, t \int_\bullet = \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \max\big(d_{\mathbb{K}}(n,m), \wr s', t' \int_\bullet\big)$$

Let $G : [0,\infty]^{p\times p} \to [0,\infty]^{p\times p}$ be the function defined by

$$G(x)_{i,j} = \max_{s_i \xrightarrow{n} s_k} \min_{s_j \xrightarrow{m} s_\ell} \max\big(d_{\mathbb{K}}(n,m), x_{k,\ell}\big)$$

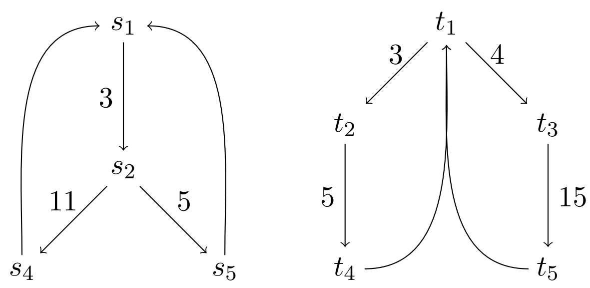

Figure 1: Example WTS

Lemma 5. *With partial order on $[0, \infty]^{p \times p}$ defined by $x \leq y$ iff $x_{i,j} \leq y_{i,j}$ for all i, j, G is (weakly) increasing.*

Proof. Trivial. $\square$

Now the Tarski fixed-point theorem allows us to conclude that G has a unique least fixed point, hence the above definition is justified.

3.3. Example

We show a computation of the two simulation distances between states s_1 and t_1 in the weighted transition system in Figure 1. Here $\mathbb{K} \subseteq \mathbb{N}$, $d(n, m) = |n - m|$, edges without specified weight have weight 0, and the discount factor is $\lambda = .90$.

Repeated application of the definition yields the following fixed-point equation for $\wr s_1, t_1 \int$ (note that there is only one transition from s_1, t_2 and t_3, respectively):

$$\begin{aligned}
\wr s_1, t_1 \int &= \min \begin{cases} |3 - 3| + .90 \wr s_2, t_2 \int \\ |3 - 4| + .90 \wr s_2, t_3 \int \end{cases} \\
&= \min \begin{cases} .90 \max \begin{cases} |11 - 5| + .90 \wr s_4, t_4 \int \\ |5 - 5| + .90 \wr s_5, t_4 \int \end{cases} \\ 1 + .90 \max \begin{cases} |11 - 15| + .90 \wr s_4, t_5 \int \\ |5 - 15| + .90 \wr s_5, t_5 \int \end{cases} \end{cases} \\
&= \min \begin{cases} .90 \max \begin{cases} 6 + .90^2 \wr s_1, t_1 \int \\ .90^2 \wr s_1, t_1 \int \end{cases} \\ 1 + .90 \max \begin{cases} 4 + .90^2 \wr s_1, t_1 \int \\ 10 + .90^2 \wr s_1, t_1 \int \end{cases} \end{cases} \\
&= \min \big(.90 \big(6 + .90^2 \wr s_1, t_1 \int \big), 1 + .90 \big(10 + .90^2 \wr s_1, t_1 \int \big) \big) \\
&= 5.4 + .90^3 \wr s_1, t_1 \int
\end{aligned}$$

Hence $\wr s_1, t_1 \int \approx 19.9$. For the pointwise distance $\wr s_1, t_1 \int_\bullet$ we have accordingly,

$$\begin{aligned}
\wr s_1, t_1 \int_\bullet &= \min \begin{cases} \max\left(|3-3|, \wr s_2, t_2 \int_\bullet\right) \\ \max\left(|3-4|, \wr s_2, t_3 \int_\bullet\right) \end{cases} \\
&= \min \begin{cases} \max\left(0, |11-5|, \wr s_4, t_4 \int_\bullet, |5-5|, \wr s_5, t_4 \int_\bullet\right) \\ \max\left(1, |11-15|, \wr s_4, t_5 \int_\bullet, |5-15|, \wr s_5, t_5 \int_\bullet\right) \end{cases} \\
&= \min \begin{cases} \max\left(6, \wr s_1, t_1 \int_\bullet\right) \\ \max\left(10, \wr s_1, t_1 \int_\bullet\right) \end{cases}
\end{aligned}$$

which has least fixed point $\wr s_1, t_1 \int_\bullet = 6$.

3.4. Properties

Proposition 6. $\wr \cdot, \cdot \int$ *and* $\wr \cdot, \cdot \int_\bullet$ *are hemimetrics on* S.

Proof. To show that $\wr s, s \int = \wr s, s \int_\bullet = 0$ is trivial. The triangle inequalities can be shown inductively; we prove the one for $\wr \cdot, \cdot \int$: For $s, t, u \in S$, we have

$$\begin{aligned}
\wr s, t \int + \wr t, u \int &= \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \left(d_{\mathbb{K}}(n, m) + \lambda \wr s', t' \int\right) \\
&\quad + \max_{t \xrightarrow{m} t'} \min_{u \xrightarrow{z} u'} \left(d_{\mathbb{K}}(m, z) + \lambda \wr t', u' \int\right) \\
&\geq \max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} \min_{u \xrightarrow{z} u'} \left(d_{\mathbb{K}}(n, m) + d_{\mathbb{K}}(m, z) + \lambda\left(\wr s', t' \int + \wr t', u' \int\right)\right) \\
&\geq \max_{s \xrightarrow{n} s'} \min_{u \xrightarrow{z} u'} \left(d_{\mathbb{K}}(n, z) + \lambda \wr s', u' \int\right) = \wr s, u \int
\end{aligned}$$

assuming the triangle inequality has been proven for the triple (s', t', u'). $\square$

In the next proposition we take the standard liberty of comparing different (weighted) transition systems by considering their disjoint union.

Proposition 7. *The weighted transition systems* $\mathbb{O} = (\{s_1\}, \emptyset)$ *and* $\mathbf{U} = (\{s_1\}, \{(s_1, n, s_1) \mid n \in \mathbb{K}\})$ *are respectively minimal and maximal elements with respect to both* $\wr \cdot, \cdot \int$ *and* $\wr \cdot, \cdot \int_\bullet$*, that is,* $\wr \mathbb{O}, A \int = \wr \mathbb{O}, A \int_\bullet = \wr A, \mathbf{U} \int = \wr A, \mathbf{U} \int_\bullet = 0$ *for any WTS* A.

Proof. For $\wr \mathbb{O}, A \int$ and $\wr \mathbb{O}, A \int_\bullet$, the maximum $\max_{s_1 \xrightarrow{n} s_1'}$ is taken over the empty set and hence is 0. For $\wr A, \mathbf{U} \int$ and $\wr A, \mathbf{U} \int_\bullet$, any transition $s \xrightarrow{n} s'$ in A can be matched by $s_1 \xrightarrow{n} s_1$ in $\mathbf{U}$, hence the distance is again 0. $\square$

Our distances are related to the usual notion of simulation in the following way. Recall [15] that a relation $R \subseteq S \times S$ is called a

- *weighted simulation* if whenever $(s,t) \in R$ and $s \xrightarrow{n} s'$, then also $t \xrightarrow{n} t'$ with $(s',t') \in R$;
- *unweighted simulation* if whenever $(s,t) \in R$ and $s \xrightarrow{n} s'$, then also $t \xrightarrow{m} t'$ with $(s',t') \in R$ and $d_{\mathbb{K}}(n,m) < \infty$.

State s is *weighted simulated*, denoted $s \le t$, respectively *unweighted simulated*, denoted $s \le_u t$, if there exists a weighted simulation relation, respectively an unweighted simulation relation, R with $(s,t) \in R$.

Proposition 8. *For s, t states in a WTS (S,T),*

- *$s \le t$ implies $\wr s,t \int = \wr s,t \int_\bullet = 0$,*
- *$\wr s,t \int = 0$ or $\wr s,t \int_\bullet = 0$ imply $s \le t$,*
- *$s \le_u t$ implies that $\wr s,t \int < \infty$ and $\wr s,t \int_\bullet < \infty$, and*
- *$\wr s,t \int < \infty$ or $\wr s,t \int_\bullet < \infty$ imply $s \le_u t$.*

Proof. For the first claim, if $(s,t) \in R$ for some weighted simulation relation R, then any $s \xrightarrow{n} s'$ can be matched by a transition $t \xrightarrow{n} t'$ with $(s',t') \in R$, hence $\wr s,t \int = 0 + \lambda \wr s',t' \int$ and $\wr s,t \int_\bullet = \max\big(0, \wr s',t' \int_\bullet\big)$, and we can proceed by induction. Conversely, if $\wr s,t \int = 0$, then $\max_{s \xrightarrow{n} s'} \min_{t \xrightarrow{m} t'} d(n,m) + \lambda \wr s',t' \int = 0$, hence $d(n,m) = 0$, and as d is a metric, $n = m$, and we can proceed by induction. The situation if $\wr s,t \int_\bullet = 0$ is similar.

For the third claim, if $(s,t) \in R$ for some unweighted simulation relation R, then any $s \xrightarrow{n} s'$ can be matched by a transition $t \xrightarrow{m} t'$ with $(s',t') \in R$, and as $d(n,m) < \infty$, we can again proceed by induction. Conversely, $\wr s,t \int < \infty$ or $\wr s,t \int_\bullet < \infty$ imply that any transition $s \xrightarrow{n} s'$ has a match $t \xrightarrow{m} t'$. □

4. Axiomatizations for Finite Weighted Processes

We now turn to a setting where our weighted transition systems are generated by finite or regular (weighted) process expressions. We construct a sound and complete axiomatization of simulation distance in a setting without recursion first and show afterwards how this may be extended to a setting with recursion.

Let $\mathcal{P}$ be the set of process expressions generated by the following grammar:

$$E ::= \mathbb{O} \mid n.E \mid E + E \mid \qquad n \in \mathbb{K}$$

Here $\mathbb{O}$ is used to denote the *empty process*, *cf.* Proposition 7.

The semantics of finite process expressions is a weighted transition system generated by the following standard SOS rules:

$$\frac{}{n.E \xrightarrow{n} E} \qquad \frac{E_1 \xrightarrow{n} E_1'}{E_1 + E_2 \xrightarrow{n} E_1'} \qquad \frac{E_2 \xrightarrow{n} E_2'}{E_1 + E_2 \xrightarrow{n} E_2'}$$

We can immediately get the following equalities

$$\begin{aligned} \wr E + \mathbb{O}, E \int &= 0 \\ \wr n.E, m.F \int &= d_{\mathbb{K}}(n,m) + \lambda \wr E, F \int && (2) \\ \wr E_1 + E_2, F \int &= \max(\wr E_1, F \int, \wr E_2, F \int) \\ \wr n.E, F_1 + F_2 \int &= \min(\wr n.E, F_1 \int, \wr n.E, F_2 \int) && (3) \end{aligned}$$

For the point-wise distance, we again need only exchange (2) with

$$\wr n.E, m.F \int_\bullet = \max(d_{\mathbb{K}}(n,m), \wr E, F \int_\bullet)$$

In order to see *e.g.* Equation (3) we simply need to apply the definitions:

$$\begin{aligned} \wr n.E, F_1 + F_2 \int &= \inf_{F_1+F_2 \xrightarrow{m} F'} d_{\mathbb{K}}(n,m) + \lambda \wr E, F' \int \\ &= \min\Big(\inf_{F_1 \xrightarrow{m} F'} d_{\mathbb{K}}(n,m) + \lambda \wr E, F' \int, \inf_{F_2 \xrightarrow{m} F'} d_{\mathbb{K}}(n,m) + \lambda \wr E, F' \int \Big) \\ &= \min\big(\wr n.E, F_1 \int, \wr n.E, F_2 \int \big) \end{aligned}$$

For Equation (2), the sup-inf expression ranges over singleton sets, hence the result is easy; the remaining equalities may shown in a similar way.

The inference system F as given in Figure 2 axiomatizes accumulating simulation distance for finite processes, as we shall prove below. Its sentences are inequalities of the form $[E, F] \bowtie r$ where $\bowtie \in \{=, \leq, \geq\}$ and $0 \leq r \leq \infty$. Whenever $[E, F] \bowtie r$ may be concluded from F, we write $\vdash_F [E, F] \bowtie r$.

In addition to reflexivity and transitivity, we will need the following standard properties of $\bowtie$ in latter proofs of soundness and completeness: Whenever $a \bowtie b$ then, for all c: $a+c \bowtie b+c$, $a \cdot c \bowtie b \cdot c$, and $\max\{a, c\} \bowtie \max\{b, c\}$.

We also remark that the left process indeed needs to be guarded in rule (R3) above, *i.e.* the following proposed rule (R3′) leads to an unsound inference system:

$$(\mathrm{R3}') \ \frac{\vdash [E, F_1] \bowtie r_1 \qquad \vdash [E, F_2] \bowtie r_2}{\vdash [E, F_1 + F_2] \bowtie r} \quad \min(r_1, r_2) \bowtie r$$

Indeed, using this rule we can derive the following (incomplete) proof tree with a contradictory conclusion; the reason behind is that with $E = E_1 + E_2$ non-deterministic as below, both F_1 and F_2 may be needed to answer the challenge posed by E:

$$
\text{(A1)}\ \frac{}{\vdash [\mathbb{O}, E] \bowtie r}\quad 0 \bowtie r \qquad\qquad \text{(A2)}\ \frac{}{\vdash [n.E, \mathbb{O}] \bowtie r}\quad \infty \bowtie r
$$

$$
\text{(R1)}\ \frac{\vdash [E, F] \bowtie r_1}{\vdash [n.E, m.F] \bowtie r}\quad d_{\mathbb{K}}(n, m) + \lambda r_1 \bowtie r
$$

$$
\text{(R2)}\ \frac{\vdash [E_1, F] \bowtie r_1 \qquad \vdash [E_2, F] \bowtie r_2}{\vdash [E_1 + E_2, F] \bowtie r}\quad \max(r_1, r_2) \bowtie r
$$

$$
\text{(R3)}\ \frac{\vdash [n.E, F_1] \bowtie r_1 \qquad \vdash [n.E, F_2] \bowtie r_2}{\vdash [n.E, F_1 + F_2] \bowtie r}\quad \min(r_1, r_2) \bowtie r
$$

Figure 2: The F proof system.

$$
\dfrac{\dfrac{\vdash [1.\mathbb{O}, 1.\mathbb{O}] \geq 0 \qquad \vdash [2.\mathbb{O}, 1.\mathbb{O}] \geq 1}{\vdash [1.\mathbb{O} + 2.\mathbb{O}, 1.\mathbb{O}] \geq 1} \qquad \dfrac{\vdash [1.\mathbb{O}, 2.\mathbb{O}] \geq 1 \qquad \vdash [2.\mathbb{O}, 2.\mathbb{O}] \geq 0}{\vdash [1.\mathbb{O} + 2.\mathbb{O}, 2.\mathbb{O}] \geq 1}}{\vdash [1.\mathbb{O} + 2.\mathbb{O}, 1.\mathbb{O} + 2.\mathbb{O}] \geq 1}
$$

Theorem 9 (Soundness). *If $\vdash_F [E, F] \bowtie r$, then $\wr E, F \int \bowtie r$.*

Proof. By an easy induction in the proof tree for $\vdash_F [E, F] \bowtie r$, with a case analysis for the applied proof rule:

(A1) follows from $\wr \mathbb{O}, E \int = 0$.

(A2) follows from $\wr n.E, \mathbb{O} \int = \infty$ which is clear by the definition of $\wr \cdot, \cdot \int$.

(R1) By induction hypothesis, $\wr E, F \int \bowtie r_1$, and as $\wr n.E, m.F \int = d_{\mathbb{K}}(n, m) + \lambda \wr E, F \int$, it follows that $\wr n.E, m.F \int = d_{\mathbb{K}}(n, m) + \lambda r_1$.

(R2) By induction hypothesis, $\wr E_1, F \int \bowtie r_1$ and $\wr E_2, F \int \bowtie r_2$, hence $\wr E_1 + E_2, F \int = \max(\wr E_1, F \int, \max\{E_2, F\}) \bowtie \max(r_1, r_2)$.

(R3) By induction hypothesis, $\wr n.E, F_1 \int \bowtie r_1$ and $\wr n.E, F_2 \int \bowtie r_2$, hence $\wr n.E, F_1 + F_2 \int = \min(\wr n.E, F_1 \int, \wr n.E, F_2 \int) \bowtie \min(r_1, r_2)$. □

Theorem 10 (Completeness). *If $\wr E, F \int \bowtie r$, then $\vdash_F [E, F] \bowtie r$.*

Proof. By an easy structural induction on E:

($E = \mathbb{O}$) We have $\wr \mathbb{O}, F \int = 0 \bowtie r$. By Axiom (A1), also $\vdash [\mathbb{O}, F] = 0$.

($E = n.E'$) We use an inner induction on F:

Subcase $F = \mathbb{O}$. Here $\wr E, F \int = \wr n.E', \mathbb{O} \int = \infty \bowtie r$. By Axiom (A2), also $\vdash [n.E', \mathbb{O}] = \infty$.

Subcase $F = m.F'$. Here $\wr E, F \int = \wr n.E', m.F' \int = d_{\mathbb{K}}(n, m) + \lambda \wr E', F' \int \bowtie r$, hence with $r' = \lambda^{-1}(r - d_{\mathbb{K}}(, nm))$, $\wr E', F' \int \bowtie r'$. By induction hypothesis it follows that $\vdash [E', F'] \bowtie r'$, and we can use Axiom (R1) to conclude that $\vdash [E, F] \bowtie r$.

Subcase $F = F_1 + F_2$. Using Equation (3), we have $\wr E, F \int = \wr n.E', F_1 + F_2 \int = \min\big(\wr n.E', F_1 \int, \wr n.E', F_2 \int\big)$. Let $\wr n.E', F_1 \int \bowtie r_1$, $\wr n.E', F_2 \int \bowtie r_2$. By the previous case, $\vdash [n.E', F_1] \bowtie r_1$. As $\min\{r_1, r_2\} \bowtie r$ it follows using (R3) that $\vdash_F [n.E, F_1 + F_2] \bowtie r$.

($E = E_1 + E_2$) By an argument similar to the one in the preceding subcase, we have $\wr E, F \int = \max\big(\wr E_1, F \int, \wr E_2, F \int\big)$. If $\wr E_1, F \int \bowtie r_1$ and $\wr E_2, F \int \bowtie r_2$ with $\max(r_1, r_2) \bowtie r$, we can use the induction hypothesis to conclude $\vdash [E_1, F] \bowtie r_1$ and $\vdash [E_1, F] \bowtie r_1$, whence $\vdash [E, F] \bowtie r$ by Axiom (R2). □

4.1. Point-wise distance

We can devise a sound and complete inference system $F^\bullet$ for point-wise distance (instead of accumulating) by replacing inference rule (R1) in System F by the rule

$$(\mathrm{R1}^\bullet)\ \frac{\vdash [E, F] \bowtie r_1}{\vdash [n.E, m.F] \bowtie r} \quad \max(d_{\mathbb{K}}(n, m), \lambda r_1) \bowtie r$$

As before, we write $\vdash_{F^\bullet} [E, F] \bowtie r$ if $[E, F] \bowtie r$ can be proven by $F^\bullet$.

Theorem 11 (Soundness & Completeness). *$\vdash_{F^\bullet} [E, F] \bowtie r$ if and only if $\wr E, F \int_\bullet \bowtie r$*

Proof. The proof is similar to the one for F. □

4.2. Simulation distance zero

We show here that for distance zero, our inference system F specializes to a sound and complete inference system for simulation. The inference system F_0 is displayed in Figure 3.

Theorem 12 (Soundness & Completeness). *$\vdash_{F_0} E \leq F$ if and only if $E \leq F$*

Proof. Soundness follows immediately from the soundness of Proof system F, and for completeness we note that the arguments one uses in the inductive proof of Theorem 10 all specialize to distance zero. □

$$(A1_0)\ \frac{}{\vdash \mathbb{O} \leq E}$$

$$(R1_0)\ \frac{\vdash E \leq F}{\vdash n.E \leq n.F}$$

$$(R2_0)\ \frac{\vdash E_1 \leq F \qquad \vdash E_2 \leq F}{\vdash E_1 + E_2 \leq F}$$

$$(R3'_0)\ \frac{\vdash n.E \leq F_1}{\vdash n.E \leq F_1 + F_2} \qquad (R3_0)\ \frac{\vdash n.E \leq F_2}{\vdash n.E \leq F_1 + F_2}$$

Figure 3: The F_0 proof system

We remark that, contrary to the situation for general distance above, we may indeed replace the guarded process $n.E$ in $(R3'_0)$ and $(R3_0)$ by a plain E without invalidating the rules. Note also that F_0 may similarly be obtained as a specialization $F_0^\bullet$ of the axiomatization $F^\bullet$ of point-wise distance above.

5. Axiomatizations for Regular Weighted Processes

Let $N = \max\{d_{\mathbb{K}}(n,m) \mid n,m \in \mathbb{K}\}$; by finiteness of $\mathbb{K}$, $N \in \mathbb{R}$. Let V be a fixed set of variables, then $\mathcal{P}^R$ is the set of process expressions generated by the following grammar:

$$E ::= \mathbf{U} \mid X \mid n.E \mid E + E \mid \mu X.E \qquad n \in \mathbb{K}, X \in V$$

Here we use $\mathbf{U}$ to denote the *universal process* recursively offering any weight in $\mathbb{K}$, *cf.* Proposition 7. Note that we do not incorporate the empty process $\mathbb{O}$. Semantically this will mean that all processes in $\mathcal{P}^R$ are non-terminating, and that the accumulating distance between any pair of processes is finite. The reason for the exchange of $\mathbb{O}$ with $\mathbf{U}$ is precisely this last property; specifically, completeness of our axiomatization (Theorem 17) can only be shown if all accumulating distances are finite.

The semantics of processes in $\mathcal{P}^R$ is given as weighted transition systems which are generated by the following standard SOS rules:

$$\frac{}{n.E \xrightarrow{n} E} \qquad \frac{}{\mathbf{U} \xrightarrow{n} \mathbf{U}}$$

$$\frac{E_1 \xrightarrow{n} E'_1}{E_1 + E_2 \xrightarrow{n} E'_1} \qquad \frac{E_2 \xrightarrow{n} E'_2}{E_1 + E_2 \xrightarrow{n} E'_2} \qquad \frac{E[\mu X.E/X] \xrightarrow{n} F}{\mu X.E \xrightarrow{n} F}$$

As usual we say that a variable X is *guarded* in an expression E if any occurrence of X in E is within a subexpression $n.E'$. Formally, we define the *guarding depth* $\mathrm{gd}(E, X)$ of variable X in expression E recursively by

$$\begin{aligned}\mathrm{gd}(\mathbf{U}, X) &= \infty \\ \mathrm{gd}(X, X) &= 0 \\ \mathrm{gd}(n.E, X) &= 1 + \mathrm{gd}(E, X) \\ \mathrm{gd}(E_1 + E_2, X) &= \min\big(\mathrm{gd}(E_1, X), \mathrm{gd}(E_2, X)\big) \\ \mathrm{gd}(\mu X.E, Y) &= \begin{cases} \mathrm{gd}(E, X) & \text{if } X \neq Y \\ \infty & \text{if } X = Y \end{cases}\end{aligned}$$

and we say that X is guarded in E if $\mathrm{gd}(E, X) \geq 1$.

Also as usual, we denote by $E[F/X]$ the expression derived from E by substituting all free occurrences of variable X in E by F, and given tuples $\bar{F} = (F_1, \ldots, F_k)$, $\bar{X} = (X_1, \ldots, X_k)$, we write $E[\bar{F}/\bar{X}] = E[F_1/X_1, \ldots, F_k/X_k]$ for the simultaneous substitution.

Our inference system for regular processes consists of the set of rules R as shown in Figure 4; whenever $[E, F] \bowtie r$ may be concluded from R, we write $\vdash_R [E, F] \bowtie r$.

Compared to inference system F for finite processes, we note that we have to include the triangle inequality (R4) as an inference rule. Also, the precongruence property of simulation distance is expressed by rules (R1), (R5), and (R6). We will need all those extra rules in the proof of Lemma 14 which again is necessary for showing completeness.

Theorem 13 (Soundness). *For closed expressions $E, F \in \mathcal{P}^R$ we have that $\vdash_R [E, F] \bowtie r$ implies $\wr E, F \int \bowtie r$.*

Proof. By an easy induction in the proof tree for $\vdash_F [E, F] \bowtie r$, using the definition of $\wr \cdot, \cdot \int$. In relation to Axiom (A3), we note that $N = \max\{d_{\mathbb{K}}(n, m) \mid n, m \in \mathbb{K}\}$ implies $\wr E, F \int \leq \sum_{i=0}^{\infty} \lambda^i N = \frac{N}{1-\lambda}$. □

Our completeness result for regular processes will be based on the following lemmas; here we call an expression $E \in \mathcal{P}^R$ *non-recursive* if it does not contain any subexpressions $\mu X.E'$:

Lemma 14. *For all $E \in \mathcal{P}^R$ and $k \in \mathbb{N}$ there exist a non-recursive expression F and tuples $\bar{E} = (E_1, \ldots, E_k)$, $\bar{X} = (X_1, \ldots, X_k)$ for which $\mathrm{gd}(F, X_i) \geq k$ for all i and*

$$\vdash_R [E, F[\bar{E}/\bar{X}]] = 0 \qquad \vdash_R [F[\bar{E}/\bar{X}], E] = 0$$

Proof. Repeated use of the unfolding axioms (A6) and (A7), the congruence rules (R1), (R5), and (R6) with $r = 0$ and of the triangle inequality (R4). □

$$\text{(A3)}\ \frac{}{\vdash [E,F] \le r}\quad \frac{N}{1-\lambda} \le r$$

$$\text{(A4)}\ \frac{}{\vdash [\mathbf{U}, \sum_{n\in\mathbb{K}} n.\mathbf{U}] = 0}$$

$$\text{(A5)}\ \frac{}{\vdash [\sum_{n\in\mathbb{K}} n.\mathbf{U}, \mathbf{U}] = 0} \qquad \text{(A6)}\ \frac{}{\vdash [\mu X.E, E[\mu X.E/X]] = 0}$$

$$\text{(A7)}\ \frac{}{\vdash [E[\mu X.E/X], \mu X.E] = 0} \qquad \text{(A8)}\ \frac{}{\vdash [E, \mathbf{U}] = 0}$$

$$\text{(R1)}\ \frac{\vdash [E,F] \bowtie r_1}{\vdash [n.E, m.F] \bowtie r}\quad d_{\mathbb{K}}(n,m) + \lambda r_1 \bowtie r$$

$$\text{(R2)}\ \frac{\vdash [E_1,F] \bowtie r_1 \qquad \vdash [E_2,F] \bowtie r_2}{\vdash [E_1 + E_2, F] \bowtie r}\quad \max(r_1,r_2) \bowtie r$$

$$\text{(R3)}\ \frac{\vdash [n.E,F_1] \bowtie r_1 \qquad \vdash [n.E,F_2] \bowtie r_2}{\vdash [n.E, F_1 + F_2] \bowtie r}\quad \min(r_1,r_2) \bowtie r$$

$$\text{(R4)}\ \frac{\vdash [E,F] \le r_1 \qquad \vdash [F,G] \le r_2}{\vdash [E,G] \le r}\quad r_1 + r_2 \le r$$

$$\text{(R5)}\ \frac{\vdash [E,F] \le r}{\vdash [E+G, F+G] \le r} \qquad \text{(R6)}\ \frac{\vdash [E,F] \le r}{\vdash [G+E, G+F] \le r}$$

Figure 4: The R proof system

Lemma 15. *Let F be a non-recursive expression and $\bar{E} = (E_1, \dots, E_k)$, $\bar{X} = (X_1, \dots, X_k)$ tuples for which $\mathrm{gd}(F, X_i) \ge k$ for all i. Then*

$$\vdash_R [F[\bar{E}/\bar{X}], F[\bar{\mathbf{U}}/\bar{X}]] = 0 \qquad \vdash_R [F[\bar{\mathbf{U}}/\bar{X}], F[\bar{E}/\bar{X}]] = \lambda^k \frac{N}{1-\lambda}$$

Proof. Repeated use of Axioms (A3) and (A8) together with the congruence rules (R1), (R5), and (R6) with $r = 0$. □

Lemma 16. *For closed non-recursive expressions E, F, $\wr E, F \int \bowtie r$ implies $\vdash_R [E,F] \bowtie r$.*

Proof. By structural induction similar to the proof of Theorem 10. □

We are now in a position to state our completeness result which enables arbitrary ϵ-close proofs in the sense below. The proof uses unfoldings of recursive expressions as in Lemma 14, and as these unfoldings are *finite* non-recursive processes, we cannot expect exact completeness.

Theorem 17 (Completeness up to ϵ). *Let E and F be closed expressions of $\mathcal{P}^R$ and $\epsilon > 0$. Then $\wr E, F \int = r$ implies $\vdash_R [E, F] \leq r + \epsilon$ and $\vdash_R [E, F] \geq r - \epsilon$.*

Proof. Assume $\wr E, F \int = r$, and choose $k \in \mathbb{N}$ such that $2\lambda^k \frac{N}{1-\lambda} \leq \epsilon$. By Lemma 14 we have non-recursive expressions E', F' and tuples $\bar{E}$, $\bar{F}$, $\bar{X}$, and $\bar{Y}$ for which $\mathrm{gd}(E', X_i) \geq k$ and $\mathrm{gd}(F', Y_i) \geq k$ for all i, and such that

$$\begin{aligned} &\vdash_R [E, E'[\bar{E}/\bar{X}]] = 0 \\ &\vdash_R [E'[\bar{E}/\bar{X}], E] = 0 \\ &\vdash_R [F, F'[\bar{F}/\bar{Y}]] = 0 \\ &\vdash_R [F'[\bar{F}/\bar{Y}], F] = 0 \end{aligned}$$

From Lemma 15 it follows that

$$\begin{aligned} &\vdash_R [E'[\bar{E}/\bar{X}], E'[\bar{\mathbf{U}}/\bar{X}]] = 0 \\ &\vdash_R [E'[\bar{\mathbf{U}}/\bar{X}], E'[\bar{E}/\bar{X}]] \leq \lambda^k \tfrac{N}{1-\lambda} = \tfrac{\epsilon}{2} \\ &\vdash_R [F'[\bar{F}/\bar{Y}], F'[\bar{\mathbf{U}}/\bar{Y}]] = 0 \\ &\vdash_R [F'[\bar{\mathbf{U}}/\bar{Y}], F'[\bar{F}/\bar{Y}]] \leq \lambda^k \tfrac{N}{1-\lambda} = \tfrac{\epsilon}{2} \end{aligned}$$

Using the triangle inequality and Theorem 13 we now have

$$\begin{aligned} \wr E'[\bar{\mathbf{U}}/\bar{X}], F'[\bar{U}/\bar{X}] \int &\leq \wr E'[\bar{\mathbf{U}}/\bar{X}], E'[\bar{E}/\bar{X}] \int + \wr E'[\bar{E}/\bar{X}], E \int \\ &\quad + \wr E, F \int + \wr F, F'[\bar{F}/\bar{Y}] \int + \wr F'[\bar{F}/\bar{Y}], F'[\bar{\mathbf{U}}/\bar{Y}] \int \\ &\leq \tfrac{\epsilon}{2} + 0 + r + 0 + 0 = r + \tfrac{\epsilon}{2} \end{aligned}$$

Only non-recursive expressions are involved here, so that we can invoke Lemma 16 to conclude

$$\vdash_R [E'[\bar{\mathbf{U}}/\bar{X}], F'[\bar{U}/\bar{X}]] \leq r + \tfrac{\epsilon}{2}$$

Now we can use the triangle inequality axiom (R4) together with the eight equations above to arrive at

$$\vdash_R [E, F] \leq r + \epsilon$$

Similar arguments show that also $\vdash_R [E, F] \geq r - \epsilon$, $\square$

5.1. Point-wise distance

Again we can easily convert our proof system R into one for point-wise (instead of accumulating) distance. In this case, we obtain $\mathrm{R}^\bullet$ by replacing inference rule (R1) by ($\mathrm{R1}^\bullet$) as we did for Proof system F, and (A3) needs to be replaced by

$$(\mathrm{A3}^\bullet)\ \frac{}{\vdash [E,F] \le N}$$

With these replacements we have a sound and ϵ-complete axiomatization of point-wise simulation distance recursive weighted processes.

Theorem 18 (Soundness & Completeness up to ϵ). *Let E and F be closed expressions of $\mathcal{P}^R$, then $\vdash_{R^\bullet} [E,F] \bowtie r$ implies $\wr E,F \int_\bullet \bowtie r$, and $\wr E,F \int_\bullet = r$ implies $\vdash_{R^\bullet} [E,F] \le r+\epsilon$ and $\vdash_{R^\bullet} [E,F] \ge r-\epsilon$ for any $\epsilon > 0$.*

Proof. The proof is similar to that for accumulated distance. □

6. Algorithmic Complexity

In this section we show that, for finite weighted transition systems, computing accumulating distance is polynomial-time equivalent to computing the value of *discounted games* (DG), hence contained in $\mathtt{NP} \cap \mathtt{coNP}$.We also give a polynomial-time algorithm for computing point-wise distance; hence the conceptually simpler point-wise distance is also computationally easier. We assume throughout this section that for all weights $n,m \in \mathbb{K}$, $d_{\mathbb{K}}(n,m)$ is polynomial-time computable.

We recall the following definition from [18]: A two-player game graph $G = (S_1, S_2, \twoheadrightarrow)$ over a finite set $W \subseteq \mathbb{R}$ of weights is a finite directed bipartite graph with vertices $S_1 \cup S_2$ (where states in S_i are said to *belong* to Player i) and edges $\twoheadrightarrow \in S_1 \times W \times S_2 \cup S_2 \times W \times S_1$, in which each vertex has at least one outgoing edge.

A memoryless Player-i strategy is a map $\xi : S_i \to W \times S_{3-i}$, and it is consistent with $\twoheadrightarrow$ if $\xi(s) = (a, s')$ implies that there exists $(s,a,s') \in \twoheadrightarrow$ (the latter written as $s \overset{a}{\twoheadrightarrow} s'$). A pair (ξ, χ) of Player-1 and Player-2 strategies admits a unique sequence (path) of edges $t = e_0 e_1 e_2 \dots$. We will sometimes write $a = w(e)$ to denote the weight of $e = (s,a,s') \in \twoheadrightarrow$.

Definition 19. *Let $G = (S_1, S_2, \twoheadrightarrow)$ be a game graph and $0 \le \lambda < 1$ a discounting factor, and $s \in S_1 \cup S_2$. The* payoff *of the discounted game on G from s is defined as $v = \lambda \sum_{i=0}^{\infty} \lambda^i w(e_i)$, where the path $t = e_0 e_1 e_2 \dots$, starting from $s \in S_1$, is induced by strategies ξ and χ of Player 1 respectively Player 2 which are such that the objective of Player 1 is to pick ξ maximizing v, whereas Player 2's objective is to pick χ minimizing v.*

The decision problem corresponding to discounted games is as follows: Given a game graph G, a starting vertex s_0, a discount factor $0 \leq \lambda < 1$, and a threshold $v \in \mathbb{R}$, can Player 1 guarantee a payoff of v or more? We use the well-known fact [18] that this decision problem is in $\texttt{NP} \cap \texttt{coNP}$, a fact obtained by reduction to *simple stochastic games* [3].

Theorem 20. *Computing accumulating distance is polynomial-time equivalent to computing the payoff for discounted games.*

We present two supporting lemmas. Note that our weighted transition systems and discounted games have different weight domains; we use the metric $d_{\mathbb{K}}$ to map between them.

Lemma 21. *For a given discount factor λ, a WTS (S, T), and $s, t \in S$, one can construct a game graph G together with a vertex s_0 such that $\wr s, t \int$ is the payoff of the discounted game on G from s_0 with discount factor $\sqrt{\lambda}$.*

Proof. We construct the game $G = (S_1, S_2, \twoheadrightarrow)$ with $S_1 = S \times S$, $S_2 = S \times S \times \mathbb{K}$, and edges

$$(s, t) \xrightarrow{0} (s', t, n) \text{ if } (s, n, s') \in T$$
$$(s, t, n) \xrightarrow{d_{\mathbb{K}}(n,m)} (s, t') \text{ if } (t, m, t') \in T$$

Note that the set of weights of the game is $W \subseteq \{d_{\mathbb{K}}(n, m) \mid n, m \in \mathbb{K}\}$, hence finite.

Now recall [18, Thm. 5.1]: For some labeling $S_1 \cup S_2 = \{s_1, \dots, s_p\}$, the payoff x_i of G from s_i is given as the unique fixed point of the set of equations

$$x_i = \begin{cases} \max\limits_{s_i \xrightarrow{c} s_j} \{c + \sqrt{\lambda} x_j\} & \text{if } s_i \in V_1 \\ \min\limits_{s_i \xrightarrow{c} s_j} \{c + \sqrt{\lambda} x_j\} & \text{if } s_i \in V_2 \end{cases}$$

Hence we can let $s_0 = (s, t)$, then $\wr s, t \int = x_0$. $\square$

Lemma 22. *For any game graph G with start vertex x there exists a WTS with states p_x and q_x such that the payoff v of the discounted game on G from x is $\wr p_x, q_x \int$.*

Proof. Let $G = (S_1, S_2, \twoheadrightarrow)$ be the game, with weight set W. Then (S, T), with (finite) weight set $\mathbb{K} = (S_1 \cup S_2) \times W$ is defined as follows: $S = \{p_x, q_x, q'_x \mid x \in S_1\} \cup \{p^b_x \mid x \in S_1, x \twoheadrightarrow b\}$, and for $x, y \in S_1$ and $b \in S_2$, T is given by:

- $p_x \xrightarrow{b,n} p^b_x$ whenever $x \xrightarrow{n} b$

– $q_x \xrightarrow{b,0} q'_y$ whenever $x \twoheadrightarrow b \twoheadrightarrow y$

– $p_x^b \xrightarrow{y,0} p_y$ whenever $b \twoheadrightarrow y$

– $q'_y \xrightarrow{y,m} q_y$ and $q'_y \xrightarrow{S_1\setminus\{y\},0} \mathbf{U}$ whenever $b \xrightarrow{m} y$

In the last item, $\mathbf{U}$ denotes the universal WTS, and the notation $q'_y \xrightarrow{S_1\setminus\{y\},0} \mathbf{U}$ means that there are transitions $q'_y \xrightarrow{z,0} \mathbf{U}$ for all $z \in S_1 \setminus \{y\}$.

The metric on $\mathbb{K}$ is defined by $d_{\mathbb{K}}\big((x,n),(y,m)\big) = |n-m|$ if $x = y$ and ∞ otherwise. The construction is sketched in Figure 5.

To see that $\wr p_x, q_x \int = v$, consider a strategy ξ for which $\xi(x) = b$ which maximizes Player 1's payoff from $x \in S_1$. The transition $p_x \xrightarrow{b,n} p_x^b$ models the choice in ξ, ensuring that minimization from q_x must match the b-label, acknowledging the move to $b \in S_2$. Doing so requires taking precisely $q_x \xrightarrow{b,0} q'_y$ for some $y \in S_1$. Whenever $\chi(b) = y'$ minimizes the payoff for Player 2 in vertex b, the corresponding $q_x \xrightarrow{b,0} q'_{y'}$ ensures

1. the correct cost for the match, *i.e.* $|n - 0| = w(x \twoheadrightarrow b)$, and

2. that the maximizing transition from p_x^b is $p_x^b \xrightarrow{y,0} p_{y'}$.

The latter implies that the choice of Player 2 is passed on to Player 1, and that Player 1 must act according to it. This latter property is obtained by the $q'_x \xrightarrow{S_1\setminus\{y\},0} \mathbf{U}$ transitions, which match any "cheating" maximization and afterwards allow Player 2 to match all possible Player 1 transitions perfectly.

After the Player 1 transition $p_x^b \xrightarrow{y,0} p_{y'}$, Player 2 must minimize the game value by choosing a $q'_{y'} \xrightarrow{y,m} q_{y'}$, thereby adding $|0 - m| = w(b \twoheadrightarrow y')$ both to the total value of G and to the total accumulating distance from p_x to q_x. We have arrived at a new configuration $p_{y'}, q_{y'}$ which models the vertex y' of the game and from which the simulation game can proceed. $\square$

Example. We shortly elaborate on the reduction in the preceding proof by considering the game in Figure 5. Assume Player 1 (with diamond-shaped vertices) has an optimal (maximizing) strategy in which $A \mapsto (n_1, B)$ and Player 2 (with square vertices) wants to play $B \mapsto (n_4, E)$ for minimizing the game value. Then the simulation from (p_A, q_A) is performed accordingly, *i.e.* the maximal choice is $p_A \xrightarrow{b,n_1} p_A^B$ signaling the label b (alternatively c or d could have been chosen). The simulating response is, of course, to match b (since taking *e.g.* $\xrightarrow{c}$ results in weight $d_{\mathbb{K}}((b,n_1)(c,0)) = \infty$); the transition corresponding to Player 2's minimizing choice $B \xrightarrow{n_4} E$ is $q_A \xrightarrow{b,0} q'_E$. The next maximizing challenge then has to be $p_A^B \xrightarrow{e,0} p_E$, as *e.g.* $p_A^B \xrightarrow{f,0} p_F$ would

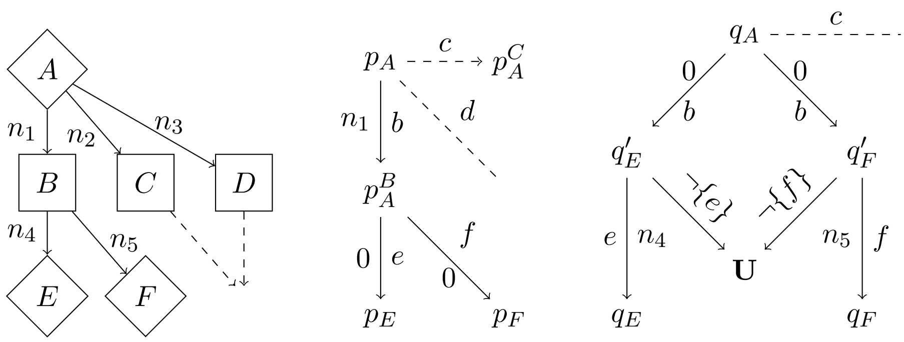

Figure 5: DG to WTS Translation (dashed lines represents omitted parts)

allow for a minimizing response $q'_E \xrightarrow{e,0} \mathbf{U}$, after which any challenge can be met with distance 0. The only minimizing transition is now $q'_E \xrightarrow{e,n_4} q_E$.

Proof of Theorem 20. The result follows directly from Lemmas 21 and 22, together with the additional fact that both reductions are clearly polynomial. $\square$

The sought-after property follows:

Corollary 23. *The decision problem corresponding to computing accumulating simulation distance of states in a WTS is contained in* $\mathtt{NP} \cap \mathtt{coNP}$*.*

6.1. Point-wise distance

To see that point-wise simulation distance is computable in polynomial time, we note that the fixed-point iteration converges in time polynomial in the size of the WTS:

Theorem 24. *For a WTS (S, T) with $|S| = p$ and $s_0, t_0 \in S$, $\wr s_0, t_0 \int_\bullet$ may be computed in p^2 steps.*

Proof. For a threshold δ, define iterated δ-simulation relations $\leq_\delta^n$, for $n \in \mathbb{N}$, by $\leq_\delta^0 = S \times S$ and

$$s \leq_\delta^{k+1} t \text{ iff } \forall s \xrightarrow{m} s' \, \exists t \xrightarrow{n} t' : d_{\mathbb{K}}(m, n) \leq \delta \text{ and } s' \leq_\delta^k t'$$

The lattice of $\leq_\delta^k$ relations such defined has at most p^2 elements, hence $\leq_\delta = \bigcap_{k=0}^\infty \leq_\delta^k$ can be computed in at most p^2 iterations. To finish the proof, we note that $\wr s_0, t_0 \int_\bullet \leq \delta$ if and only if $s_0 \leq_\delta t_0$. $\square$

Corollary 25. *The decision problem corresponding to computing point-wise simulation distance is contained in* $\mathtt{P}$*.*

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