Graph-based Field Automata for Modeling of Sliding Mode Systems

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Abstract— **A novel hybrid automaton admitting the modeling of both conventional and modern sliding mode systems is presented. A scheme for defining hybrid-automaton executions beyond Zeno points is proposed. Conventional and Filippov-like executions of the hybrid automaton are introduced and studied.**

I. INTRODUCTION

Sliding mode control is an important branch of modern control theory attracting an interest of both research and practicing engineer communities [1], [2]. Sliding mode control systems are usually modeled by ordinary differential equations with discontinuous right hand sides [3]. In this case, system dynamics is governed by a control feedback that is discontinuous with respect to the current state. Frequently, it is important to consider a wider class of control laws which depend on both current and previous states of the system. In particular, such control laws appear for time-delay systems with sliding modes, see for example, [4]. Another example is presented in the paper [5], which introduces two control functions and a switching logic between them in order to stabilize the triple integrator. The resulting control law is a function of the state and its prehistory. It ensures finite-time reaching and existence of the 3rd order sliding mode at the origin of the closed-loop system. The sliding mode is realized by infinitely fast switching between the control functions. Expansion of the same ideas to integrator chains had required to introduce a special finite automaton describing the logic of switchings between several control functions [6].

Systems combining continuous and discrete dynamics are known as hybrid automata [7]. Although hybrid automata are widely studied, modeling of systems with sliding motion under this framework is a challenging problem [8], [9], [10]. The main obstacle here is Zeno behavior: infinite number of switchings in a finite time interval. According to standard notion [11], executions of hybrid automata are not defined beyond Zeno points. In fact, in the majority of works, Zeno behavior is considered pathological and avoided [12]. However, Zeno-like behavior of sliding mode is one of the main working principles of VSS [13].

In this paper we propose a systematic approach for defining hybrid-automaton executions beyond Zeno points, which is applicable to a wide class of hybrid automata, namely, automata with the identity reset relation.

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The paper is organized as follows. The second section gives well-known definitions of hybrid automata, nondeterministic finite automata and their executions. In the third section some examples of Zeno hybrid automata are considered. It is shown that the set of switching instances can have a complicated structure preventing straightforward step-by-step prolongation of executions. The fourth section introduces a novel representation of non-resettable hybrid automata called graph-based field automata. The fifth section deals with graph-based field initial value problems which are discrete analogs for the Cauchy problem. In the sixth section, executions of graph-based field automata are defined. Peano-like existence theorem for executions of graph-based field automata is stated.

II. HYBRID AND NONDETERMINISTIC FINITE AUTOMATA

A. Hybrid automata

Let us recall well-known definitions concerning hybrid systems [7], [11].

By $P(S)$ denote the powerset of a set S, i.e., the set of all subsets of S. Throughout the paper, $A \subset B$ means that A is a subset of B including the case $A = B$. By $|S|$ denote the number of elements of S.

Definition 1: A *hybrid automaton* $\mathscr{H}$ is a tuple $(Q, X, Init, Dom, f, E, G, R)$, where

- $Q \subset \mathbb{N}$ is a set of discrete states or locations;
- $X = \mathbb{R}^n$ is a set of continuous states;
- $Init \subset Q \times X$ is a set of initial states;
- $Dom(\cdot) : Q \to P(X)$ is a domain;
- $f(\cdot,\cdot) : Q \times X \to \mathbb{R}^n$ is a vector field;
- $E \subset Q \times Q$ is a set of edges;
- $G(\cdot) : E \to P(X)$ is a guard condition;
- $R(\cdot,\cdot) : E \times X \to P(X)$ is a reset map.

Elements of the tuple $\mathscr{H}$ have the following meaning. A domain $Dom(q)$ and continuous dynamics described by $f(q,\cdot)$ are assigned to each location $q \in Q$. Usually, the functions $f(q,\cdot) : X \to \mathbb{R}^n$ are Lipschitz continuous in X. But we don't address the uniqueness issues in this paper and f are only supposed to be continuous in X. Also suppose that Q is finite. The pair (Q,E) is a directed graph that defines possible discrete state transitions. A guard condition $G(e)$ is assigned to each edge $e \in E$. The current state of the hybrid automaton is defined by a pair (q,x), where $q \in Q$ and $x \in X$. $\mathscr{H}$ starts evolving at some state $(q_0,x_0) \in Init$. Then a series of evolution steps follows. Each of these steps is either a continuous evolution of x or an instant discrete state transition. If $x \in G(e)$, then the transition defined by e is allowed. If x leaves $Dom(q)$, then the transition is obligatory.

Definition 2: A *hybrid time trajectory* $\tau = \{I_i\}_{i=0}^N$ is a finite or infinite sequence of intervals $I_i \subset \mathbb{R}$ such that

- for all $0 \le i < N$, $I_i = [\tau_i, \tau_{i+1}]$ with $\tau_i \le \tau_{i+1}$;
- if $N < \infty$, either $I_N = [\tau_N, \tau_{N+1}]$ with $\tau_N \le \tau_{N+1} < \infty$, or $I_N = [\tau_N, \tau_{N+1})$ with $\tau_N < \tau_{N+1} \le \infty$.

Definition 3: An *execution* of a hybrid automaton $\mathscr{H}$ is a tuple (τ, q, x), where

- $\tau = \{I_i\}_{i=0}^N$ is a hybrid time trajectory;
- $q = \{q_i\}_{i=0}^N$ is a sequence of $q_i \in Q$;
- $x = \{x_i(\cdot)\}_{i=0}^N$ is a sequence of $x_i(\cdot) : I_i \to X$;
- $(q_0, x_0(\tau_0)) \in Init$;
- for all $i < N$,
  1) $(q_i, q_{i+1}) \in E$,
  2) $x_i(\tau_{i+1}) \in G(q_i, q_{i+1})$,
  3) $x_{i+1}(\tau_{i+1}) \in R(q_i, q_{i+1}, x_i(\tau_{i+1}))$;
- for all i with $\tau_i < \tau_{i+1}$,
  1) for all $t \in I_i$, $\dot{x}_i(t) = f(q_i, x_i(t))$,
  2) for all $t \in [\tau_i, \tau_{i+1})$, $x_i(t) \in Dom(q_i)$.

An execution (τ, q, x) is called *infinite* if τ is an infinite sequence or $\sum_{i=0}^N (\tau_{i+1} - \tau_i) = \infty$. An execution (τ, q, x) is called *Zeno* if τ is an infinite sequence and $\sum_{i=0}^\infty (\tau_{i+1} - \tau_i) < \infty$. The instant $\tau_\infty = \tau_0 + \sum_{i=0}^\infty (\tau_{i+1} - \tau_i)$ is called a *Zeno point*. A Zeno execution (τ, q, x) is called *chattering Zeno* if there exists a number $C \in N$ such that $\tau_i = \tau_{i+1}$ for all $i > C$. Otherwise, (τ, q, x) is called *genuinely Zeno*.

The following new definitions are introduced for the needs of this paper. Let $D = \bigcup_{q \in Q} Dom(q)$. A hybrid automaton $\mathscr{H}$ is called *uniformly non-blocking* if $Init = Q \times D$ and for every $(q', x') \in Init$ there exists an infinite execution (τ, q, x) of $\mathscr{H}$ such that $q_0 = q'$ and $x_0(\tau_0) = x'$. A hybrid automaton $\mathscr{H}$ is called *non-resettable* if for every $e \in E$ and $x \in X$, $R(e, x) = \{x\}$. A hybrid automaton $\mathscr{H}$ is called *closed* if all sets $Dom(q)$ and $G(e)$ are closed in D.

B. Nondeterministic finite automata

Hybrid automata are based on nondeterministic finite automata, which will be exploited further in the paper.

Definition 4: A *nondeterministic finite automaton* is a tuple $(Q, \Delta, Init)$, where

- Q is a finite set of states;
- $\Delta : Q \to P(Q)$ is a transition relation;
- $Init \subset Q$ is a set of initial states.

The transition relation Δ defines a graph (Q, γ), where Q is the set of vertices and γ is the set of edges. Let $(p, q) \in \gamma$ iff $q \in \Delta(p)$. Assume that $P(q)$ is non-empty for every $q \in Q$.

Definition 5: A sequence $\{q_i\}_{i=0}^N$, where $q_i \in Q$, is called an *execution* of a nondeterministic finite automaton $(Q, \Delta, Init)$ if

1) $q_0 \in Init$;
2) for all $i < N$, $(q_i, q_{i+1}) \in \gamma$.

It is natural to consider multivalued executions for nondeterministic finite automata.

Definition 6: A sequence $\{Q_i\}_{i=0}^N$, where $Q_i \subset Q$ and $Q_i \ne \emptyset$, is called a *multivalued execution* of a nondeterministic finite automaton $(Q, \Delta, Init)$ if

1) $Q_0 \subset Init$;
2) for all $i < N$ and for every $p \in Q_i$, there exists $q \in Q_{i+1}$ such that $(p, q) \in \gamma$,
3) for all $i < N$ and for every $q \in Q_{i+1}$, there exists $p \in Q_i$ such that $(p, q) \in \gamma$.

Obviously, any multivalued execution $\{Q_i\}_{i=0}^N$ is a union of single-valued executions $\{q_i\}_{i=0}^N$ such that $q_0 \in Q_0$.

III. MOTIVATING EXAMPLES

Let us consider some examples of uniformly non-blocking non-resettable hybrid automata that model well-known sliding mode systems, which behavior has been already studied. Therefore we know how executions of the considered automata should be prolonged beyond Zeno points.

A. Chattering system

Consider a hybrid automaton $\mathscr{H}_C$ with $Q = \{1, 2\}$, $X = \mathbb{R}$, $Init = Q \times X$, $Dom(1) = [0, +\infty)$, $Dom(2) = (-\infty, 0]$, $f(1, x) = -1$, $f(2, x) = 1$, $E = \{(1,2), (2,1)\}$, $G(1,2) = Dom(2)$, $G(2,1) = Dom(1)$, $R(e, x) = \{x\}$ (see Fig. 1). One can see that any infinite execution of $\mathscr{H}_C$ is chattering Zeno. E.g., suppose $a > 0$ and (τ, q, x) is an infinite execution of $\mathscr{H}_C$ such that $\tau_0 = 0$, $x_0(0) = a$, and $q_0 = 2$. The first time interval of τ is $I_0 = [0, 0]$ and it corresponds to the instant transition from $q = 2$ to $q = 1$. After that, $q_1 = 1$ and continuous state x evolves on $I_1 = [0, a]$: $x_1(t) = a - t$. As x reaches the origin, only instant transitions between $q = 1$ and $q = 2$ are possible. Hence $q_{2j} = 2$, $q_{2j+1} = 1$, $j = 0, 1, \ldots$, and $I_i = [a, a]$, $i = 2, 3, \ldots$. Time in the hybrid time trajectory τ stops progressing and gets stuck at a Zeno point $\tau_\infty = a$.

The automaton $\mathscr{H}_C$ is a model for the following differential equation with a discontinuous right-hand side:

$$\dot{x} = -\operatorname{sgn} x, \tag{1}$$

which has a solution $x \equiv 0$ in the Filippov sense [3]. The automaton $\mathscr{H}_C$ can be augmented by an additional discrete state 3 with $f(3, x) = 0$ [17], [8]. In that case, executions of the augmented automaton can be continued beyond τ_∞ by the transition to the state 3.

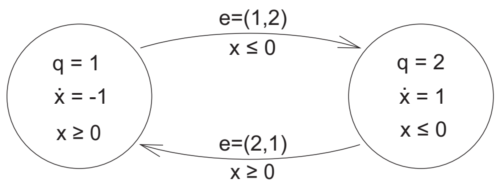

Fig. 1. Chattering hybrid automaton

B. Twisting system

Consider the twisting controller [14] described by

$$\ddot{x} = -\frac{b+s}{2} \operatorname{sgn} x - \frac{b-s}{2} \operatorname{sgn} \dot{x}, \tag{2}$$

where $b > s > 0$ are positive constants. It can be modeled by a hybrid automaton $\mathscr{H}_T$ with $Q = \{1, 2, 3, 4\}$, $X = \mathbb{R}^2 = \{x, \dot{x}\}$, $Init = Q \times X$, $Dom(1) = \{(x, \dot{x}) : x \ge 0, \dot{x} \ge 0\}$, $Dom(2) = \{(x, \dot{x}) : x \ge 0, \dot{x} \le 0\}$, $Dom(3) = \{(x, \dot{x}) : x \le 0, \dot{x} \le 0\}$, $Dom(4) = \{(x, \dot{x}) : x \le 0, \dot{x} \ge 0\}$, $f(1, x, \dot{x}) = (\dot{x}, -b)$,

$f(2,x,\dot{x}) = (\dot{x},-s)$, $f(3,x,\dot{x}) = (\dot{x},b)$, $f(4,x,\dot{x}) = (\dot{x},s)$, $E = \{(i,j) : i \in Q, j \in Q, i \neq j\}$, $G(i,j) = Dom(j)$, $R(e,x,\dot{x}) = \{(x,\dot{x})\}$.

For any initial state $(q_0,x_0,\dot{x}_0)$, where $x_0 \neq 0$ or $\dot{x}_0 \neq 0$, an infinite execution $(\tau,q,x,\dot{x})$ of $\mathscr{H}_T$ is genuinely Zeno (see Fig. 2). Discrete state q changes in a loop $1 \to 2 \to 3 \to 4 \to 1$. Due to the homogeneity properties of (2), the lengths of the intervals $I_i = [\tau_i,\tau_{i+1}]$ form a decreasing geometric progression: $\tau_{i+2} - \tau_i = \alpha(\tau_i - \tau_{i-2})$, $i = 3,4,\ldots$, where $\alpha = s/b < 1$. Hence $\tau_i \to \tau_\infty < \infty$ as $i \to \infty$. In addition, $x \to 0$ and $\dot{x} \to 0$ as $t \to \tau_\infty$. After $t = \tau_\infty$, $x \equiv 0$ is a second order sliding mode solution for (2). Similar to the example of the chattering system, an auxiliary discrete state (a Zeno state [15]) z can be added to $\mathscr{H}_T$, but that is not enough for prolongation of executions beyond τ_∞. One should amend the definitions of the hybrid time trajectory and execution to allow transitions to Zeno states after an *infinite* sequence of time intervals that converges to $\tau_\infty < \infty$.

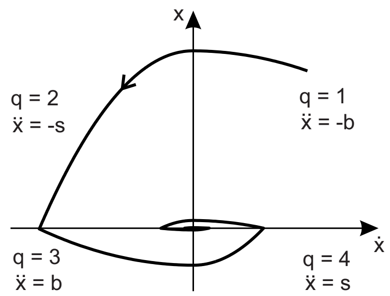

Fig. 2. Phase portrait of the twisting hybrid automaton

IV. GRAPH-BASED FIELD AUTOMATA

In this section we introduce a different representation of non-resettable hybrid automata that is helpful in the study of executions beyond Zeno points.

Definition 7: A *graph-based field automaton H* is a tuple $(\mathbb{N}_m,\mathbb{R}^n,A,Init,f,\gamma)$, where

- $\mathbb{N}_m = \{1,2,\ldots,m\}$ is a finite set of discrete states;
- $\mathbb{R}^n$ is a set of continuous states;
- $A \subset \mathbb{R} \times \mathbb{R}^n$ is a set of points (t,x) (a domain);
- $Init \subset \mathbb{N}_m \times A$ is a set of initial states;
- $f(\cdot,\cdot) : \mathbb{N}_m \times A \to \mathbb{R}^n$ is a vector field;
- $\gamma(\cdot) : A \to P(\mathbb{N}_m \times \mathbb{N}_m)$ is a graph-based field assigning a set of edges to each point $(t,x) \in A$.

Notice that non-resettable hybrid automata and graph-based field automata are not the same technically. The former uses one graph (Q,E) and the latter exploits many graphs: a graph $(\mathbb{N}_m,\gamma(t,x))$ for each point $(t,x) \in A$. A non-resettable hybrid automaton requires guard conditions to specify the area $G(e)$ for each edge $e \in E$, where the corresponding transition can occur. In graph-based field automata, these conditions are defined implicitly by a proper choice of the graph-based field γ. If a graph-based field automaton does not depend on time, the two notions are equivalent as we will see.

If $A = \mathbb{R} \times A_x$, where $A_x \subset \mathbb{R}^n$, and functions $f(s,t,x)$ and $\gamma(t,x)$ do not depend on t, then H is called an *autonomous graph-based field automaton*.

Let us define a bijection assigning an autonomous graph-based field automaton H to every non-resettable hybrid automaton $\mathscr{H}$. Recall definition 1 of $\mathscr{H}$ and set $m = |Q|$. Let $s \in \mathbb{N}_m$ correspond to $q_s \in Q$. Let $A = \mathbb{R} \times \bigcup_{q\in Q} Dom(q)$. Finally, for every $(t,x) \in A$, define $\gamma(t,x)$ as follows: $(i,j) \in \gamma(t,x)$, $i \neq j$, iff $x \in G(q_i,q_j)$; and $(i,i) \in \gamma(t,x)$ iff $x \in Dom(q_i)$.

The left and the right limits of a function $s(t)$ at a point t are denoted by $s(t-0)$ and $s(t+0)$.

Definition 8: A pair (s,x), where $s : [a,b) \to \mathbb{N}_m$ and $x : [a,b) \to \mathbb{R}^n$, is called a *classical execution* of a graph-based field automaton H if

1) s is a piecewise continuous on every $[a,c] \subset [a,b)$;
2) x is a piecewise differentiable on every $[a,c] \subset [a,b)$;
3) $(s(a),a,x(a)) \in Init$;
4) for all $t \in [a,b)$, $(t,x(t)) \in A$;
5) for all $t \in [a,b)$ except discontinuity points of $\dot{x}(t)$ and $s(t)$, $\dot{x}(t) = f(s(t),t,x(t))$;
6) at the points $t \in [a,b)$ that are continuity points of $s(t)$, the edge $(s(t),s(t))$ belongs to $\gamma(t,x(t))$;
7) at the points $t \in (a,b)$ that are discontinuity points of $s(t)$, there is a path from $s(t-0)$ to $s(t+0)$ in $\gamma(t,x(t))$ and either $s(t) = s(t-0)$ or $s(t) = s(t+0)$;
8) at the point $t = a$, if it is discontinuity point of $s(t)$, there is a path from $s(a)$ to $s(a+0)$ in $\gamma(a,x(a))$.

Classical executions of graph-based field automata are introduced to model executions of non-resettable hybrid automata. When discrete state q changes in a non-resettable hybrid automaton, continuous state x doesn't jump. Hence we can consider one continuous function $x(t)$ instead of the set $\{x_i(t)\}_{i=0}^N$ of continuous functions defined on a hybrid time trajectory. Conditions 1 and 2 of definition 8 are chosen especially to model hybrid time trajectories. In hybrid automata, several instant transitions can happen at the same moment. As continuous state x doesn't change in non-resettable hybrid automata during a sequence of instant transitions, we are only interested in the first and the last discrete states of this sequence. That is why conditions 7 and 8 deal with paths in the graph $\gamma(t,x(t))$.

Classical executions of autonomous graph-based field automata and non-resettable hybrid automata are equivalent in the following sense. Let H correspond to $\mathscr{H}$ by the constructed bijection. We can assign an execution $(\tau,q,x) = (\{I_i\},\{q_i\},\{x_i(t)\})$ of $\mathscr{H}$ to every execution (s,x) of H. Indeed, consider a set Θ of discontinuity points of $s(t)$. Obviously, Θ is finite or countable, $\Theta = \{\theta_k\}_{k=1}^M$, $a \leq \theta_k < \theta_{k+1} < b$ for all $k = 1,2,\ldots,M-1$, and $\theta_k \to b$ as $k \to \infty$ if $M = \infty$. To each interval $J_0 = [a,\theta_1]$, $J_1 = [\theta_1,\theta_2]$, $\ldots$, $J_M = [\theta_M,b)$ (if $M < \infty$) we assign an interval $I_{i_k} = J_k$ of the hybrid time trajectory τ. Let $q_{i_k} = s(t_k)$, where t_i is an interior point of J_k. Let $x_{i_k} : I_{i_k} \to \mathbb{R}^n$ be defined as follows: $x_{i_k}(t) = x(t)$. To each $\theta \in \Theta$ we assign several (may be 0) intervals $[\tau_i,\tau_{i+1}]$, $[\tau_{i+1},\tau_{i+2}]$, $\ldots$, $[\tau_{i+l},\tau_{i+l+1}]$ of τ such that $\tau_i = \tau_{i+1} = \ldots = \tau_{i+l+1}$. Let these intervals correspond to vertices of the path from $s(\theta-0)$ (or $s(a)$ if $\theta=a$) to $s(\theta+0)$ in $\gamma(\theta,x(\theta))$. Hence we can construct an execution (τ,q,x) of $\mathscr{H}$. Similarly, we can assign an execution (s,x) of H to every execution (τ,q,x) of $\mathscr{H}$ such that $\sum_{i=0}^{N}(\tau_{i+1}-\tau_i)>0$.

Thus, autonomous graph-based field automata with classical executions are equivalent to non-resettable hybrid automata. Of course, definition 8 doesn't allow sliding modes as definition 3 doesn't define executions that go beyond Zeno points. In the sequel, we will amend definition 8 in order to allow executions of graph-based field automata being prolonged beyond Zeno points.

Let us revisit the example of the chattering system. Consider a graph-based field automaton $H_C=(\mathbb{N}_2,\mathbb{R},A,Init,f,\gamma)$ corresponding to $\mathscr{H}_C$. Here, $A=\mathbb{R}\times\mathbb{R}$, $Init=\mathbb{N}_2\times A$, $f(1,t,x)=-1$, $f(2,t,x)=1$, $\gamma(t,0)=\gamma_0$, $\gamma(t,x)=\gamma_1$ when $x>0$, and $\gamma(t,x)=\gamma_2$ when $x<0$, where $\gamma_0=\{(1,1),(1,2),(2,1),(2,2)\}$, $\gamma_1=\{(1,1),(2,1)\}$, and $\gamma_2=\{(2,2),(1,2)\}$. In the sliding mode solution $x\equiv 0$ of (1), $\dot{x}$ belongs to the convex hull of 1 and -1 by the definition of Filippov [3], which infers that the discrete state s should also be some "combination" of states 1 and 2. Consider a nondeterministic finite automaton $(Q,\Delta,Init)$, where $Q=\mathbb{N}_2$ and Δ is constructed by γ_0: $\Delta(1)=\Delta(2)=\{1,2\}$. Notice that the automaton accepts multivalued executions of the form $Q_i=\{1,2\}$, $i=0,1,\ldots$. Further on, the idea of using multivalued executions will be exploited in the general definition of graph-based field automata executions.

V. GRAPH-BASED FIELD INITIAL VALUE PROBLEMS

Recall definition 8 and notice that conditions 1, 6, 7, and 8 describe the form of $s(t)$. Suppose that $x(t)$ is already known and fixed, then a graph-based field $\gamma(t)=\gamma(t,x(t))$ is well defined. Given $\gamma(t)$, conditions 1, 6, 7, and 8 can be considered separately from the other conditions of definition 8. Thus we have an interval $[a,b)$, a graph-based field $\gamma:[a,b)\to P(\mathbb{N}_m\times\mathbb{N}_m)$, and some set of initial states $Init_s\subset\mathbb{N}_m$, where $s(a)\in Init_s$. We need to determine $s(t)$ on $[a,b)$. That looks like an initial value problem for ordinary differential equations. In this section we will investigate the properties of graph-based field initial value problems.

Notice that the condition "$s(t)$ is a piecewise continuous on every $[a,c]\subset[a,b)$" implies that b is the only accumulation point of the set Θ of discontinuity points of $s(t)$. This restriction on the form of $s(t)$ prevents going beyond Zeno points. Therefore we should consider a wider class of functions $s(t)$. Secondly, the example of the graph-based field automaton H_C infers that we should consider multivalued state functions $S(t)$. Thirdly, definition 8 works with paths in the graph $(\mathbb{N}_m,\gamma)$, and hence it is convenient to use the transitive closure Γ instead of γ. I.e., $(p,q)\in\Gamma$ iff there exists a path from vertex p to vertex q in γ: $(p,s_1),(s_1,s_2),\ldots,(s_j,q)\in\gamma$.

We call a number $s\in N_m$ a partial limit of a multivalued function $S:T\to P(\mathbb{N}_m)$ at a point $t\in T$ iff there exists a sequence $\{t_i\}$, $i=1,2,\ldots$, such that $t_i\to t$ as $i\to\infty$, $t_i\neq t$, and $s\in S(t_i)$ for for every $i=1,2,\ldots$. If $t_i<t$, then s is called a left partial limit. Similarly, if $t_i>t$, then s is called a right partial limit. By $\mathscr{P}_S(t)$ denote the set of all partial limits of a function S at the point t. Similarly, by $\mathscr{P}_S(t-0)$ and $\mathscr{P}_S(t+0)$ denote the sets of left and right partial limits of a function S at the point t.

Definition 9: Let $T\subset\mathbb{R}$ be an interval. A function $S:T\to P(\mathbb{N}_m)$ is called an *admissible function* for a graph-based field $\Gamma:T\to P(\mathbb{N}_m\times\mathbb{N}_m)$ if

1) for any $t\in T$, $S(t)$ is not empty;
2) for any $t\in T$, $\mathscr{P}_S(t)\subset S(t)$;
3) for any interior point $t\in T$ and for any $s\in S(t)$, $s\in\mathscr{P}_S(t)$ or there exist $p\in\mathscr{P}_S(t)$ and $q\in\mathscr{P}_S(t)$ such that $(p,s)\in\Gamma(t)$ and $(s,q)\in\Gamma(t)$;
4) for any interior point $t\in T$ and for any $s_-\in\mathscr{P}_S(t-0)$, there exists $s_+\in\mathscr{P}_S(t+0)$ such that $(s_-,s_+)\in\Gamma(t)$;
5) for any interior point $t\in T$ and for any $s_+\in\mathscr{P}_S(t+0)$, there exists $s_-\in\mathscr{P}_S(t-0)$ such that $(s_-,s_+)\in\Gamma(t)$;
6) if $t\in T$ is the left endpoint of T, then for any $s\in S(t)$, there exists $s_+\in\mathscr{P}_S(t+0)$ such that $(s,s_+)\in\Gamma(t)$;
7) if $t\in T$ is the right endpoint of T, then for any $s\in S(t)$, there exists $s_-\in\mathscr{P}_S(t-0)$ such that $(s_-,s)\in\Gamma(t)$.

Condition 2 of definition 9 means that $S(t)$ is a β-continuous multivalued function [3]. Notice that conditions 4 and 5 are analogous to conditions 2 and 3 of definition 6 of multivalued executions of nondeterministic finite automata.

By the initial value problem for a graph-based field on an interval $T=[a,b]$ (or $T=[a,b)$) we mean looking for an admissible function that satisfies the initial condition $S(a)=S_0$.

Let $P\subset\mathbb{N}_m$ and $Q\subset\mathbb{N}_m$ be two disjoint sets such that $P\cup Q=\mathbb{N}_m$. We say that a graph-based field $\Gamma:T\to P(\mathbb{N}_m\times\mathbb{N}_m)$ *disallows transitions* from the set P to the set Q at the point $t\in T$ iff there are no edges (p,q) such that $(p,q)\in\Gamma(t)$, $p\in P$, and $q\in Q$.

Lemma 1: Let a function $S:T\to P(\mathbb{N}_m)$ be admissible for a graph-based field $\Gamma:T\to P(\mathbb{N}_m\times\mathbb{N}_m)$. Let Γ disallow transitions from P to Q in a neighborhood O of the point $t\in T$. Then:

1) if $S(t)\cap Q\neq\emptyset$, then $S(\tau)\cap Q\neq\emptyset$ for any $\tau\leq t$, $\tau\in O$;
2) if $S(t)\subset Q$, then $S(\tau)\subset Q$ for any $\tau\leq t$, $\tau\in O$;
3) if $S(t)\cap P\neq\emptyset$, then $S(\tau)\cap P\neq\emptyset$ for any $\tau\geq t$, $\tau\in O$;
4) if $S(t)\subset P$, then $S(\tau)\subset P$ for any $\tau\geq t$, $\tau\in O$.

Let us define an analog of single-valued admissible function for a graph-based field, namely a connected admissible function.

Definition 10: An admissible function $S:T\to P(\mathbb{N}_m)$ for a graph-based field $\Gamma:T\to P(\mathbb{N}_m\times\mathbb{N}_m)$ is called *connected* if

1) for any $t\in T$ except the left endpoint of T and for any $p_-\in\mathscr{P}_S(t-0)$ and $q_-\in\mathscr{P}_S(t-0)$, $(p_-,q_-)\in\Gamma(t)$ and $(q_-,p_-)\in\Gamma(t)$;
2) for any $t\in T$ except the right endpoint of T and for any $p_+\in\mathscr{P}_S(t+0)$ and $q_+\in\mathscr{P}_S(t+0)$, $(p_+,q_+)\in\Gamma(t)$ and $(q_+,p_+)\in\Gamma(t)$.

Notice that if there exists a set $C\subset\mathbb{N}_m$ such that for any $p\in C$ and $q\in C$, $(p,q)\in\Gamma(t)$ for all $t\in T$, then $S(t)=C$ is a connected admissible function for Γ. Sets like C are called strongly connected components. Suppose there are two sets of vertices P and Q that are disconnected by $\Gamma(t)$ and there is an admissible function S that has values from both of P and Q at some point $t \in T$. Then S can be decomposed into two admissible function S_1 and S_2 such that $S_1(t) \subset P$ and $S_2(t) \subset Q$. Connected admissible functions are admissible functions decomposed up to strongly connected components.

For some graph-based fields, there may be no admissible functions at all. Now we propose sufficient conditions for a graph-based field to have an admissible function.

A graph-based field $\Gamma : T \to P(\mathbb{N}_m \times \mathbb{N}_m)$ is called *closed* if for any $t \in T$ the following conditions hold:

1) for any sequence $\{t_i\}$, $i = 1,2,\ldots$, such that $t_i \to t$ as $i \to \infty$ and $(p,q) \in \Gamma(t_i)$ for every $i = 1,2,\ldots$, the inclusion $(p,q) \in \Gamma(t)$ holds,
2) $\Gamma(t)$ coincides with its transitive closure.

A graph-based field $\Gamma : T \to P(\mathbb{N}_m \times \mathbb{N}_m)$ is called *non-blocking* if for any $t \in T$ and for any $p \in \mathbb{N}_m$, there exists $(p,q) \in \Gamma(t)$.

Consider a non-resettable hybrid automaton $\mathscr{H}$ and a continuous function $x : T \to \mathbb{R}^n$. Let a graph-based field automaton $H = (\mathbb{N}_m, \mathbb{R}^n, A, Init, f, \gamma)$ correspond to $\mathscr{H}$. By $\Gamma(t,x)$ denote the transitive closure of $\gamma(t,x)$. If $\mathscr{H}$ is closed, then $\Gamma(t,x(t))$ is closed. If $\mathscr{H}$ is uniformly non-blocking, then $\Gamma(t,x(t))$ is non-blocking.

It turns out that an initial value problem for a closed non-blocking graph-based field is feasible in connected admissible functions. In order to show that, we consider sequences of approximate admissible functions.

Let $\{\Gamma_i(t)\}$, $i = 1,2,\ldots$, be a sequence of graph-based fields, where $\Gamma_i : T \to P(\mathbb{N}_m \times \mathbb{N}_m)$. Let us construct a graph $\Gamma'(t)$ as follows: $(p,q) \in \Gamma'(t)$ iff there exist two sequnces $\{t_k\}$ and $\{i_k\}$, $k = 1,2,\ldots$, such that $t_k \to t$ and $i_k \to \infty$ as $k \to \infty$, and $(p,q) \in \Gamma_{i_k}(t_k)$ for every $k = 1,2,\ldots$. By $\Gamma(t)$ denote the transitive closure of $\Gamma'(t)$. We call $\Gamma(t)$ a *limit graph-based field* of $\{\Gamma_i(t)\}$.

Lemma 2: A limit graph-based field is a closed graph-based field.

Let $\{S_i(t)\}$, $i = 1,2,\ldots$, be a sequence of admissible functions for graph-based fields $\Gamma_i(t)$, where $S_i : T \to P(\mathbb{N}_m)$. Let $\Gamma(t)$ be the limit graph-based field of $\{\Gamma_i(t)\}$. Let us construct a function $S'(t)$ as follows: $s \in S'(t)$ iff there exist two sequences $\{t_k\}$ and $\{i_k\}$, $k = 1,2,\ldots$, such that $t_k \to t$ and $i_k \to \infty$ as $k \to \infty$, and $s \in S_{i_k}(t_k)$ for every $k = 1,2,\ldots$. Given the function $S'(t)$, construct a function $S(t)$: $s \in S(t)$ iff $s \in S'(t)$ or there exist $p \in S'(t)$ and $q \in S'(t)$ such that $(p,s) \in \Gamma(t)$ and $(s,q) \in \Gamma(t)$. We call $S(t)$ a *limit function* of $\{S_i(t)\}$.

Theorem 1: Let $S(t)$ be the limit function of a sequence $\{S_i(t)\}$, $i = 1,2,\ldots$, of admissible functions for graph-based fields $\Gamma_i(t)$. Let $\Gamma(t)$ be the limit graph-based field of $\{\Gamma_i(t)\}$. Then $S(t)$ is admissible function for $\Gamma(t)$.

Theorem 2: Let $S(t)$ be the limit function of a sequence $\{S_i(t)\}$, $i = 1,2,\ldots$, of connected admissible functions for graph-based fields $\Gamma_i(t)$. Let $\Gamma(t)$ be the limit graph-based field of $\{\Gamma_i(t)\}$. Then there exists a connected admissible function $S_c(t) \subset S(t)$ for $\Gamma(t)$.

Let $\delta > 0$ be a positive constant and $\Gamma : T \to P(\mathbb{N}_m \times \mathbb{N}_m)$ be a graph-based field. By $\Gamma'^{\delta}(t)$ denote the graph-based field $\bigcup_{\tau \in [t-\delta, t+\delta], t \in T} \Gamma(\tau)$. By $\Gamma^{\delta}(t)$ denote the transitive closure of $\Gamma'^{\delta}(t)$.

Lemma 3: Let T be an interval $[a,b]$ or $[a,b)$ and $\Gamma : T \to P(\mathbb{N}_m \times \mathbb{N}_m)$ be a non-blocking graph-based field. For any $\delta > 0$ and any initial condition $s_0 \in \mathbb{N}_m$, there exists a connected admissible function $S_\delta(t)$ for $\Gamma^\delta(t)$ such that $s_0 \in S_\delta(a)$.

Lemma 4: Let $\Gamma : T \to P(\mathbb{N}_m \times \mathbb{N}_m)$ be a closed graph-based field and $\{\delta_i\}$, $i = 1,2,\ldots$, be a sequence of positive numbers such that $\delta_i \to 0$ as $i \to \infty$. By $\Gamma^*(t)$ denote the limit graph-based field of $\{\Gamma^{\delta_i}(t)\}$. Then $\Gamma^* = \Gamma$.

Theorem 3: Let T be an interval $[a,b]$ or $[a,b)$ and $\Gamma : T \to P(\mathbb{N}_m \times \mathbb{N}_m)$ be a closed non-blocking graph-based field. Then for any initial condition $s_0 \in \mathbb{N}_m$, there exists a connected admissible function $S(t)$ for $\Gamma(t)$ such that $s_0 \in S(a)$.

VI. EXECUTION EXISTENCE FOR GRAPH-BASED FIELD AUTOMATA

By $\mathrm{co}\,V$ denote the convex hull of a set of vectors $V \subset \mathbb{R}^n$.

The following definition extends definition 8 of classical executions for graph-based field automata. *It plays the same role for non-resettable hybrid automata as the Filippov regularization does for differential equations with discontinuous right hand sides.*

Definition 11: Let $T \subset \mathbb{R}$ be an interval. A pair (S,x), where $S : T \to P(\mathbb{N}_m)$ and $x : T \to \mathbb{R}^n$, is called an *execution* of a graph-based field automaton $H = (\mathbb{N}_m, \mathbb{R}^n, A, Init, f, \gamma)$ if:

1) $x(t)$ is absolutely continuous on T;
2) for all $t \in T$, $(t, x(t)) \in A$;
3) almost everywhere on T,
$$\dot{x}(t) \in \mathrm{co} \bigcup_{s \in S(t)} f(s,t,x(t)); \tag{3}$$
4) $S(t)$ is a connected admissible function for the graph-based field $\Gamma(t)$, where $\Gamma(t)$ is the transitive closure of $\gamma(t,x(t))$.

By the initial value problem for a graph-based field automaton on an interval $T = [a,b]$ (or $T = [a,b)$) we mean looking for an execution that satisfies the initial conditions: $x(a) = x_0$ and $s_0 \in S(a)$. We exploit Euler's polygonal lines to prove that the initial value problem for a closed non-blocking graph-based field automaton has a solution.

By $\rho(x,y)$ denote the Euclidean distance between two points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. By definition, $\rho(X,Y) = \inf_{x \in X, y \in Y} \rho(x,y)$ for any sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$. By M^δ we denote a closed δ-neighborhood of a set $M \subset \mathbb{R}^n$, i.e., the set of all points $x \in \mathbb{R}^n$ such that $\rho(x,M) \le \delta$. Similarly, by $F(\tau^\delta, \xi^\delta)$ we denote the union of all values of $F(t,x)$ where $t \in \tau^\delta$ and $x \in \xi^\delta$, i.e., $t \in [\tau - \delta, \tau + \delta]$ and $\rho(x,\xi) \le \delta$.

Definition 12: Let $\delta > 0$ be a positive constant. A pair (S,x), where $S : T \to P(\mathbb{N}_m)$ and $x : T \to \mathbb{R}^n$, is called a δ-*execution* of a graph-based field automaton H (or an *approximate execution with an accuracy of* δ) if:

1) $x(t)$ is absolutely continuous on T;
2) for all $t \in T$, $(t^{\delta}, x(t)^{\delta}) \subset A$;
3) almost everywhere on T,

$$\dot{x}(t) \in \left(\text{co} \bigcup_{s \in S(t^{\delta})} f(s, t^{\delta}, x(t)^{\delta}) \right)^{\delta};$$

4) $S(t)$ is a connected admissible function for the graph-based field $\Gamma^{\delta}(t)$, where $\Gamma^{\delta}(t)$ is the transitive closure of $\gamma(t^{\delta}, x(t)^{\delta})$.

Theorem 4: Let H be a closed graph-based field automaton. Let $\{(S_i, x_i)\}$, $i = 1, 2, \ldots$, be a sequence of δ_i-executions of H on an interval $[a, b]$, where $\delta_i \to 0$ as $i \to \infty$. Let $x_i(t)$ uniformly converges to $x(t)$ on $[a, b]$, and $(t, x(t)) \in A$ for any $t \in [a, b]$. Let $S(t)$ be the limit function of $\{S_i(t)\}$. Then there exists a connected admissible function $S_c(t) \subset S(t)$ such that (S_c, x) is an execution of H on $[a, b]$.

Theorem 5: Let $H = (\mathbb{N}_m, \mathbb{R}^n, A, Init, f, \gamma)$ be a closed non-blocking graph-based field automaton. For any interior point $(t_0, x_0) \in A$ and $s_0 \in \mathbb{N}_m$, there exists an execution (S, x) of H such that $x(t_0) = x_0$ and $s_0 \in S(t_0)$.

If A contains a cylinder $Z = \{(t, x) : t_0 \le t \le t_0 + a, \rho(x, x_0) \le b\}$, then the execution exists at least on the interval $[t_0, t_0 + d]$, where

$$d = \min\left\{a, \frac{b}{m}\right\}, \quad m = \sup_{\mathbb{N}_m, Z} |f(s, t, x)|.$$

Recall that a classical execution (s, x) of H can be assigned to every execution (τ, q, x) of non-resettable hybrid automaton $\mathscr{H}$ such that $\sum_{i=0}^{N}(\tau_{i+1} - \tau_i) > 0$.

Lemma 5: Let the pair (s, x) be a classical execution of a graph-based field automaton H. Then the pair (S, x), where $S(t) = \{s(t-0), s(t), s(t+0)\}$, is an execution of H.

Theorem 6: Let $A \subset \mathbb{R} \times \mathbb{R}^n$ be a bounded set and $f : A \to \mathbb{R}^n$ be a piecewise continuous function on A [3]. Consider a differential equation with a discontinuous right-hand side

$$\dot{x} = f(t, x). \tag{4}$$

There exists a graph-based field automaton H with the following properties:

- for any solution $x(t)$ of (4), there exists $S(t)$ such that (S, x) is an execution of H;
- for any execution (S, x) of H, $x(t)$ is a solution of (4).

Recall the example of the graph-based field automaton H_C that models the chattering system. Consider an initial problem for H_C on the interval $[0, +\infty)$ with arbitrary $x_0 \in \mathbb{R}$ and $s_0 \in \mathbb{N}_2$. For every x_0 and s_0, there exists an execution (S, x) of H_C such that $x(0) = x_0$ and $s_0 \in S(0)$. Moreover, $x(t)$ is defined for all $t \ge 0$ and unique. E.g., suppose $x_0 > 0$ and $s_0 = 1$. Then (S, x) has the following form: $x(t) = x_0 - t$ when $t \in [0, x_0]$, $x(t) = 0$ when $t \in [x_0, +\infty)$, $S(t) = \{1\}$ when $t \in [0, x_0)$, and $S(t) = \{1, 2\}$ when $t \in [x_0, +\infty)$. Notice that $S(t)$ is multivalued on $[x_0, +\infty)$ indicating sliding motion.

Consider a graph-based field automaton $H_T = (\mathbb{N}_4, \mathbb{R}^2, A, Init, f, \gamma)$ corresponding to the twisting hybrid automaton $\mathscr{H}_T$, where $A = \mathbb{R} \times \mathbb{R}^2$ and $Init = \mathbb{N}_4 \times A$. Graph-based field γ has the following form: $(i, j) \in \gamma(t, x, \dot{x})$ iff $(x, \dot{x}) \in Dom(j)$. Similarly, for every initial x_0, $\dot{x}_0$, and s_0, there exists a unique execution (S, x) of H_T on $[x_0, +\infty)$. As was shown previously, the twisting hybrid automaton goes through an infinite sequence of transitions by a genuine Zeno point τ_∞. After $t = \tau_\infty$, the pair (S, x), where $S(t) = \{1, 2, 3, 4\}$ and $x(t) = 0$, is a sliding mode execution of H_T.

VII. CONCLUSION

Execution prolongation beyond Zeno points by means of regularization was proposed in [16]. Filippov hybrid automata with additional sliding states applicable for modeling of first order sliding motion were introduced in [17], [8]. This paper describes a regularization scheme for general non-resettable hybrid automata capable of higher order sliding mode control modeling.

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