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# Hausdorff, Large Deviation and Legendre Multifractal Spectra of Lévy Multistable Processes

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## Abstract

We compute the Hausdorff multifractal spectrum of two versions of multistable Lévy motions. These processes extend classical Lévy motion by letting the stability exponent  $\alpha$  evolve in time. The spectra provide a decomposition of  $[0, 1]$  into an uncountable disjoint union of sets with Hausdorff dimension one. We also compute the increments-based large deviations multifractal spectrum of the independent increments multistable Lévy motion. This spectrum turns out to be concave and thus coincides with the Legendre multifractal spectrum, but it is different from the Hausdorff multifractal spectrum. The independent increments multistable Lévy motion thus provides an example where the strong multifractal formalism does not hold.

## 1 Introduction and background

Multifractal analysis gives a fairly complete description of the singularity structure of measures, functions or stochastic processes. Various versions of multifractal analysis exist, which include the determinations of the so-called Hausdorff, large deviation, and Legendre multifractal spectra [20]. Multifractal analysis has been performed for various measures [1, 7], functions [15], and stochastic processes [4, 5, 8, 9, 16]. In the case of Lévy processes, substantially finer results have been obtained in [3] using 2-microlocal analysis.

This article deals with the multifractal analysis of extensions of Lévy stable motions known as *multistable Lévy motions*. Generally speaking, multistable processes extend the well-known stable processes (see, *e.g.* [23]) by letting the stability index  $\alpha$  evolve in “time”. These processes have been introduced in [13] and have been studied for instance in [2, 6, 14, 18, 19, 22]. They provide useful models in various applications where the data display jumps with varying intensity, such as financial records, EEG or natural terrains: indeed, multistability is one practical way to deal with (increments-) non-stationarities

observed in various real-world phenomena, since a multistable process  $X$  is tangent, at each time  $u$ , to a stable process  $Z_u$  in the following sense [11, 12]:

$$\lim_{r \rightarrow 0} \frac{X(u + rt) - X(u)}{r^h} = Z_u(t) \quad (1)$$

for a suitable  $h$  (the limit (1) is taken either in finite dimensional distributions or, when  $X$  has a version with càdlàg paths, in distribution - one then speaks of strong localisability).

Without loss of generality, we shall consider our processes on  $[0, 1]$ . We will need the following ingredients:

- $\alpha : [0, 1] \rightarrow (1, 2)$  is a  $\mathcal{C}^1$  function.
- $(\Gamma_i)_{i \geq 1}$  is a sequence of arrival times of a Poisson process with unit arrival time.
- $(V_i)_{i \geq 1}$  is a sequence of i.i.d. random variables with uniform distribution on  $[0, 1]$ .
- $(\gamma_i)_{i \geq 1}$  is a sequence of i.i.d. random variables with distribution  $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$ .

The three sequences  $(\Gamma_i)_{i \geq 1}$ ,  $(V_i)_{i \geq 1}$ , and  $(\gamma_i)_{i \geq 1}$  are independent. We denote  $c = \inf_{u \in [0, 1]} \alpha(u)$ ,  $d = \sup_{u \in [0, 1]} \alpha(u)$ ,

We shall consider two versions of Lévy multistable processes: the first one has independent but non stationary increments. It admits the following representation:

$$B(t) = \sum_{i=1}^{+\infty} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} \Gamma_i^{-1/\alpha(V_i)} \mathbf{1}_{(V_i \leq t)}, \quad (2)$$

while the second one has correlated non stationary increments and reads:

$$D(t) = C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0, t]}(V_i), \quad (3)$$

where  $C_u = \left( \int_0^\infty x^{-u} \sin x \, dx \right)^{-1}$ . Both processes are semi-martingales and are tangent, at each time  $t$ , to  $\alpha(t)$ -stable Lévy motion. See [19] and the references therein for more details on these processes.

We shall also denote:

$$Y(t) = \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0, t]}(V_i).$$

## 2 Hausdorff multifractal spectra

Let  $h_Y(t)$  denote the pointwise Hölder exponent of  $Y$  at  $t$ . The Hausdorff multifractal analysis of  $Y$  consists in measuring the Hausdorff dimension (denoted  $\dim_H$ ) of the sets  $F_h = \{t \in [0, 1] : h_Y(t) = h\}$ . The Hausdorff multifractal spectrum is the function  $h \mapsto f_H(h) := \dim_H F_h$ .

We will use the following notations:  $S = \cup_i \{V_i\}$ ,  $\mathcal{S} = S^{\mathbf{N}}$  and  $\mathcal{R}_t = \{(r_n)_{n \in \mathbf{N}} \in \mathcal{S} : r_n \rightarrow t\}$ . If  $(r_n)_n \in \mathcal{R}_t$ , we put  $V_{\phi(n)} = r_n$ . Finally, define the positive function  $\delta$  for  $t \notin S$ ,

$$\delta(t) = \inf_{(V_{\phi(n)})_n \in \mathcal{R}_t} \liminf_{i \rightarrow \infty} - \frac{\log \phi(i)}{\log |V_{\phi(i)} - t|}.$$

## 2.1 Main result

The Hausdorff multifractal spectra of both  $B$  and  $D$  are described by the following theorem:

**Theorem 1.** *With probability one, the common Hausdorff multifractal spectrum  $f_H$  of  $B$  and  $D$  satisfies:*

$$f_H(h) = \begin{cases} -\infty & \text{for } h < 0; \\ hd & \text{for } h \in [0, \frac{1}{d}]; \\ 1 & \text{for } h \in (\frac{1}{d}, \frac{1}{c}); \\ \dim_H(\{t \in [0, 1] : \alpha(t) = c\}) & \text{for } h = \frac{1}{c}; \\ -\infty & \text{for } h > \frac{1}{c}. \end{cases} \quad (4)$$

Theorem 1 follows from a series of lemmas that are proven in the next section:

**Lemma 2.** *Almost surely,  $t \mapsto Y(t)$  is càdlàg.*

**Lemma 3.** *Almost surely,  $\forall t \in [0, 1] \setminus S$ ,  $\delta(t) \leq 1$ .*

**Lemma 4.** *Almost surely,  $\forall t \in [0, 1] \setminus S$ ,  $h_Y(t) \leq \frac{\delta(t)}{\alpha(t)}$ .*

**Lemma 5.** *Let  $g : [0, 1] \rightarrow \mathbf{R}$  be a càdlàg function, and  $f$  be the function defined on  $[0, 1]$  by  $f(t) = \int_0^t g(x)dx$ . The pointwise Hölder exponent  $h_f$  of  $f$  verifies:  $\forall t \in (0, 1)$ ,*

$$h_f(t) \geq 1.$$

**Lemma 6.** *Almost surely,  $\forall t \in [0, 1] \setminus S$ ,  $h_Y(t) \geq \frac{\delta(t)}{\alpha(t)}$ .*

**Lemma 7.** *Almost surely,  $\forall h < 0$ ,  $f_H(h) = -\infty$ .*

**Lemma 8.** *Almost surely,  $f_H(0) = 0$ .*

**Lemma 9.** *Almost surely,  $\forall h \in (0, \frac{1}{d}]$ ,  $f_H(h) = hd$ .*

**Lemma 10.** *Almost surely,  $\forall h \in (\frac{1}{d}, \frac{1}{c})$ ,  $f_H(h) = 1$ .*

**Lemma 11.** *Almost surely,  $f_H(\frac{1}{c}) = \dim_H(\{t \in [0, 1] : \alpha(t) = c\})$ .*

**Lemma 12.** *Almost surely,  $\forall h > \frac{1}{c}$ ,  $f_H(h) = -\infty$ .*

## 2.2 Proofs of the lemmas

**Proof of Lemma 2:**

Set  $Y_N(t) = \sum_{i=1}^N \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0,t]}(V_i)$ . Lemma 8 in [19] states that, almost surely,  $Y_N$  converges to  $Y(t)$  uniformly on  $[0, 1]$ .

Fix  $\varepsilon > 0$  and choose  $N_0 \in \mathbf{N}$  such that,  $\forall N \geq N_0$ ,

$$\sup_{t \in [0,1]} |Y_N(t) - Y(t)| \leq \frac{\varepsilon}{3}. \quad (5)$$

1<sup>st</sup> case :  $t \in S$ .

Let  $i_0 \in \mathbf{N}$  be such that  $t = V_{i_0}$ , and  $N_1 = \max(i_0, N_0)$ . Then, for  $h \in \mathbf{R}$ ,

$$Y(t) - Y(t+h) = Y(t) - Y_{N_1}(t) + Y_{N_1}(t) - Y_{N_1}(t+h) + Y_{N_1}(t+h) - Y(t+h).$$

Since  $\lim_{h \rightarrow 0^+} Y_{N_1}(t) - Y_{N_1}(t+h) = 0$ , there exists  $h_0 > 0$  such that  $\forall h \in (0, h_0)$ ,

$$|Y_{N_1}(t) - Y_{N_1}(t+h)| \leq \frac{\varepsilon}{3}.$$

As a consequence,  $\forall h \in (0, h_0)$ ,  $|Y(t) - Y(t+h)| \leq \varepsilon$  and thus  $\lim_{h \rightarrow 0^+} Y(t) - Y(t+h) = 0$ .

$$\lim_{h \rightarrow 0^-} Y_{N_1}(t) - Y_{N_1}(t+h) = \lim_{h \rightarrow 0^-} \sum_{i=1}^{N_1} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{(t+h, t]}(V_i) = \gamma_{i_0} \Gamma_{i_0}^{-1/\alpha(V_{i_0})}, \text{ thus}$$

$$\lim_{h \rightarrow 0^-} Y_{N_1}(t) - Y_{N_1}(t+h) - \gamma_{i_0} \Gamma_{i_0}^{-1/\alpha(V_{i_0})} = 0.$$

Choose  $h_0 < 0$  such that  $\forall h \in (h_0, 0)$ ,

$$|Y(t) - Y(t+h) - \gamma_{i_0} \Gamma_{i_0}^{-1/\alpha(V_{i_0})}| \leq \varepsilon.$$

Thus,

$$\lim_{h \rightarrow 0^-} Y(t) - Y(t+h) = \gamma_{i_0} \Gamma_{i_0}^{-1/\alpha(V_{i_0})}.$$

2<sup>nd</sup> case :  $t \notin S$

Since  $\lim_{h \rightarrow 0} |Y_{N_0}(t+h) - Y_{N_0}(t)| = 0$ , there exists  $h_0 > 0$  such that  $\forall |h| < h_0$ ,

$$|Y_{N_0}(t) - Y_{N_0}(t+h)| \leq \frac{\varepsilon}{3}.$$

Using (5), one thus has, for  $|h| < h_0$ ,  $|Y(t) - Y(t+h)| \leq \varepsilon$ , and thus  $\lim_{h \rightarrow 0} |Y(t+h) - Y(t)| = 0$

■

**Note 1.** We have shown precisely that  $Y$  is càdlàg with set of jump points exactly equal to  $S$ . The jump at point  $V_i$  is of size  $\gamma_i \Gamma_i^{-1/\alpha(V_i)}$ .

### Proof of Lemma 3:

For  $j \geq 1$ ,  $k = 1, \dots, 2^j$ , let  $E_{k,j}$  denote the interval

$$E_{k,j} = \left[ \frac{k-1}{2^j} - \frac{1}{2^{1+(j+1)(1-\frac{1}{\sqrt{j}})}}, \frac{k-1}{2^j} + \frac{1}{2^{1+(j+1)(1-\frac{1}{\sqrt{j}})}} \right).$$

Let us show that  $\liminf_{j \rightarrow \infty} \bigcap_{k=1}^{2^j} \bigcup_{i=2^j}^{2^{j+1}-1} \{V_i \in E_{k,j}\} \subset \{\forall t \notin S, \delta(t) \leq 1\}$  first, and then that

$$\mathbb{P} \left( \liminf_{j \rightarrow \infty} \bigcap_{k=1}^{2^j} \bigcup_{i=2^j}^{2^{j+1}-1} \{V_i \in E_{k,j}\} \right) = 1. \text{ We denote } a_j = \frac{1}{2^{1+(j+1)(1-\frac{1}{\sqrt{j}})}}.$$

Assume that there exists  $J_0 \in \mathbf{N}$  such that for all  $j \geq J_0$ , and all  $k = 1, \dots, 2^j$ , we can fix  $i(k,j) \in [2^j, 2^{j+1} - 1]$  with  $V_{i(k,j)} \in E_{k,j}$ . Let  $t \notin S$ . For all  $j \geq J_0$ , there exists  $k(j) \in [1, 2^j]$  such that  $t \in E_{k(j),j}$ , because  $2a_j \geq \frac{1}{2^j}$ .

As a consequence,

$$|t - V_{i(k(j),j)}| \leq \frac{1}{2^{(j+1)(1-\frac{1}{\sqrt{j}})}} \leq \frac{1}{i(k(j),j)^{1-\frac{1}{\sqrt{j}}}}.$$

Hence  $-\frac{\log i(k(j),j)}{\log |t - V_{i(k(j),j)}|} \leq \frac{1}{1-\frac{1}{\sqrt{j}}}$ . This entails  $\liminf_{j \rightarrow \infty} -\frac{\log i(k(j),j)}{\log |t - V_{i(k(j),j)}|} \leq 1$ , and, since  $V_{i(k(j),j)}$  tends to  $t$ ,  $\delta(t) \leq 1$ .

Finally, distinguishing the cases  $k = 1, k = 2, \dots, 2^j - 1$  and  $k = 2^j$ , one estimates

$$\begin{aligned} \mathbb{P} \left( \bigcup_{k=1}^{2^j} \bigcap_{i=2^j}^{2^{j+1}-1} \{V_i \notin E_{k,j}\} \right) &\leq \sum_{k=1}^{2^j} \mathbb{P} \left( \bigcap_{i=2^j}^{2^{j+1}-1} \{V_i \notin E_{k,j}\} \right) \\ &\leq a_j + \left(1 - \left(\frac{2^j - 1}{2^j} - a_j\right)\right) + \sum_{k=2}^{2^j-1} (\mathbb{P}(\{V_1 \notin E_{k,j}\}))^{2^j} \\ &\leq 2a_j + \frac{1}{2^j} + (2^j - 2)(1 - 2a_j)^{2^j}. \end{aligned}$$

Since  $\sum_{j=1}^{+\infty} (2^j - 2)(1 - 2a_j)^{2^j} < +\infty$ , Borel-Cantelli lemma allows us to conclude. ■

#### Proof of Lemma 4:

Recall Note 1. Lemma 1 of [15] entails that, for all sequences  $V_{\phi(i)} \in \mathcal{R}_t$ , and all  $t \notin S$ ,

$$\begin{aligned} h_Y(t) &\leq \liminf_{i \rightarrow +\infty} \frac{\log |\Gamma_{\phi(i)}^{-\frac{1}{\alpha(V_{\phi(i)})}}|}{\log |V_{\phi(i)} - t|} \\ &= \liminf_{i \rightarrow +\infty} -\frac{1}{\alpha(V_{\phi(i)})} \frac{\log |\Gamma_{\phi(i)}|}{\log |V_{\phi(i)} - t|}. \end{aligned}$$

Since  $\alpha$  is continuous,  $\phi(i)$  tends to infinity, the sequences  $(V_{\phi(i)})_i$  converges to  $t$ , and almost surely  $(\frac{\Gamma_i}{i})_i$  tends to 1 when  $i$  tends to infinity, one obtains

$$h_Y(t) \leq \frac{1}{\alpha(t)} \liminf_{i \rightarrow +\infty} -\frac{\log |\phi(i)|}{\log |V_{\phi(i)} - t|}.$$

This inequality holds for all sequences  $V_{\phi(i)} \in \mathcal{R}_t$ , and thus,  $\forall t \notin S$ ,  $h_Y(t) \leq \frac{\delta(t)}{\alpha(t)}$  ■

#### Proof of Lemma 5:

Since  $g$  is càdlàg,  $h_g$  is non negative for all  $t$ . Integration increases pointwise regularity by at least one, and thus  $h_f(t) \geq 1$  for all  $t$ . An alternative direct proof goes as follows: let  $t \in (0, 1)$ , and  $h > 0$ . One computes

$$\begin{aligned} \frac{f(t+h) - f(t)}{h} - g(t^+) &= \frac{1}{h} \int_t^{t+h} (g(x) - g(t^+)) dx \\ &= \int_0^1 (g(t+sh) - g(t^+)) ds. \end{aligned}$$

$\forall s \in (0, 1), \lim_{h \rightarrow 0^+} (g(t + sh) - g(t^+)) = 0$ . Since  $g$  is càdlàg, it is bounded and thus:

$$\lim_{h \rightarrow 0^+} \frac{f(t + h) - f(t)}{h} = g(t^+).$$

Likewise, for  $h < 0$ ,

$$\frac{f(t + h) - f(t)}{h} - g(t^-) = \int_0^1 (g(t + sh) - g(t^-)) ds,$$

and

$$\lim_{h \rightarrow 0^-} \frac{f(t + h) - f(t)}{h} = g(t^-).$$

This entails  $h_f(t) \geq 1$  ■

**Proof of Lemma 6:**

Theorem 7 of [19] states that:

$$D(t) = A(t) + B(t),$$

where

$$A(t) = \int_0^t \sum_{i=1}^{+\infty} \gamma_i \frac{d\left(C_{\alpha(\cdot)}^{1/\alpha(\cdot)} \Gamma_i^{-1/\alpha(\cdot)}\right)}{dt}(s) \mathbf{1}_{[0,s]}(V_i) ds.$$

In addition, Lemma 8 of [19] entails that  $\left(\sum_{i=1}^N \gamma_i \frac{d\left(C_{\alpha(\cdot)}^{1/\alpha(\cdot)} \Gamma_i^{-1/\alpha(\cdot)}\right)}{dt}(s) \mathbf{1}_{[0,s]}(V_i)\right)_N$  converges to  $\sum_{i=1}^{+\infty} \gamma_i \frac{d\left(C_{\alpha(\cdot)}^{1/\alpha(\cdot)} \Gamma_i^{-1/\alpha(\cdot)}\right)}{dt}(s) \mathbf{1}_{[0,s]}(V_i)$  uniformly on  $[0, 1]$ . The same proof as the one of Lemma 2 shows that  $s \mapsto \sum_{i=1}^{+\infty} \gamma_i \frac{d\left(C_{\alpha(\cdot)}^{1/\alpha(\cdot)} \Gamma_i^{-1/\alpha(\cdot)}\right)}{dt}(s) \mathbf{1}_{[0,s]}(V_i)$  is càdlàg. Lemma 5 then entails that  $h_A(t) \geq 1$ . Since  $c > 1$ , it is thus sufficient to show that  $h_B(t) \geq \frac{\delta(t)}{\alpha(t)}$ .

Write  $B(t) = W(t) + Z(t)$  where

$$W(t) = \sum_{i=1}^{+\infty} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} \left( \Gamma_i^{-1/\alpha(V_i)} - i^{-1/\alpha(V_i)} \right) \mathbf{1}_{(V_i \leq t)} \quad (6)$$

and

$$Z(t) = \sum_{i=1}^{+\infty} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{(V_i \leq t)}. \quad (7)$$

Set  $I_{k,m} = \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right)$ ,  $N_{m,j,k} = \text{Card} \{V_i, i = 2^j, \dots, 2^{j+1} - 1, V_i \in I_{k,m}\}$ ,  $d_{k,m} = \max_{u \in I_{k,m}} \alpha(u)$

and  $C_0 = \max_{t \in [0,1]} C_{\alpha(t)}^{1/\alpha(t)}$ . Define

$$M_t^{m,j,k} = \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[0,t] \cap I_{k,m}}(V_i).$$

It is easily seen that

$$\sup_{(s,t) \in I_{k,m}^2} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t)}(V_i) \right| \leq 2 \sup_{t \in [0,1]} \left| M_t^{m,j,k} \right|. \quad (8)$$

For  $t \in (0,1)$ , let  $\alpha_n(t) = \sqrt{n} \left[ \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{V_i \leq t} - t \right]$  denote the empirical process, and  $w_n(a) = \sup_{|t-s| \leq a} |\alpha_n(t) - \alpha_n(s)|$  denote the oscillation modulus of  $\alpha_n$ . We apply Lemma 2.4 of [25] with

$$m \geq 3, a = 2^{-m}, s = m^{1/4} \sqrt{j}, n = 2^j, \delta = \frac{1}{2}.$$

This yields that there exists  $M_0 \in \mathbf{N}$  such that

$$\forall m \geq M_0, \exists j(m) \text{ with } m \leq j(m) \leq 2m \text{ such that } \forall j \geq j(m) : m^{1/4} \sqrt{j} \leq \sqrt{2^{j-m}}$$

and

$$\mathbf{P} \left( \sup_{0 \leq |t-s| \leq \frac{1}{2^m}} |\alpha_{2^j}(t) - \alpha_{2^j}(s)| > \frac{m^{1/4} \sqrt{j}}{\sqrt{2^m}} \right) \leq 256 \times 2^m e^{-\frac{j\sqrt{m}}{64}}. \quad (9)$$

We need to estimate  $N_{m,j,k}$  and  $\sup_{(s,t) \in I_{k,m}^2} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t)}(V_i) \right|$  for  $k = 0, \dots, 2^m - 1$  and  $j \geq m$ .

• Study of  $N_{m,j,k}$  for  $m \leq j \leq j(m) - 1$ :

Set  $X_i = \mathbf{1}_{\frac{k}{2^m} \leq V_i \leq \frac{k+1}{2^m}}$  and  $n = 2^j$ .  $(X_i)_i$  is an i.i.d. sequence of Bernoulli random variables with parameter  $p = \frac{1}{2^m}$ . For  $m \leq j \leq j(m) - 1$ , one has  $\sqrt{m}j \geq 2^{j-m} = np$ , and thus, for  $a > 0$ , using a classical bound on the sum of i.i.d. Bernoulli random variables,

$$\begin{aligned} \mathbf{P}(N_{m,j,k} > a\sqrt{m}j) &\leq \mathbf{P} \left( \sum_{i=2^j}^{2^{j+1}-1} X_i \geq a2^{j-m} \right) \\ &= \mathbf{P} \left( \sum_{i=2^j}^{2^{j+1}-1} X_i \geq anp \right) \\ &\leq \frac{1}{a^n} \\ &\leq \frac{1}{a^j}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \mathbf{P} \left( \bigcup_{j=m}^{j(m)-1} \bigcup_{k=0}^{2^m-1} \{N_{m,j,k} > a\sqrt{m}j\} \right) &\leq \sum_{j=m}^{2^m-1} \sum_{k=0}^{2^m-1} \frac{1}{a^j} \\ &\leq 2^m \sum_{j=m}^{+\infty} a^{-j} \\ &\leq \frac{a}{a-1} \left( \frac{2}{a} \right)^m. \end{aligned}$$



Choosing  $a = 3$ , Borel Cantelli lemma entails that

$$\mathbb{P} \left( \limsup_{m \rightarrow +\infty} \bigcup_{j=m}^{j(m)-1} \bigcup_{k=0}^{2^m-1} \{N_{m,j,k} > 3\sqrt{mj}\} \right) = 0.$$

- Study of  $N_{m,j,k}$  for  $j \geq j(m)$ :

$$\begin{aligned} \mathbb{P}(N_{m,j,k} > 2 \cdot 2^{j-m}) &= \mathbb{P} \left( \sqrt{2^j} \left( \alpha_{2^j} \left( \frac{k+1}{2^m} \right) - \alpha_{2^j} \left( \frac{k}{2^m} \right) \right) + 2^{j-m} \geq 2 \cdot 2^{j-m} \right) \\ &\leq \mathbb{P} \left( \sqrt{2^j} w_{2^j} \left( \frac{1}{2^m} \right) \geq 2^{j-m} \right) \\ &= \mathbb{P} \left( w_{2^j} \left( \frac{1}{2^m} \right) \geq \frac{\sqrt{2^{j-m}}}{\sqrt{2^m}} \right) \\ &\leq \mathbb{P} \left( w_{2^j} \left( \frac{1}{2^m} \right) \geq \frac{m^{1/4} \sqrt{j}}{\sqrt{2^m}} \right) \\ &\leq 256 \times 2^m e^{-\frac{j\sqrt{m}}{64}} \end{aligned}$$

using (9). As a consequence,

$$\begin{aligned} \mathbb{P} \left( \bigcup_{j=j(m)}^{+\infty} \bigcup_{k=0}^{2^m-1} \{N_{m,j,k} > 2 \cdot 2^{j-m}\} \right) &\leq \sum_{j=m}^{+\infty} 2^m (256 \times 2^m e^{-\frac{j\sqrt{m}}{64}}) \\ &\leq 256 \cdot 4^m \cdot \frac{e^{-\frac{m\sqrt{m}}{64}}}{1 - e^{-\frac{\sqrt{m}}{64}}}. \end{aligned}$$

Borel Cantelli lemma yields

$$\mathbb{P} \left( \limsup_{m \rightarrow +\infty} \bigcup_{j=j(m)}^{+\infty} \bigcup_{k=0}^{2^m-1} \{N_{m,j,k} \geq 2 \cdot 2^{j-m}\} \right) = 0.$$

- Study of  $\sup_{(s,t) \in I_{k,m}^2} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right|$  for  $m \leq j \leq j(m) - 1$ :

Almost surely, there exists  $m_0$  such that, for  $m \geq m_0$  and for  $m \leq j \leq j(m) - 1$ ,  $\forall k = 0, \dots, 2^m - 1$ ,  $N_{m,j,k} \leq 3\sqrt{mj}$ , and  $\forall (s, t) \in I_{k,m}^2$ ,  $\forall i = 2^j, \dots, 2^{j+1} - 1$ ,

$$\left| \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right| \leq \frac{C_0}{2^{\frac{j}{d_{k,m}}}}$$

thus

$$\sup_{(s,t) \in I_{k,m}^2} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right| \leq \frac{3\sqrt{mj}C_0}{2^{\frac{j}{d_{k,m}}}.$$

- Study of  $\sup_{(s,t) \in I_{k,m}^2} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} t_i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right|$  for  $j \geq j(m)$ :

We consider the events

$$E_{m,j,k} = \{N_{m,j,k} \leq 2 \cdot 2^{j-m}\},$$

$$F_{m,j,k} = \left\{ \sup_{(s,t) \in I_{k,m}^2} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} t_i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right| \geq \frac{j C_0 \sqrt{2 \cdot 2^{j-m}}}{2.2^{\frac{j}{d_{k,m}}}} \right\}$$

and

$$G_{m,j,k} = \left\{ \sup_{t \in [0,1]} |M_t^{m,j,k}| \geq \frac{j C_0 \sqrt{2 \cdot 2^{j-m}}}{2.2^{\frac{j}{d_{k,m}}}} \right\}.$$

Relation (8) entails that  $\mathbb{P}(F_{m,j,k} \cap E_{m,j,k}) \leq \mathbb{P}(G_{m,j,k} \cap E_{m,j,k})$ .

In the following computation,  $l$  corresponds to the number of terms  $V_i$  belonging to  $I_{k,m}$ , and  $n$  corresponds to the number of those  $V_i$  among them that contribute to the supremum of  $G_{m,j,k}$ .

$$G_{m,j,k} \cap E_{m,j,k} \subset \bigcup_{l=1}^{2 \cdot 2^{j-m}} \bigcup_{l_1, \dots, l_l \in [2^j, 2^{j+1}-1]} \left[ G_{m,j,k} \cap \left( \bigcap_{i \in \{l_1, \dots, l_l\}} V_i \in I_{k,m} \right) \cap \left( \bigcap_{i \notin \{l_1, \dots, l_l\}} V_i \notin I_{k,m} \right) \right].$$

Using independence of the  $V_i$ ,

$$\mathbb{P}(G_{m,j,k} \cap E_{m,j,k}) \leq \sum_{l=1}^{2 \cdot 2^{j-m}} \sum_{l_1, \dots, l_l \in [2^j, 2^{j+1}-1]} \mathbb{P} \left( G_{m,j,k} \cap \bigcap_{i \in \{l_1, \dots, l_l\}} V_i \in I_{k,m} \right) \mathbb{P} \left( \bigcap_{i \notin \{l_1, \dots, l_l\}} V_i \notin I_{k,m} \right).$$

Now,  $\mathbb{P} \left( \bigcap_{i \notin \{l_1, \dots, l_l\}} V_i \notin I_{k,m} \right) = (1 - \frac{1}{2^m})^{2^j - l}$ . Let us fix an order on the  $V_i$  belonging to  $I_{k,m}$ : there are  $l!$  possibilities, all equiprobable, and thus

$$\mathbb{P} \left( G_{m,j,k} \cap \bigcap_{i \in \{l_1, \dots, l_l\}} V_i \in I_{k,m} \right) = (l!) \mathbb{P} \left( G_{m,j,k} \cap \left( \frac{k}{2^m} < V_{l_1} < \dots < V_{l_l} < \frac{k+1}{2^m} \right) \right).$$

Let  $A_{k,m}^l$  denote the event  $\{\frac{k}{2^m} < V_{l_1} < \dots < V_{l_l} < \frac{k+1}{2^m}\}$ . Then

$$\begin{aligned} \mathbb{P}(G_{m,j,k} \cap A_{k,m}^l) &\leq \sum_{n=1}^l \mathbb{P} \left( A_{k,m}^l \cap \left| \sum_{i=1}^n \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} t_i^{-1/\alpha(V_i)} \right| \geq \frac{j C_0 \sqrt{2 \cdot 2^{j-m}}}{2.2^{\frac{j}{d_{k,m}}}} \right) \\ &\leq \sum_{n=1}^l \int_{\frac{k}{2^m} < x_1 < \dots < x_l < \frac{k+1}{2^m}} \mathbb{P} \left( \left| \sum_{i=1}^n \gamma_i C_{\alpha(x_i)}^{1/\alpha(x_i)} t_i^{-1/\alpha(x_i)} \right| \geq \frac{j C_0 \sqrt{n}}{2.2^{\frac{j}{d_{k,m}}}} \right) dx_1 \dots dx_l. \end{aligned}$$

The probability inside the integral above may be estimated with the help of Lemma 1.5 in [17]:

$$\mathbb{P} \left( \left| \sum_{i=1}^n \gamma_{l_i} \frac{C^{1/\alpha(x_i)}}{C_0} \frac{2^{\frac{j}{d_{k,m}}}}{l_i^{1/\alpha(x_i)}} \right| \geq \frac{j\sqrt{n}}{2} \right) \leq 2e^{-\frac{j^2}{8}}.$$

One then computes

$$\begin{aligned} \mathbb{P}(G_{m,j,k} \cap E_{m,j,k}) &\leq \sum_{l=1}^{2 \cdot 2^{j-m}} \sum_{l_1, \dots, l_l \in [2^j, 2^{j+1}-1]} \left(1 - \frac{1}{2^m}\right)^{2^j-l} (l!) \sum_{n=1}^l 2e^{-\frac{j^2}{8}} \int_{\frac{k}{2^m} < x_1 < \dots < x_l < \frac{k+1}{2^m}} dx_1 \dots dx_l \\ &\leq \sum_{l=1}^{2 \cdot 2^{j-m}} \frac{(2^j)!}{(l!)(2^j-l)!} \left(1 - \frac{1}{2^m}\right)^{2^j-l} (l!) 2le^{-\frac{j^2}{8}} \frac{1}{2^{ml}} \frac{1}{l!} \\ &\leq 2e^{-\frac{j^2}{8}} \sum_{l=1}^{2 \cdot 2^{j-m}} l \frac{(2^j)!}{(l!)(2^j-l)!} \left(1 - \frac{1}{2^m}\right)^{2^j-l} \frac{1}{2^{ml}} \\ &\leq 4 \cdot 2^{j-m} e^{-\frac{j^2}{8}} \sum_{l=1}^{2 \cdot 2^{j-m}} \frac{(2^j)!}{(l!)(2^j-l)!} \left(1 - \frac{1}{2^m}\right)^{2^j-l} \frac{1}{2^{ml}} \\ &\leq 4 \cdot 2^{j-m} e^{-\frac{j^2}{8}}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \mathbb{P} \left( \bigcup_{j=j(m)}^{+\infty} \bigcup_{k=0}^{2^m-1} (F_{m,j,k} \cap E_{m,j,k}) \right) &\leq 4 \sum_{j=m}^{+\infty} 2^j e^{-\frac{j^2}{8}} \\ &\leq \sum_{j=m}^{+\infty} e^{-j} \end{aligned}$$

for  $m \geq 100$ . Borel Cantelli lemma entails that

$$\mathbb{P} \left( \limsup_{m \rightarrow +\infty} \bigcup_{j=j(m)}^{+\infty} \bigcup_{k=0}^{2^m-1} F_{m,j,k} \right) = 0.$$

• Computation of the Hölder exponent:

Let  $t \in (0, 1)$ ,  $t \notin S$  and let  $U$  be an open interval of  $(0, 1)$  containing  $t$ . Denote  $d_U = \max_{t \in U} \alpha(t)$ . If  $\delta(t) = 0$ , then  $h_Y(t) = 0$  and the formula holds. Suppose now  $\delta(t) > 0$ . Let  $\varepsilon > 0$  be such that  $\delta(t) > \varepsilon$  and  $\frac{1}{d_U} - \frac{1}{2} > \varepsilon$ . There exists  $i_0 \in \mathbf{N}$  such that  $\forall i \geq i_0$ ,  $|t - V_i| \geq \frac{1}{i^{\delta(t)-\varepsilon}}$ . Choose  $m$  large enough so that  $\frac{1}{2^m} < \min\{|t - V_i|, i = 1, \dots, i_0\}$ . Let  $j_0 = [m(\delta(t) - \varepsilon)]$ . Increasing  $m$  if necessary, we may and will assume that  $i_0 \leq 2^{j_0}$ , and  $\forall j \geq j_0$ ,  $j \leq 2^{j\varepsilon}$ . Let  $s \in (0, 1)$  be such that  $\frac{1}{2^{m+2}} \leq |t - s| < \frac{1}{2^{m+1}}$ . There exists  $k \in \{0, \dots, 2^m - 1\}$  such that  $(t, s) \in I_{k,m}^2$ . Increasing again  $m$  if necessary, we may assume that  $I_{k,m} \subset U$ . Then  $d_{k,m} \leq d_U$ , and, for  $i \leq 2^{j_0}$ ,  $\mathbf{1}_{[s,t)}(V_i) = 0$ . One computes:

$$\begin{aligned}
|Z(t) - Z(s)| &= \left| \sum_{i=1}^{+\infty} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right| \\
&= \left| \sum_{j=j_0}^{+\infty} \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right| \\
&\leq \sum_{j=j_0}^{j(m)} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right| + \sum_{j=j(m)}^{+\infty} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} i^{-1/\alpha(V_i)} \mathbf{1}_{[s,t]}(V_i) \right| \\
&\leq \sum_{j=j_0}^{j(m)} \frac{3\sqrt{m}j}{2^{\frac{j}{d_{k,m}}}} + \sum_{j=j(m)}^{+\infty} \frac{jC_0\sqrt{2 \cdot 2^{j-m}}}{2^{\frac{j}{d_{k,m}}}} \\
&\leq 2(3 + \sqrt{2}C_0)\sqrt{m} \sum_{j=j_0}^{+\infty} \frac{j\sqrt{2^{j-m}}}{2^{\frac{j}{d_{k,m}}}} \\
&\leq \frac{6(1 + C_0)\sqrt{m}}{\sqrt{2^m}} \sum_{j=j_0}^{+\infty} 2^{j(\varepsilon + \frac{1}{2} - \frac{1}{d_U})} \\
&\leq \frac{6(1 + C_0)\sqrt{m}}{\sqrt{2^m}} \frac{2^{j_0(\varepsilon + \frac{1}{2} - \frac{1}{d_U})}}{1 - 2^{\varepsilon + \frac{1}{2} - \frac{1}{d_U}}} \\
&\leq K_{U,\varepsilon,C_0}\sqrt{m}2^m \left[ -\frac{1}{2} + (\delta(t) - \varepsilon)(\varepsilon + \frac{1}{2} - \frac{1}{d_U}) \right] \\
&\leq K_{U,\varepsilon,C_0}\sqrt{|\log|t - s||} |t - s|^{(\delta(t) - \varepsilon)(\frac{1}{d_U} - \varepsilon) + \frac{1}{2} - \frac{1}{2}(\delta(t) - \varepsilon)}.
\end{aligned}$$

Since  $\frac{1}{2} - \frac{1}{2}(\delta(t) - \varepsilon) > 0$ ,

$$|Z(t) - Z(s)| \leq K_{U,\varepsilon,C_0}\sqrt{|\log|t - s||} |t - s|^{(\delta(t) - \varepsilon)(\frac{1}{d_U} - \varepsilon)}.$$

Let us now study  $W$  (recall (6)): there exists  $i_0$  such that, for all  $i \geq i_0$ ,

$$\begin{aligned}
\left| \Gamma_i^{-1/\alpha(V_i)} - i^{-1/\alpha(V_i)} \right| &\leq \frac{K_{c,d}}{i^{\frac{1}{d_{k,m}} + \frac{1}{2}}} \\
&\leq \frac{K_{c,d}}{i^{\frac{1}{d_U} + \frac{1}{2}}}.
\end{aligned}$$

One then computes:

$$\begin{aligned}
|W(t) - W(s)| &= \left| \sum_{j=j_0}^{+\infty} \sum_{i=2^j}^{2^{j+1}-1} \gamma_i C_{\alpha(V_i)}^{1/\alpha(V_i)} (\Gamma_i^{-1/\alpha(V_i)} - i^{-1/\alpha(V_i)}) \mathbf{1}_{[s,t]}(V_i) \right| \\
&\leq \sum_{j=j_0}^{j(m)} \frac{3K_{c,d} C_0 \sqrt{m} j}{2^{j(\frac{1}{2} + \frac{1}{d_U})}} + \sum_{j=j(m)}^{+\infty} \frac{K_{c,d} C_0 2 \cdot 2^{j-m}}{2^{j(\frac{1}{2} + \frac{1}{d_U})}} \\
&\leq K \sqrt{m} \sum_{j=j_0}^{+\infty} 2^{j(\varepsilon - \frac{1}{2} - \frac{1}{d_U})} + \frac{K}{2^m} \sum_{j=j_0}^{+\infty} 2^{j(\frac{1}{2} - \frac{1}{d_U})} \\
&\leq K \sqrt{|\log |t - s||} \frac{2^{m(\delta(t) - \varepsilon)(\varepsilon - \frac{1}{2} - \frac{1}{d_U})}}{1 - 2^{\varepsilon - \frac{1}{2} - \frac{1}{d_U}}} + K |t - s| \frac{2^{m(\delta(t) - \varepsilon)(\frac{1}{2} - \frac{1}{d_U})}}{1 - 2^{\frac{1}{2} - \frac{1}{d_U}}} \\
&\leq K \sqrt{|\log |t - s||} |t - s|^{(\delta(t) - \varepsilon)(\frac{1}{2} + \frac{1}{d_U} - \varepsilon)} + K |t - s|^{1 + (\delta(t) - \varepsilon)(\frac{1}{d_U} - \frac{1}{2})}.
\end{aligned}$$

Gathering our results, we have shown that:

$$|B(t) - B(s)| \leq K \sqrt{|\log |t - s||} |t - s|^{(\delta(t) - \varepsilon)(\frac{1}{d_U} - \varepsilon)}.$$

In other words,  $h_B(t) \geq \frac{\delta(t)}{d_U}$  for any open interval  $U$  containing  $t$ . Letting the diameter of  $U$  go to 0, one gets  $h_B(t) \geq \frac{\delta(t)}{\alpha(t)}$ . ■

**Proof of Lemma 7:**

Lemma 2 entails that  $Y$  is almost surely a càdlàg process. Thus, for all  $t \in (0, 1)$ ,  $h_Y(t) \geq 0$ . ■

**Proof of Lemma 8:**

We seek to compute the Hausdorff dimension of  $F_0 = \{t \in [0, 1] \setminus S : \delta(t) = 0\} \cup S$ . Let  $E_\gamma = \limsup_{j \rightarrow +\infty} \bigcup_{i=2^j}^{2^{j+1}-1} \left[ V_i - i^{-\frac{1}{\gamma \alpha(V_i)}}, V_i + i^{-\frac{1}{\gamma \alpha(V_i)}} \right]$ . Since  $E_\gamma \subset \{t : h_Y(t) \leq \gamma\}$ ,

$$\{t \in [0, 1] \setminus S : \delta(t) = 0\} \subset \bigcap_{\gamma > 0} E_\gamma.$$

Now,  $\left( \bigcup_{i \geq 1} \left[ V_i - i^{-\frac{1}{\gamma d}}, V_i + i^{-\frac{1}{\gamma d}} \right] \right)_i$  is a covering of  $E_\gamma$ , and thus  $\dim_H(E_\gamma) \leq \gamma d$ . As a consequence,  $\dim_H(\{t \in [0, 1] \setminus S : \delta(t) = 0\}) = 0$ . Since  $\dim_H(S) = 0$ , we find that  $f_H(0) = 0$ .

**Proof of Lemma 9:**

Following [4], set  $\lambda_i = \frac{1}{1 + \Gamma_i}$ . For the system of points  $\mathcal{P} = \{(V_i, \lambda_i)\}_{i \geq 1}$  and  $t \in [0, 1]$ , define the approximation rate of  $t$  by  $\mathcal{P}$  as

$$\delta_t(\mathcal{P}) = \sup\{\delta \geq 1 : t \text{ belongs to an infinite number of balls } B(V_i, \lambda_i^\delta)\}.$$

Let us show that  $\delta_t(\mathcal{P}) = \frac{1}{\delta(t)}$ .

In that view, note first that, since  $\lim_{i \rightarrow +\infty} \frac{\Gamma_i}{i} = 1$  almost surely,

$$\delta(t) = \inf_{(V_{\phi(n)})_n \in \mathcal{R}_t} \liminf_{i \in \mathbf{N}} -\frac{\log |1 + \Gamma_{\phi(i)}|}{\log |V_{\phi(i)} - t|}.$$

Let  $\delta \geq 1$  be such that  $t$  belongs to an infinite number of balls  $B(V_i, \lambda_i^\delta)$ . There exists  $\phi : \mathbf{N} \rightarrow \mathbf{N}$  such that  $(V_{\phi(n)})_n \in \mathcal{R}_t$  and  $|t - V_{\phi(i)}| \leq \lambda_{\phi(i)}^\delta$ , *i.e.*  $-\frac{\log |1 + \Gamma_{\phi(i)}|}{\log |V_{\phi(i)} - t|} \leq \frac{1}{\delta}$ . As a consequence  $\delta(t) \leq \frac{1}{\delta}$ . Taking the supremum over all admissible  $\delta$ , one gets  $\delta_t(\mathcal{P}) \leq \frac{1}{\delta(t)}$ .

For the reverse inequality, consider two cases:

- $\delta(t) = 1$ : since  $\delta_t(\mathcal{P}) \geq 1$ , one gets  $\delta_t(\mathcal{P}) \geq \frac{1}{\delta(t)}$ ;
- $\delta(t) < 1$ : choose  $\varepsilon > 0$  such that  $\delta(t) + \varepsilon \leq 1$ . There exists  $\phi : \mathbf{N} \rightarrow \mathbf{N}$  such that for all large enough  $i \in \mathbf{N}$ ,  $-\frac{\log |1 + \Gamma_{\phi(i)}|}{\log |V_{\phi(i)} - t|} \leq \delta(t) + \varepsilon$ , *i.e.*  $|t - V_{\phi(i)}| \leq \frac{1}{(1 + \Gamma_{\phi(i)})^{\delta(t) + \varepsilon}}$ . By definition, this entails  $\delta_t(\mathcal{P}) \geq \frac{1}{\delta(t) + \varepsilon}$ , and finally  $\delta_t(\mathcal{P}) \geq \frac{1}{\delta(t)}$  by letting  $\varepsilon$  go to 0.

We now apply Theorem 21 of [4]: since  $hd \leq 1$ , one has  $h\alpha(t) \leq 1$  for all  $t \in (0, 1)$ . The function  $t \mapsto \frac{1}{h\alpha(t)}$  is continuous and thus

$$\dim_H(\{t \in (0, 1) : \delta_t(\mathcal{P}) = \frac{1}{h\alpha(t)}\}) = \sup\{h\alpha(t), t \in (0, 1)\} = hd$$

*i.e.*  $f_H(h) = hd$  ■

### Proof of Lemma 10:

Since  $hc < 1 < hd$  and  $\alpha$  is  $C^1$ , there exist  $(t_0, t_1) \in (0, 1)^2$  such that  $h\alpha(t_0) = 1$  and  $h\alpha(t) \leq 1$  for all  $t \in I := (t_1, t_0)$  or  $(t_0, t_1)$ .

Define  $g : t \mapsto \min(1, \frac{1}{h\alpha(t)})$ . Theorem 21 in [4] yields:

$$\begin{aligned} \dim_H(\{t \in I : \delta_t(\mathcal{P}) = g(t)\}) &= \sup\{\frac{1}{g(t)}, t \in I\} \\ &= \sup\{h\alpha(t), t \in I\} \\ &= h\alpha(t_0) \\ &= 1. \end{aligned}$$

One gets that  $f_H(h) \geq \dim_H(\{t \in I : \delta(t) = h\alpha(t)\}) = 1$ , *i.e.*  $f_H(h) = 1$ . ■

### Proof of Lemma 11:

By definition and Lemma 3,

$$F_{1/c} = \{t \in [0, 1] : \delta(t) = \frac{\alpha(t)}{c}\} = \{t \in [0, 1] : \alpha(t) = c\} \cap \{t \in [0, 1] : \delta(t) = 1\}.$$

Set  $E = \{t \in [0, 1] : \alpha(t) = c\}$ ,  $E_0 = \{t \in [0, 1] : \delta(t) < 1\}$  and  $E_1 = \{t \in [0, 1] : \delta(t) = 1\}$ . Then  $[0, 1] = E_0 \cup E_1$ ,  $F_{1/c} = E \cap E_1$  and thus  $f_H(\frac{1}{c}) \leq \dim_H(E)$ .

Now, if  $\dim_H(E) = 0$ , the lemma holds true since  $F_{1/c}$  is not empty and thus  $f_H(1/c) \geq 0$ . Suppose then that  $\dim_H(E) > 0$ . Choose  $s < \dim_H(E)$ . This implies that  $\mathcal{H}^s(E) = +\infty$ , and Theorem 4.10 in [10] entails that there exist a compact set  $E_c \subset E$  such that  $0 < \mathcal{H}^s(E_c) < +\infty$ .

Set  $\mu_s(\cdot) = \mathcal{H}^s(E_c \cap \cdot)$ . This is a finite and positive Borel measure on  $[0, 1]$ . Theorem 3.7 in [18], along with Lemmas 4 and 6, entail that for all  $t \in (0, 1)$ ,  $\mathbf{P}(t \in E_1) = 1$  and thus  $\mathbf{P}(t \in E_c \cap E_1) = \mathbf{1}_{t \in E_c}$ .

One computes:

$$\begin{aligned} \mathbf{E}[\mu_s(E_0)] &= \mathbf{E}\left[\int_0^1 \mathbf{1}_{t \in E_0} \mu_s(dt)\right] \\ &= \int_0^1 \mathbf{E}[\mathbf{1}_{t \in E_0}] \mu_s(dt) \\ &= \int_{E_c} \mathbf{P}(t \in E_0) \mu_s(dt) \\ &= 0. \end{aligned}$$

Thus,  $\mu_s(E_0)$  is a positive random variable with vanishing expectation: almost surely,  $\mu_s(E_0) = 0$ . Since  $\mu_s(E_c) = \mu_s(E_c \cap E_0) + \mu_s(E_c \cap E_1)$ , one obtains that, almost surely,  $\mu_s(E_c) = \mu_s(E_c \cap E_1)$ .

Now,  $\mathcal{H}^s(E \cap E_1) \geq \mathcal{H}^s(E_c \cap E_1) = \mathcal{H}^s(E_c) > 0$ . Thus  $\dim_H(E \cap E_1) \geq s$ ,  $\forall s \leq \dim_H(E)$  and  $\dim_H(E \cap E_1) \geq \dim_H(E)$ . ■

### Proof of Lemma 12:

For all  $t \in [0, 1]$ ,  $\alpha(t) \geq c$  and almost surely, for all  $t \in [0, 1]$ ,  $\delta(t) \leq 1$ , thus, almost surely, for all  $t \in [0, 1]$ ,  $h_Y(t) \leq \frac{1}{c}$ . As a consequence, for  $h > \frac{1}{c}$ ,  $F_h = \emptyset$  and  $f_H(h) = -\infty$ . ■

## 3 Large deviation and Legendre multifractal spectra

We compute in this section the large deviation and Legendre multifractal spectra of the process  $B$  on an interval. Recall that we consider the process on  $[0, 1]$ , and that the large deviation multifractal spectrum of a process  $X$  on  $[0, 1]$  is the (random) function  $f_g$  defined on  $\mathbf{R}$  by

$$f_g(\beta) = \lim_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n},$$

where, for a positive integer  $n$  and  $\varepsilon > 0$ ,

$$N_n^\varepsilon(\beta) = \#\{j \in \{0, \dots, n-1\} : \beta - \varepsilon \leq \frac{\log |X(\frac{j+1}{n}) - X(\frac{j}{n})|}{-\log n} \leq \beta + \varepsilon\}.$$

Other large deviation multifractal spectra can be defined by replacing the increments  $X(\frac{j+1}{n}) - X(\frac{j}{n})$  by other measures of the variation of  $X$ , such as its oscillations, but we will not consider these in this work. We do not recall the definition of the Legendre multifractal spectrum, and refer the reader to [10, 20] instead.

We shall denote

$$\begin{aligned} P_n^j &= \mathbf{P}\left(\beta - \varepsilon \leq \frac{\log |Y(\frac{j+1}{n}) - Y(\frac{j}{n})|}{-\log n} \leq \beta + \varepsilon\right) \\ &= \mathbf{P}\left(\frac{1}{n^{\beta+\varepsilon}} \leq |Y(\frac{j+1}{n}) - Y(\frac{j}{n})| \leq \frac{1}{n^{\beta-\varepsilon}}\right). \end{aligned}$$

Set also

$$X_j = \mathbf{1}_{\{\frac{1}{n\beta+\varepsilon} \leq |B(\frac{j+1}{n}) - B(\frac{j}{n})| \leq \frac{1}{n\beta-\varepsilon}\}}$$

which follows a Bernoulli law with parameter  $P_n^j$ . Clearly,

$$N_n^\varepsilon(\beta) = \sum_{j=1}^n X_j.$$

For  $U$  an open interval of  $(0, 1)$ , we write

$$J_n(U) = \{j : \frac{j}{n} \in U\}.$$

There exists a constant  $K_U > 0$  such that, for  $n$  large enough,

$$\#J_n(U) \geq K_U n.$$

Finally, we will make use of the characteristic function of  $B$ , which reads [19]:

$$\mathbb{E} \left( \exp \left( i \sum_{j=1}^m \theta_j B(t_j) \right) \right) = \exp \left( - \int \left| \sum_{j=1}^m \theta_j 1_{[0, t_j]}(s) \right|^{\alpha(s)} ds \right). \quad (10)$$

where  $m \in \mathbb{N}$ ,  $(\theta_1, \dots, \theta_m) \in \mathbf{R}^m$ ,  $(t_1, \dots, t_m) \in \mathbf{R}^m$ .

### 3.1 Main result

The large deviation and Legendre multifractal spectra of  $B$  are described by the following theorem:

**Theorem 13.** *With probability one, the large deviation and Legendre multifractal spectra of  $B$  satisfy:*

$$f_g(\beta) = f_l(\beta) = \begin{cases} -\infty & \text{for } \beta < 0; \\ \beta d & \text{for } \beta \in [0, \frac{1}{d}]; \\ 1 & \text{for } \beta \in (\frac{1}{d}, \frac{1}{c}]; \\ 1 + \frac{1}{c} - \beta & \text{for } \beta \in (\frac{1}{c}, 1 + \frac{1}{c}]; \\ -\infty & \text{for } \beta > 1 + \frac{1}{c}. \end{cases} \quad (11)$$

The fact that  $f_l = f_g$  stems from the general result that  $f_l$  is always the concave hull of  $f_g$  when the set  $\{\beta : f_g(\beta) \geq 0\}$  is bounded. The part concerning  $f_g$  in Theorem 13 follows from a series of lemmas that are proven in the next sections.

We note in passing that, comparing with Theorem 1, we see that the weak multifractal formalism holds for  $B$ , but the strong one does not, that is,  $f_H \leq f_g$  and  $f_H \neq f_g$ . The decreasing part with slope -1 for “large” exponents present in  $f_g$  but not in  $f_H$  is a common phenomenon when variations are measured with increments.

In order to prove Theorem 13, we will first show in each case of (11) that the equality holds true for any given  $\beta$  with probability one. Permuting “for all  $\beta$ ” and “almost surely” will then often be achieved thanks to the two following general simple but useful lemmas on the large deviation spectrum, which are of independent interest.



**Lemma 14.** *The large deviation spectrum of any real function is an upper semicontinuous function.*

*Proof.* Let  $f_g$  be the large deviation spectrum of a real function. Consider  $\beta \in \mathbf{R}$ ,  $(x_j)_{j \geq 1}$  a sequence such that  $\lim_{j \rightarrow +\infty} x_j = \beta$ , and set  $\varepsilon_j = \sup_{k \geq j} |\beta - x_k|$ .

Note first that

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} = \lim_{j \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \frac{\log N_n^{\varepsilon_j}(\beta)}{\log n}.$$

For all  $j \geq 1$  and all  $l \geq j$ ,  $N_n^{2\varepsilon_j}(\beta) \geq N_n^{\varepsilon_j}(x_j) \geq N_n^{\varepsilon_l}(x_j)$ . As a consequence,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^{2\varepsilon_j}(\beta)}{\log n} \geq \liminf_{n \rightarrow +\infty} \frac{\log N_n^{\varepsilon_l}(x_j)}{\log n}.$$

Letting  $l$  tend to infinity, one gets  $\liminf_{n \rightarrow +\infty} \frac{\log N_n^{2\varepsilon_j}(\beta)}{\log n} \geq f_g(x_j)$  and letting  $j$  tend to infinity one finally obtains

$$f_g(\beta) \geq \limsup_{j \rightarrow +\infty} f_g(x_j).$$

□

**Lemma 15.** *Assume that there exist four functions  $\underline{h}, \bar{h}, \underline{g}, \bar{g}$  with  $\lim_{u \rightarrow 0} \underline{g}(u) = \lim_{u \rightarrow 0} \bar{g}(u) = 0$  such that, for all  $\beta$  in some interval  $I$  and all sufficiently small  $\varepsilon > 0$ , almost surely*

$$\underline{h}(\beta) + \underline{g}(\varepsilon) \leq \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq \bar{h}(\beta) + \bar{g}(\varepsilon).$$

*Then, almost surely, for all  $\beta$  in  $I$ ,  $\underline{h}(\beta) \leq f_g(\beta) \leq \bar{h}(\beta)$ .*

*Proof.* Define

$$M_n(\beta_1, \beta_2) = \#\{j \in \{0, \dots, n-1\} : \beta_1 \leq \frac{\log |X(\frac{j+1}{n}) - X(\frac{j}{n})|}{-\log n} \leq \beta_2\}.$$

Then, for  $\beta_1 < \beta < \beta_2$ ,

$$N_n^{(\beta_2 - \beta) \vee (\beta - \beta_1)}(\beta) \leq M_n(\beta_1, \beta_2) \leq N_n^{(\beta_2 - \beta) \wedge (\beta - \beta_1)}(\beta)$$

and

$$f_g(\beta) = \inf_{(\beta_1, \beta_2) \in \mathbb{Q}^2, \beta_1 < \beta < \beta_2} \liminf_{n \rightarrow +\infty} \frac{\log M_n(\beta_1, \beta_2)}{\log n}.$$

The set

$$A = \bigcup_{(\beta_1, \beta_2) \in \mathbb{Q}^2} M(\beta_1, \beta_2)$$

is countable, and, thus  $f_g$  is obtained by a countable infimum for all  $\beta \in I$ . This yields the result.

□

## 3.2 Preliminary lemmas

### 3.2.1 Statements

Define  $H(t) = H_{\lambda,p}(t) = \lambda t - \log(1 - p + pe^t)$ .

**Lemma 16.** *If  $0 < p < \lambda < 1$ , then*

$$\sup_{t>0} H(t) = \lambda \log\left(\frac{\lambda}{p}\right) + (1 - \lambda) \log\left(\frac{1 - \lambda}{1 - p}\right).$$

**Lemma 17.** *If  $0 < \lambda < p < 1$ , then*

$$\sup_{t<0} H(t) = \lambda \log\left(\frac{\lambda}{p}\right) + (1 - \lambda) \log\left(\frac{1 - \lambda}{1 - p}\right).$$

**Lemma 18.** *If  $p = p(n) = Kn^b$  and  $\lambda = \lambda(n) = n^a$ , where  $K > 0$  and  $0 > a > b$ , then there exists  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$ ,*

$$\sup_{t>0} H(t) \geq \frac{(a - b)}{2} n^a \log n.$$

**Lemma 19.** *If  $p = p(n) = K_1 n^b$  and  $\lambda = \lambda(n) = K_2 n^a$ , where  $K_1 > 0$ ,  $K_2 > 0$  and  $0 > b > a$ , then there exists  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$ ,*

$$\sup_{t<0} H(t) \geq \frac{K_1}{4} n^b.$$

**Lemma 20.** *If  $p = p(n) = Kn^b$  and  $\lambda = \lambda(n) = n^a$ , where  $K > 0$  and  $0 > a > b$ , then there exists  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$ ,*

$$\inf_{t>0} e^{-\lambda t n} (1 - p + pe^t)^n \leq e^{-\frac{(a-b)}{2} n^{1+a} \log n}.$$

**Lemma 21.** *If  $p = p(n) = K_1 n^b$  and  $\lambda = \lambda(n) = K_2 n^a$ , where  $K_1 > 0$ ,  $K_2 > 0$  and  $0 > b > a$ , then there exists  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$ ,*

$$\inf_{t<0} e^{-\lambda t n} (1 - p + pe^t)^{K_U n} \leq e^{-\frac{K_1 K_U}{4} n^{1+b}}.$$

**Lemma 22.** *Assume there exist  $b \in (-1, 0)$  and  $K > 0$  such that, for all  $n \geq n_0$  and for all  $j \in \llbracket 1, n \rrbracket$ ,  $P_n^j \leq Kn^b$ . Then, almost surely,*

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq 1 + b.$$

**Lemma 23.** *Assume there exist an open interval  $U$ , a real  $b \in (-1, 0)$  and  $K > 0$  such that, for all  $n \geq n_0$  and for all  $j \in J_n(U)$ ,  $P_n^j \geq Kn^b$ . Then, almost surely,*

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \geq 1 + b.$$

### 3.2.2 Proofs

*Proof of Lemma 16.*  $\sup_{t>0} H(t) = H(t_0)$  where  $t_0 = \log \left( \frac{\lambda(1-p)}{p(1-\lambda)} \right)$ . □

*Proof of Lemma 17.*  $\sup_{t<0} H(t) = H(t_0)$  where  $t_0 = \log \left( \frac{\lambda(1-p)}{p(1-\lambda)} \right)$ . □

*Proof of Lemma 18.* With  $t_0 = \log \left( \frac{\lambda(1-p)}{p(1-\lambda)} \right)$ , one has, for  $n$  large enough,

$$\begin{aligned} H(t_0) &= n^a \log \left( \frac{n^{a-b}}{K} \right) + (1 - n^a) \log \left( \frac{1 - n^a}{1 - Kn^b} \right) \\ &= n^a \log \left( \frac{n^{a-b}}{K} \right) + (1 - n^a) \log \left( 1 + \frac{Kn^b - n^a}{1 - Kn^b} \right) \\ &\geq n^a \log \left( \frac{n^{a-b}}{K} \right) + 2(1 - n^a) \left( \frac{Kn^b - n^a}{1 - Kn^b} \right) \\ &= (a - b)n^a \log n + n^a \left[ 2(1 - n^a) \frac{Kn^{b-a} - 1}{1 - Kn^b} - \log K \right]. \end{aligned}$$

Since  $\lim_{n \rightarrow +\infty} 2(1 - n^a) \frac{Kn^{b-a} - 1}{1 - Kn^b} - \log K = -2 - \log K$ , there exists  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$ ,

$$\frac{n^{-a} H(t_0)}{\log n} \geq (a - b) + \frac{1}{\log n} \left[ 2(1 - n^a) \frac{Kn^{b-a} - 1}{1 - Kn^b} - \log K \right] \geq \frac{(a - b)}{2}. \quad \square$$

*Proof of Lemma 19.* With  $t_0 = \log \left( \frac{\lambda(1-p)}{p(1-\lambda)} \right)$ , one has, for  $n$  large enough,

$$\begin{aligned} H(t_0) &= K_2 n^a \log \left( \frac{K_2 n^{a-b}}{K_1} \right) + (1 - K_2 n^a) \log \left( 1 + \frac{K_1 n^b - K_2 n^a}{1 - K_1 n^b} \right) \\ &\geq K_2 n^a \log \left( \frac{K_2 n^{a-b}}{K_1} \right) + \frac{1}{2} (1 - K_2 n^a) \left( \frac{K_1 n^b - K_2 n^a}{1 - K_1 n^b} \right) \\ &= n^b \left[ K_2 n^{a-b} \log \left( \frac{K_2 n^{a-b}}{K_1} \right) + \frac{1}{2} (1 - K_2 n^a) \frac{K_1 - K_2 n^{a-b}}{1 - K_1 n^b} \right] \end{aligned}$$

which yields the result since  $\lim_{n \rightarrow +\infty} K_2 n^{a-b} \log \left( \frac{K_2 n^{a-b}}{K_1} \right) + \frac{1}{2} (1 - K_2 n^a) \frac{K_1 - K_2 n^{a-b}}{1 - K_1 n^b} = \frac{K_1}{2}$ . □

*Proof of Lemma 20.* One has

$$\begin{aligned} \inf_{t>0} e^{-\lambda t n} (1 - p + p e^t)^n &= \inf_{t>0} e^{-n H(t)} \\ &= e^{-n \sup_{t>0} H(t)}. \end{aligned}$$

Lemma 18 then implies that, for  $n \geq n_0$ ,

$$\inf_{t>0} e^{-\lambda t n} (1 - p + p e^t)^n \leq e^{-\frac{(a-b)}{2} n^{1+a} \log n}. \quad \square$$

*Proof of Lemma 21.* Write

$$\inf_{t < 0} e^{-\lambda t n} (1 - p + pe^t)^{K_U n} = e^{-K_U n \sup_{t < 0} H(t)}$$

where  $H(t) = \frac{\lambda}{K_U} t - \log(1 - p + pe^t)$ . Lemma 19 ensures that, for  $n \geq n_0$ ,

$$\inf_{t < 0} e^{-\lambda t n} (1 - p + pe^t)^n \leq e^{-\frac{K_1 K_U}{4} n^{1+b}}.$$

□

*Proof of Lemma 22.* Fix  $a \in (b, 0)$ . Then, for all  $t > 0$ ,

$$\begin{aligned} \mathbf{P}(N_n^\varepsilon(\beta) \geq n^{1+a}) &= \mathbf{P}\left(e^{\sum_{j=1}^n X_j} \geq e^{tn^{1+a}}\right) \\ &\leq e^{-nt\lambda} \mathbf{E}\left[\prod_{j=1}^n e^{tX_j}\right] \end{aligned}$$

where  $\lambda = n^a$ . The  $X_j$  are independent and  $\mathbf{E}[e^{tX_j}] = 1 - P_n^j + P_n^j e^t$ , thus

$$\mathbf{P}(N_n^\varepsilon(\beta) \geq n^{1+a}) \leq e^{-nt\lambda} \prod_{j=1}^n (1 - P_n^j + P_n^j e^t), \quad \forall t > 0.$$

For  $t > 0$ , the function  $p \mapsto 1 - p + pe^t$  is increasing and so, by assumption on  $P_n^j$ ,

$$\mathbf{P}(N_n^\varepsilon(\beta) \geq n^{1+a}) \leq e^{-nt\lambda} (1 - p + pe^t)^n,$$

where  $p = Kn^b$ . Minimizing over  $t > 0$  and using Lemma 20, one gets

$$\forall n \geq n_0, \mathbf{P}(N_n^\varepsilon(\beta) \geq n^{1+a}) \leq e^{-\frac{(a-b)}{2} n^{1+a} \log n}$$

and thus

$$\sum_{n \in \mathbf{N}} \mathbf{P}(N_n^\varepsilon(\beta) \geq n^{1+a}) < +\infty.$$

The Borel-Cantelli lemma then ensures that, almost surely,

$$\exists n_0 \in \mathbf{N}, \forall n \geq n_0, N_n^\varepsilon(\beta) \leq n^{1+a}$$

or

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq 1 + a.$$

Since this inequality holds true for any  $a \in (b, 0)$  one has indeed that, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq 1 + b.$$

□

*Proof of Lemma 23.* Fix  $a < b$ . For all  $t < 0$ ,

$$\begin{aligned} \mathbf{P} \left( N_n^\varepsilon(\beta) \leq n^{1+a} \right) &\leq \mathbf{P} \left( \sum_{j \in J_n(U)} X_j \leq n^{1+a} \right) \\ &= \mathbf{P} \left( e^{\sum_{j \in J_n(U)} X_j} \geq e^{tn^{1+a}} \right) \\ &\leq e^{-tn^{1+a}} \prod_{j \in J_n(U)} (1 - P_n^j + P_n^j e^t). \end{aligned}$$

When  $t < 0$ , the function  $p \mapsto 1 - p + pe^t$  is decreasing. As a consequence, by assumption on  $P_n^j$  and with  $p = Kn^b$ ,  $\lambda = n^a$ , one has, for  $n$  large enough,

$$\begin{aligned} \mathbf{P} \left( N_n^\varepsilon(\beta) \leq n^{1+a} \right) &\leq e^{-nt\lambda} (1 - p + pe^t)^{\#J_n(U)} \\ &\leq e^{-nt\lambda} (1 - p + pe^t)^{K_U n}. \end{aligned}$$

Minimizing over  $t < 0$  and using Lemma 21, one gets

$$\forall n \geq n_0, \mathbf{P} \left( N_n^\varepsilon(\beta) \leq n^{1+a} \right) \leq e^{-\frac{K_1 K_U}{4} n^{1+b}}$$

and thus

$$\sum_{n \in \mathbf{N}} \mathbf{P} \left( N_n^\varepsilon(\beta) \leq n^{1+a} \right) < +\infty.$$

As in the proof of Lemma 22, this leads to

$$\forall a < b, \quad \text{almost surely,} \quad \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \geq 1 + a$$

and finally, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \geq 1 + b.$$

□

### 3.3 Estimates of $P_n^j$

For  $U$  an open interval, denote  $c_U = \inf_{t \in U} \alpha(t)$  and  $d_U = \sup_{t \in U} \alpha(t)$ . Set also  $t_j = \frac{j}{n}$ .

#### 3.3.1 Lemmas

**Lemma 24.** Assume  $\beta < \frac{1}{d}$ . Then,  $\exists K > 0$ ,  $\exists n_0 \in \mathbf{N}$  such that  $\forall n \geq n_0$ ,  $\forall j \in \llbracket 1, n \rrbracket$ ,

$$P_n^j \leq K n^{\alpha(t_j)\beta + \alpha(t_j)\varepsilon - 1}.$$

**Lemma 25.** Assume  $\beta < \frac{1}{d_U}$ . Then,  $\exists K > 0$ ,  $\exists n_0 \in \mathbf{N}$  such that  $\forall n \geq n_0$ ,  $\forall j \in J_n(U)$ ,

$$P_n^j \geq K n^{\alpha(t_j)\beta - \alpha(t_j)\varepsilon - 1}.$$

**Lemma 26.** Assume  $\beta > \frac{1}{c}$  and  $\varepsilon \in (0, \beta - \frac{1}{c})$ . Then,  $\exists K > 0$ ,  $\exists n_0 \in \mathbf{N}$  such that  $\forall n \geq n_0$ ,  $\forall j \in \llbracket 1, n \rrbracket$ ,

$$P_n^j \leq K n^{\frac{1}{\alpha(t_j)} + \varepsilon - \beta}.$$

**Lemma 27.** Assume  $\beta > \frac{1}{c_U}$  and  $\varepsilon \in (0, \beta - \frac{1}{c_U})$ . Then,  $\exists K > 0$ ,  $\exists n_0 \in \mathbf{N}$  such that  $\forall n \geq n_0$ ,  $\forall j \in J_n(U)$ ,

$$P_n^j \geq K n^{\frac{1}{\alpha(t_j)} + \varepsilon - \beta}.$$

### 3.3.2 Proofs

*Proof of Lemma 24.* Set  $\mu_j = \alpha(t_j)\beta + \alpha(t_j)\varepsilon - 1$ . Using the truncation inequality [21, Section 13, p. 209], one computes

$$\begin{aligned}
P_n^j &= \mathbb{P}\left(\frac{1}{n^{\beta+\varepsilon}} \leq |Y(t_{j+1}) - Y(t_j)| \leq \frac{1}{n^{\beta-\varepsilon}}\right) \\
&\leq \mathbb{P}\left(|Y(t_{j+1}) - Y(t_j)| \geq \frac{1}{n^{\beta+\varepsilon}}\right) \\
&\leq \frac{7}{n^{\beta+\varepsilon}} \int_0^{n^{\beta+\varepsilon}} \left(1 - e^{-\int_{t_j}^{t_j+1} |\xi|^{\alpha(x)} dx}\right) d\xi \\
&= 7 \int_0^1 \left(1 - e^{-\int_{t_j}^{t_j+1} n^{(\beta+\varepsilon)\alpha(x)} |v|^{\alpha(x)} dx}\right) dv \\
&\leq 7 \int_0^1 \int_{t_j}^{t_j+1} n^{(\beta+\varepsilon)\alpha(x)} |v|^{\alpha(x)} dx dv \\
&\leq 7 \int_{t_j}^{t_j+1} n^{(\beta+\varepsilon)\alpha(x)} dx \\
&= 7n^{\mu_j} \int_{t_j}^{t_j+1} n^{1+(\beta+\varepsilon)(\alpha(x)-\alpha(t_j))} dx.
\end{aligned}$$

Since  $\alpha$  is  $C^1$ , there exists a constant  $K$  such that, for all  $x \in (t_j, t_{j+1})$ ,

$$|\beta + \varepsilon| |\alpha(x) - \alpha(t_j)| \leq K|x - t_j| \leq \frac{K}{n}.$$

As a consequence,

$$\begin{aligned}
P_n^j &\leq 7n^{\mu_j} \int_{t_j}^{t_j+1} n^{1+\frac{K}{n}} dx \\
&= 7n^{\mu_j} n^{\frac{K}{n}} \\
&\leq K_1 n^{\mu_j}
\end{aligned}$$

for a constant  $K_1$ . □

*Proof of Lemma 25.* Set  $\mu = \frac{1}{2} \left(\frac{1}{n^{\beta+\varepsilon}} + \frac{1}{n^{\beta-\varepsilon}}\right)$  and  $\sigma = \frac{1}{2} \left(\frac{1}{n^{\beta-\varepsilon}} - \frac{1}{n^{\beta+\varepsilon}}\right)$ .

Choose a function  $\varphi$  that satisfies the following properties:

1.  $\text{supp}(\varphi) \subset [-1, 1]$ .

2.  $\varphi$  is even.
3.  $\varphi$  is  $\mathcal{C}^4$ .
4.  $\forall x \in \mathbf{R}, 0 \leq \varphi(x) \leq 1$ .
5.  $\varphi$  is not identically 0.

These properties imply in particular that  $\hat{\varphi}$  is real and even. In addition, for all  $x \in \mathbf{R}$ :

$$\varphi\left(-\frac{x+\mu}{\sigma}\right) + \varphi\left(\frac{x-\mu}{\sigma}\right) \leq \mathbf{1}_{[-n-\beta+\varepsilon, -n-\beta-\varepsilon]}(x) + \mathbf{1}_{[n-\beta-\varepsilon, n-\beta+\varepsilon]}(x).$$

Since the Fourier transform of  $\varphi\left(\frac{x-\mu}{\sigma}\right)$  is  $\sigma \exp(-i\mu\xi)\hat{\varphi}(\sigma\xi)$ , Parseval formula yields

$$\begin{aligned} P_n^j &\geq \int_{\mathbf{R}} \left( \varphi\left(-\frac{x+\mu}{\sigma}\right) + \varphi\left(\frac{x-\mu}{\sigma}\right) \right) \mathbf{P}(Y(t_{j+1}) - Y(t_j) \in dx) \\ &\geq \frac{1}{\pi} \sigma \int_{\mathbf{R}} \cos(\mu\xi) \hat{\varphi}(\sigma\xi) e^{-\int_{t_j}^{t_{j+1}} |\xi|^{\alpha(x)} dx} d\xi \\ &= \frac{2}{\pi} \int_0^{+\infty} \cos\left(\frac{\mu}{\sigma}\eta\right) \hat{\varphi}(\eta) e^{-\int_{t_j}^{t_{j+1}} \left|\frac{\eta}{\sigma}\right|^{\alpha(x)} dx} d\eta. \end{aligned}$$

Now,

$$\int_{\mathbf{R}} \cos\left(\frac{\mu}{\sigma}\eta\right) \hat{\varphi}(\eta) d\eta = \varphi\left(\frac{\mu}{\sigma}\right) = 0$$

since  $\mu > \sigma$ . One may thus write:

$$\begin{aligned} P_n^j &\geq \frac{2}{\pi} \int_0^{+\infty} \cos\left(\frac{\mu}{\sigma}\eta\right) \hat{\varphi}(\eta) \left( e^{-\int_{t_j}^{t_{j+1}} \left|\frac{\eta}{\sigma}\right|^{\alpha(x)} dx} - 1 \right) d\eta \\ &=: \frac{2}{\pi} n^{\alpha(t_j)\beta - \alpha(t_j)\varepsilon - 1} I_n^j, \end{aligned}$$

where  $I_n^j = I_{n,1}^j + I_{n,2}^j + I_{n,3}^j$  with

$$\begin{aligned} I_{n,1}^j &= \int_0^{+\infty} \hat{\varphi}(\eta) \left( \cos\left(\frac{\mu}{\sigma}\eta\right) - \cos(\eta) \right) n^{1-\alpha(t_j)\beta + \varepsilon\alpha(t_j)} \left( e^{-\int_{t_j}^{t_{j+1}} \left|\frac{\eta}{\sigma}\right|^{\alpha(x)} dx} - 1 \right) d\eta, \\ I_{n,2}^j &= \int_0^{+\infty} \hat{\varphi}(\eta) \cos(\eta) n^{1-\alpha(t_j)\beta + \varepsilon\alpha(t_j)} \left( e^{-\int_{t_j}^{t_{j+1}} \left|\frac{\eta}{\sigma}\right|^{\alpha(x)} dx} - 1 + \int_{t_j}^{t_{j+1}} \left|\frac{\eta}{\sigma}\right|^{\alpha(x)} dx \right) d\eta, \\ I_{n,3}^j &= - \int_0^{+\infty} \hat{\varphi}(\eta) \cos(\eta) n^{1-\alpha(t_j)\beta + \varepsilon\alpha(t_j)} \int_{t_j}^{t_{j+1}} \left|\frac{\eta}{\sigma}\right|^{\alpha(x)} dx d\eta. \end{aligned}$$

Let us show that there exists  $K_U > 0$  such that, for all  $j \in J_n(U)$ ,

$$n^{1-\alpha(t_j)\beta+\varepsilon\alpha(t_j)} \int_{t_j}^{t_{j+1}} \left| \frac{\eta}{\sigma} \right|^{\alpha(x)} dx \leq K_U(\eta^c + \eta^d).$$

Since  $\frac{1}{\sigma} = \frac{2n^{\beta+\varepsilon}}{n^{2\varepsilon}-1}$  and, there exists  $K$  such that, for all  $x \in (t_j, t_{j+1})$ ,  $|\alpha(x) - \alpha(t_j)| \leq \frac{K}{n}$ , one has  $|\sigma^{\alpha(t_j)-\alpha(x)}| \leq K$  and

$$\begin{aligned} n^{1-\alpha(t_j)\beta+\varepsilon\alpha(t_j)} \int_{t_j}^{t_{j+1}} \left| \frac{\eta}{\sigma} \right|^{\alpha(x)} dx &= \frac{n^{-\alpha(t_j)\beta+\varepsilon\alpha(t_j)}}{\sigma^{\alpha(t_j)}} n \int_{t_j}^{t_{j+1}} |\eta|^{\alpha(x)} \sigma^{\alpha(t_j)-\alpha(x)} dx \\ &\leq 2^{\alpha(t_j)} \frac{n^{2\varepsilon\alpha(t_j)}}{(n^{2\varepsilon}-1)^{\alpha(t_j)}} (\eta^c + \eta^d) n \int_{t_j}^{t_{j+1}} \sigma^{\alpha(t_j)-\alpha(x)} dx \\ &\leq K(\eta^c + \eta^d). \end{aligned}$$

As a consequence,

$$\begin{aligned} |I_{n,1}^j| &\leq \int_0^{+\infty} |\hat{\varphi}(\eta)| \left| \frac{\mu}{\sigma} - 1 \right| n^{1-\alpha(t_j)\beta+\varepsilon\alpha(t_j)} \int_{t_j}^{t_{j+1}} \left| \frac{\eta}{\sigma} \right|^{\alpha(x)} dx d\eta \\ &\leq K \left| \frac{\mu}{\sigma} - 1 \right| \int_0^{+\infty} |\hat{\varphi}(\eta)| (\eta^c + \eta^d) d\eta. \end{aligned}$$

One finally obtains that  $\sup_{j \in J_n(U)} |I_{n,1}^j| \leq K \left| \frac{2}{n^{2\varepsilon}-1} \right|$  and  $\lim_{n \rightarrow +\infty} \left( \sup_{j \in J_n(U)} |I_{n,1}^j| \right) = 0$ .

Let us now deal with  $I_{n,2}^j$ .

$$\begin{aligned} |I_{n,2}^j| &\leq \int_0^{+\infty} |\hat{\varphi}(\eta)| n^{1-\alpha(t_j)\beta+\varepsilon\alpha(t_j)} \frac{1}{2} \left( \int_{t_j}^{t_{j+1}} \left| \frac{\eta}{\sigma} \right|^{\alpha(x)} dx \right)^2 d\eta \\ &\leq \frac{K_U^2}{2} \int_0^{+\infty} |\hat{\varphi}(\eta)| n^{1-\alpha(t_j)\beta+\varepsilon\alpha(t_j)} (\eta^c + \eta^d)^2 n^{2\alpha(t_j)\beta-2-2\varepsilon\alpha(t_j)} d\eta \\ &\leq K_U n^{\alpha(t_j)(\beta-\varepsilon-\frac{1}{\alpha(t_j)})} \\ &\leq K_U n^{d_U(\beta-\varepsilon-\frac{1}{d_U})} \end{aligned}$$

and thus  $\lim_{n \rightarrow +\infty} \left( \sup_{j \in J_n(U)} |I_{n,2}^j| \right) = 0$ .

Fubini's theorem implies that

$$\begin{aligned} I_{n,3}^j &= \int_{t_j}^{t_{j+1}} \frac{n^{1-\alpha(t_j)\beta+\varepsilon\alpha(t_j)}}{\sigma^{\alpha(x)}} \left( - \int_0^{+\infty} \hat{\varphi}(\eta) \cos(\eta) \eta^{\alpha(x)} d\eta \right) dx \\ &= \int_{t_j}^{t_{j+1}} \frac{n^{1-\alpha(t_j)\beta+\varepsilon\alpha(t_j)}}{\sigma^{\alpha(x)}} F(\alpha(x)) dx \end{aligned}$$



where  $F(\delta) = - \int_0^{+\infty} \hat{\varphi}(\eta) \cos(\eta) \eta^\delta d\eta$ .

The appendix contains a proof that  $\min_{\delta \in [c, d]} F(\delta) > 0$ . As a consequence,

$$\begin{aligned} I_{n,3}^j &\geq \left( \min_{\delta \in [c, d]} F(\delta) \right) \frac{n^{-\alpha(t_j)\beta + \varepsilon \alpha(t_j)}}{\sigma^{\alpha(t_j)}} n \int_{t_j}^{t_{j+1}} \sigma^{\alpha(t_j) - \alpha(x)} dx \\ &= \left( \min_{\delta \in [c, d]} F(\delta) \right) \left[ \frac{n^{2\varepsilon}}{n^{2\varepsilon} - 1} \right]^{\alpha(t_j)} n \int_{t_j}^{t_{j+1}} e^{(\alpha(t_j) - \alpha(x)) \log \sigma} dx \\ &\geq K_U \left[ \frac{n^{2\varepsilon}}{n^{2\varepsilon} - 1} \right]^c \left[ n \int_{t_j}^{t_{j+1}} (e^{(\alpha(t_j) - \alpha(x)) \log \sigma} - 1) dx + 1 \right]. \end{aligned}$$

Now  $\lim_{n \rightarrow +\infty} \frac{n^{2\varepsilon}}{n^{2\varepsilon} - 1} = 1$  and  $\lim_{n \rightarrow +\infty} \left( \sup_{j \in J_n(U)} |n \int_{t_j}^{t_{j+1}} (e^{(\alpha(t_j) - \alpha(x)) \log \sigma} - 1) dx| \right) = 0$ . As a consequence,

$$\exists K_U > 0, \exists n_0 \in \mathbf{N} : \forall n \geq n_0, \inf_{j \in J_n(U)} I_{n,3}^j \geq K_U > 0$$

and thus

$$\inf_{j \in J_n(U)} (n^{1 - \alpha(t_j)\beta + \varepsilon \alpha(t_j)} P_n^j) > 0.$$

□

*Proofs of Lemma 26 and Lemma 27.* Parseval formula yields

$$P_n^j = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\sin(\frac{\xi}{n^{\beta - \varepsilon}}) - \sin(\frac{\xi}{n^{\beta + \varepsilon}})}{\xi} e^{-\int_{t_j}^{t_{j+1}} |\xi|^{\alpha(x)} dx} d\xi.$$

Set  $\mu_j = \frac{1}{\alpha(t_j)} + \varepsilon - \beta$  with  $\varepsilon \in (0, \beta - \frac{1}{c_U})$ . Using the change of variable  $\xi = n^{1/\alpha(t_j)} v$ , one gets

$$P_n^j = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\sin(n^{\mu_j} v) - \sin(n^{\mu_j - 2\varepsilon} v)}{v} e^{-\int_{t_j}^{t_{j+1}} |v|^{\alpha(x)} n^{\frac{\alpha(x)}{\alpha(t_j)}} dx} dv.$$

Define

$$I_n^j = \int_{\mathbf{R}} e^{-\int_{t_j}^{t_{j+1}} |v|^{\alpha(x)} n^{\frac{\alpha(x)}{\alpha(t_j)}} dx} dv.$$

Since  $\alpha$  is Lipschitz and  $c \leq \alpha(x) \leq d$ , one deduces that

$$\exists K_U^1 > 0, \exists K_U^2 > 0, \exists n_0 \in \mathbf{N}, \quad K_U^1 \leq \inf_{j \in J_n(U)} I_n^j \leq \sup_{j \in J_n(U)} I_n^j \leq K_U^2.$$

Now,

$$\begin{aligned} P_n^j &= n^{\mu_j} I_n^j \left( 1 + \frac{1}{I_n^j} \int_{\mathbf{R}} \left( \frac{\sin(n^{\mu_j} v) - \sin(n^{\mu_j - 2\varepsilon} v)}{n^{\mu_j} v} - 1 \right) e^{-\int_{t_j}^{t_{j+1}} |v|^{\alpha(x)} n^{\frac{\alpha(x)}{\alpha(t_j)}} dx} dv \right) \\ &= n^{\mu_j} I_n^j (1 + L_n^j). \end{aligned}$$

One computes:

$$\begin{aligned}
|L_n^j| &\leq \frac{1}{K_U^1} \int_{\mathbf{R}} \left[ \left| \frac{\sin(n^{\mu_j} v)}{n^{\mu_j} v} - 1 \right| + \left| \frac{\sin(n^{\mu_j - 2\varepsilon} v)}{n^{\mu_j} v} \right| \right] e^{-\int_{t_j}^{t_{j+1}} |v|^{\alpha(x)} n^{\frac{\alpha(x)}{\alpha(t_j)}} dx} dv \\
&\leq \frac{1}{K_U^1} \int_{\mathbf{R}} \left( \frac{1}{6} n^{2\mu_j} v^2 + \frac{1}{n^{2\varepsilon}} \right) e^{-\int_{t_j}^{t_{j+1}} |v|^{\alpha(x)} n^{\frac{\alpha(x)}{\alpha(t_j)}} dx} dv \\
&\leq \frac{1}{6K_U^1} n^{2(\frac{1}{c_U} + \varepsilon - \beta)} \int_{\mathbf{R}} v^2 e^{-\int_{t_j}^{t_{j+1}} |v|^{\alpha(x)} n^{\frac{\alpha(x)}{\alpha(t_j)}} dx} dv + \frac{K_U^2}{K_U^2} \frac{1}{n^{2\varepsilon}}.
\end{aligned}$$

There exist  $K_U^3 \in \mathbf{R}$  and  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$ ,

$$\sup_{j \in J_n(U)} \int_{\mathbf{R}} v^2 e^{-\int_{t_j}^{t_{j+1}} |v|^{\alpha(x)} n^{\frac{\alpha(x)}{\alpha(t_j)}} dx} dv \leq K_U^3 < +\infty.$$

As a consequence,  $\lim_{n \rightarrow +\infty} \left( \sup_{j \in J_n(U)} |L_n^j| \right) = 0$ .

□

### 3.4 Estimates for the number of increments and determination of the spectrum

#### 3.4.1 Lemmas

**Lemma 28.** *Almost surely,  $\forall \beta < 0$ ,  $f_g(\beta) = -\infty$ .*

**Lemma 29.** *Almost surely,  $\forall \beta \in (0, \frac{1}{d})$ ,  $f_g(\beta) = \beta d$ .*

**Lemma 30.** *Almost surely,  $f_g(0) = 0$ .*

**Lemma 31.** *Almost surely,  $\forall \beta \in [\frac{1}{d}, \frac{1}{c}]$ ,  $f_g(\beta) = 1$ .*

**Lemma 32.** *Almost surely,  $\forall \beta \in (\frac{1}{c}, 1 + \frac{1}{c})$ ,  $f_g(\beta) = 1 + \frac{1}{c} - \beta$ .*

**Lemma 33.** *Almost surely,  $f_g(1 + \frac{1}{c}) = 0$ .*

**Lemma 34.** *Almost surely,  $\forall \beta > 1 + \frac{1}{c}$ ,  $f_g(\beta) = -\infty$ .*

#### 3.4.2 Proofs

*Proof of Lemma 28.* Fix  $\beta < 0$ . Denote  $E_\beta = (0, -\beta)$ . If  $\varepsilon \in E_\beta$ , then

$$\begin{aligned}
\mathbf{P}(N_n^\varepsilon(\beta) \geq 1) &= 1 - \mathbf{P}(N_n^\varepsilon(\beta) = 0) \\
&= 1 - \prod_{j=1}^n (1 - P_n^j).
\end{aligned}$$

Lemma 24 implies that, for  $n$  large enough,

$$\begin{aligned} \mathbf{P}(N_n^\varepsilon(\beta) \geq 1) &\leq 1 - (1 - Kn^{d\beta+d\varepsilon-1})^n \\ &\leq 1 - e^{-\frac{3}{2}Kn^{d\beta+d\varepsilon}} \\ &\leq \frac{3}{2}Kn^{d\beta+d\varepsilon}, \end{aligned}$$

and thus  $\lim_{n \rightarrow +\infty} \mathbf{P}(N_n^\varepsilon(\beta) \geq 1) = 0$ . Since  $N_n^\varepsilon(\beta)$  tends to 0 in probability when  $n$  tends to infinity, there exists a subsequence  $\sigma(n)$  such that  $N_{\sigma(n)}^\varepsilon(\beta)$  tends to 0 almost surely. This implies that, almost surely,  $\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} = -\infty$ . We have proved that:

$$\forall \beta < 0, \forall \varepsilon \in E_\beta, \text{ almost surely, } \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} = -\infty. \quad (12)$$

Let  $\Omega_{\beta,\varepsilon} = \{\omega : (12) \text{ holds}\}$ ,  $\Omega_\beta = \bigcap_{\varepsilon \in E_\beta \cap \mathbb{Q}} \Omega_{\beta,\varepsilon}$  and  $\Omega = \bigcap_{\beta \in (-\infty, 0) \cap \mathbb{Q}} \Omega_\beta$ . Note that  $\Omega$  has probability 1, and consider  $\omega \in \Omega$ . Choose  $\beta < 0$  and  $j \in \mathbf{N}$ . Set  $\beta_j = \frac{[\beta j]}{j}$ . For  $j$  large enough,  $N_n^{1/j}(\beta) \leq N_n^{2/j}(\beta_j)$ . In addition  $\liminf_{n \rightarrow \infty} \frac{\log N_n^{2/j}(\beta_j)}{\log n} = -\infty$ . As a consequence,  $\liminf_{n \rightarrow \infty} \frac{\log N_n^{1/j}(\beta)}{\log n} = -\infty$  and, almost surely, for all  $\beta < 0$ ,  $f_g(\beta) = -\infty$ .  $\square$

*Proof of Lemma 29.* Fix  $\beta \in (0, \frac{1}{d})$ . Denote  $E_\beta = \{\varepsilon \in (0, \min(\beta, \frac{1}{d} - \beta)) \text{ such that } (d - \varepsilon)(\beta - \varepsilon) \in (0, 1)\}$ . Choose  $\varepsilon \in E_\beta$ . By Lemma 24, there exists  $K > 0$  and  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$  and all  $j \in \llbracket 1, n \rrbracket$ ,

$$P_n^j \leq Kn^{d\beta+d\varepsilon-1}.$$

Lemma 22 then implies that, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq d\beta + d\varepsilon. \quad (13)$$

There exists an open interval  $U$  such that, for all  $t \in U$ ,  $\alpha(t) \geq d - \varepsilon$ . Using Lemma 25, there exist  $K > 0$  and  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$  and all  $j \in J_n(U)$ ,

$$P_n^j \geq Kn^{(d-\varepsilon)(\beta-\varepsilon)-1},$$

and Lemma 23 then implies that, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \geq (d - \varepsilon)(\beta - \varepsilon).$$

We thus have proved that, for all  $\beta \in (0, \frac{1}{d})$  and all  $\varepsilon \in E_\beta$ , almost surely,

$$(d - \varepsilon)(\beta - \varepsilon) \leq \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq d\beta + d\varepsilon. \quad (14)$$

Then Lemma 15 ensures that almost surely, for all  $\beta \in (0, \frac{1}{d})$ ,  $f_g(\beta) = d\beta$ .  $\square$

*Proof of Lemma 30.* We obtain Inequality (13) for  $\beta = 0$  by applying Lemma 24 and Lemma 22. This yields that, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(0)}{\log n} \leq d\varepsilon.$$

As a consequence,  $f_g(0) \leq 0$ .

Then apply Lemma 14 and Lemma 29 to obtain

$$\begin{aligned} f_g(0) &\geq \limsup_{j \rightarrow +\infty} f_g\left(\frac{1}{j}\right) \\ &= \limsup_{j \rightarrow +\infty} \frac{d}{j} \\ &= 0. \end{aligned}$$

□

*Proof of Lemma 31.* Let  $\beta \in (\frac{1}{d}, \frac{1}{c})$  and  $\varepsilon \in (0, 1)$ .

Choose an open interval  $U$  such that  $\beta > \frac{1}{c_U}$  and  $\beta < \frac{1}{d_U} + 2\varepsilon$ . Lemma 27 then ensures that there exist  $K > 0$  and  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$  and all  $j \in J_n(U)$ ,

$$\begin{aligned} P_n^j &\geq Kn^{\frac{1}{\alpha(t_j)} + \varepsilon - \beta} \\ &\geq Kn^{\frac{1}{d_U} + \varepsilon - \beta} \\ &\geq Kn^{-\varepsilon}. \end{aligned}$$

By Lemma 23, for all  $\beta \in (\frac{1}{d}, \frac{1}{c})$ , almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \geq 1 - \varepsilon.$$

We conclude by applying Lemma 15.

For  $\beta = \frac{1}{c}$ , we apply Lemma 14 to obtain

$$\begin{aligned} f_g\left(\frac{1}{c}\right) &\geq \limsup_{j \rightarrow +\infty} f_g\left(\frac{[\beta j]}{j}\right) \\ &= \limsup_{j \rightarrow +\infty} 1. \end{aligned}$$

For  $\beta = \frac{1}{d}$ , Lemma 14 and Lemma 29 lead to

$$\begin{aligned} f_g\left(\frac{1}{d}\right) &\geq \limsup_{j \rightarrow +\infty} f_g\left(\frac{[\beta j]}{j}\right) \\ &= \limsup_{j \rightarrow +\infty} d \frac{[\beta j]}{j} \\ &= 1. \end{aligned}$$

□

*Proof of Lemma 32.* Let  $\beta \in (\frac{1}{c}, 1 + \frac{1}{c})$ . Denote  $E_\beta = (0, \min(\beta - \frac{1}{c}, 1 + \frac{1}{c} - \beta))$ . Fix  $\varepsilon \in E_\beta$ . By Lemma 26, there exist  $K > 0$  and  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$  and all  $j \in \llbracket 1, n \rrbracket$ ,

$$P_n^j \leq Kn^{\frac{1}{c} + \varepsilon - \beta}.$$

Lemma 22 then implies that, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq 1 + \frac{1}{c} + \varepsilon - \beta. \quad (15)$$

Choose an open interval  $U$  such that  $\frac{1}{c} - 2\varepsilon \leq \frac{1}{d_U} \leq \beta - \varepsilon$ . By Lemma 27, there exist  $K > 0$  and  $n_0 \in \mathbf{N}$  such that, for all  $n \geq n_0$  and all  $j \in J_n(U)$ ,

$$P_n^j \geq Kn^{\frac{1}{d_U} + \varepsilon - \beta}.$$

Lemma 23 then implies that, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \geq 1 + \frac{1}{d_U} + \varepsilon - \beta \geq 1 + \frac{1}{c} - \varepsilon - \beta.$$

We have proved that, for all  $\beta \in (\frac{1}{c}, 1 + \frac{1}{c})$  and all  $\varepsilon \in E_\beta$ , almost surely,

$$1 + \frac{1}{c} - \varepsilon - \beta \leq \liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} \leq 1 + \frac{1}{c} + \varepsilon - \beta. \quad (16)$$

The result then follows from Lemma 15.  $\square$

*Proof of Lemma 33.* We obtain Inequality (15) for  $\beta = 1 + \frac{1}{c}$  by applying Lemma 26 and Lemma 22. This yields that, almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(1 + \frac{1}{c})}{\log n} \leq d\varepsilon.$$

As a consequence,  $f_g(1 + \frac{1}{c}) \leq 0$ .

Then, Lemma 14 and Lemma 32 lead to

$$\begin{aligned} f_g(1 + \frac{1}{c}) &\geq \limsup_{j \rightarrow +\infty} f_g\left(\frac{[(1 + \frac{1}{c})j]}{j}\right) \\ &= \limsup_{j \rightarrow +\infty} \left(1 + \frac{1}{c} - \frac{[(1 + \frac{1}{c})j]}{j}\right) \\ &= 0. \end{aligned}$$

$\square$

*Proof of Lemma 34.* Let  $\beta > 1 + \frac{1}{c}$  and denote  $E_\beta = (0, \beta - 1 - \frac{1}{c})$ . For  $\varepsilon \in E_\beta$ ,

$$\begin{aligned} \mathbf{P}(N_n^\varepsilon(\beta) \geq 1) &= 1 - \mathbf{P}(N_n^\varepsilon(\beta) = 0) \\ &= 1 - \prod_{j=1}^n (1 - P_n^j). \end{aligned}$$

Lemma 26 ensures that, for  $n$  large enough,

$$\begin{aligned} \mathbb{P}(N_n^\varepsilon(\beta) \geq 1) &\leq 1 - (1 - Kn^{\frac{1}{c} + \varepsilon - \beta})^n \\ &\leq \frac{3}{2}Kn^{1 + \frac{1}{c} + \varepsilon - \beta}. \end{aligned}$$

Thus,  $\lim_{n \rightarrow +\infty} \mathbb{P}(N_n^\varepsilon(\beta) \geq 1) = 0$ , which implies that  $N_n^\varepsilon(\beta)$  tends to 0 in probability when  $n$  tends to infinity. There exists a subsequence  $\sigma(n)$  such that  $N_{\sigma(n)}^\varepsilon(\beta)$  tends to 0 almost surely. As a consequence, for all  $\beta > 1 + \frac{1}{c}$  and all  $\varepsilon \in E_\beta$ , almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{\log N_n^\varepsilon(\beta)}{\log n} = -\infty. \quad (17)$$

We conclude as in the last part of the proof of Lemma 28.  $\square$

## Appendix

The following result is due to R. Schelling [24]:

**Lemma 35.** *For all  $\beta > 0$ ,*

$$F(\beta) = - \int_0^\infty \eta^\beta \cos(\eta) \widehat{\varphi}(\eta) d\eta > 0. \quad (18)$$

*Proof.* The Lévy–Khintchine formula yields

$$|\eta|^\beta = \int_{y \neq 0} (1 - \cos(y\eta)) \nu_\beta(dy) \quad \text{with} \quad \nu_\beta(dy) = \frac{c_\beta dy}{|y|^{1+\beta}}.$$

By Fubini's theorem,

$$\begin{aligned} &\int_0^\infty \eta^\beta \cos(\eta) \widehat{\varphi}(\eta) d\eta \\ &= \int_{y \neq 0} \int_0^\infty (\cos(\eta) - \cos(y\eta) \cos(\eta)) \widehat{\varphi}(\eta) d\eta \nu_\beta(dy) \\ &= \int_{y \neq 0} \int_0^\infty (\cos(\eta) - \frac{1}{2} \cos(1+y)\eta - \frac{1}{2} \cos(1-y)\eta) \widehat{\varphi}(\eta) d\eta \nu_\beta(dy) \\ &= c \int_{y \neq 0} (\varphi(1) - \frac{1}{2} \varphi(1+y) - \frac{1}{2} \varphi(1-y)) \nu_\beta(dy), \end{aligned}$$

where  $c$  is a positive constant. By definition,  $\varphi$  is smooth and supported on  $[-1, 1]$ , thus  $\varphi(1) = 0$ . As a consequence, we find that

$$\int_0^\infty \eta^\beta \cos(\eta) \widehat{\varphi}(\eta) d\eta = -\frac{c}{2} \int_{y \neq 0} (\varphi(1+y) + \varphi(1-y)) \nu_\beta(dy) < 0.$$

This is inequality (18).  $\square$

It is easy to see that the function  $\beta \mapsto F(\beta)$  is continuous. As a consequence,  $\min_{\delta \in [c, d]} F(\delta) > 0$ .

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