Label-free Modular Systems for Classical and Intuitionistic Modal Logics

Sonia Marin, Lutz Straßburger

Label-free Modular Systems for Classical and Intuitionistic Modal Logics

Sonia Marin

ENS, Paris, France

Lutz Straßburger

Inria, Palaiseau, France

Abstract

In this paper we show for each of the modal axioms d, t, b, 4, and 5 an equivalent set of inference rules in a nested sequent system, such that, when added to the basic system for the modal logic K, the resulting system admits cut elimination. Then we show the same result also for intuitionistic modal logic. We achieve this by combining structural and logical rules.

Keywords: Modal logic, cut elimination, nested sequents, Hilbert axioms

1 Introduction

It very often happens that a new logic is introduced in terms of axioms in a Hilbert system. It is then a tedious task for proof theorists to find a cut-free deductive system in a sequent-like calculus. This is usually done by "trial and error" since there is no general method. It should be a goal of structural proof theory to automate this process, and to find general criteria for determining when a set of Hilbert axioms can be transformed into an equivalent set of inference rules such that cut elimination is preserved.

Recently, this goal has been achieved for substructural logics: In [4] it has been unveiled which classes of axioms can be transformed into equivalent structural rules in the sequent calculus, respectively hypersequent calculus, such that the resulting system admits cut elimination. In [5] a similar result has been obtained in the display calculus.

It is a natural question to ask whether this can also be done for modal logics. The work in [9] shows how certain classes of axioms in modal-tense logics can be transformed into logical rules in the display calculus and in nested sequents. Unfortunately, the established correspondence between axioms and logical rules works well only in the presence of the tense modalities. For modal logics without tense modalities, nested sequents have been used to give cut-free deductive systems for all logics in the classical modal S5-cube [2], as well as

d: $\Box A \supset \Diamond A$
t: $(A \supset \Diamond A) \wedge (\Box A \supset A)$
b: $(A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset A)$
$\mathsf{4}$: $(\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A)$
$\mathsf{5}$: $(\Diamond A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset \Box A)$

Fig. 1. Modal axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$

for all logics in the intuitionistic modal S5-cube [18]. This concerns the modal axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$, shown in Figure 1. In classical logic only one of the two conjuncts in each axiom shown in that Figure is needed because the other follows from De Morgan duality. However, in the intuitionistic setting both conjuncts are needed. With these five axioms one can, *a priori*, obtain 32 logics but some coincide, such that there are only 15, which can be arranged in a cube as shown in Figure 2. This cube has the same shape in the classical as well as in the intuitionistic setting.

However, the two papers [2] and [18] have one drawback: Although they provide cut-free systems for all logics in the cube, they do not provide cut-free systems for all possible combinations of axioms. For example, the logic $\mathsf{S5}$ can be obtained by adding b and $\mathsf{4}$, or by adding t and $\mathsf{5}$, to the modal logic K, but a complete cut-free system could only be obtained by adding rules for b, $\mathsf{4}$, and $\mathsf{5}$, or for t, $\mathsf{4}$, and $\mathsf{5}$ (in both the classical and the intuitionistic case).

This might be sufficient for someone interested in a cut-free system for a particular logic, but it is not sufficient for our goal—we do not want different rules for axioms t and $\mathsf{5}$, depending on whether we have only one or both of them in the system.

The works in [9,2,18] all use logical rules for the $\Diamond$-modality. An alternative route is taken in [3] where the authors use structural rules, which is closer in spirit to the work in substructural logics [4], mentioned above. However, the work in [3] does not cover all possible axiom combinations either (although it claims to do so).

In the present paper we achieve full modularity, for classical and intuitionistic modal logic, by using the logical rules of [2] and [18] together with the structural rules of [3]. Interestingly, the structural rules are the same in the classical and the intuitionistic setting.

This paper is organized as follows. In the next section we recall the nested sequent system for classical modal logic presented in [2]. Then, in Section 3, we show the structural rules of [3] and discuss the mistake in that paper. In Section 4, we then show our modularity result for classical modal logics. Section 5 recalls how nested sequents can be used for intuitionistic modal logics, as done in [18]. Finally, in Section 6, we show our modularity result for intuitionistic modal logics.

2 Nested Sequents for Classical Modal Logics

For simplicity, we consider here only formulas in negation normal form, generated by the grammar:

$$A, B, \ldots ::= p \mid \bar{p} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

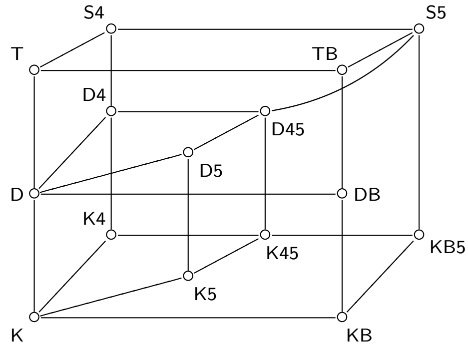

Fig. 2. The modal S5-cube

where p stands for a propositional variable and $\bar{p}$ its dual. Then the negation $\bar{A}$ of a formula A is defined in the usual way using the De Morgan duality, and implication $A \supset B$ is an abbreviation for $\bar{A} \vee B$.

Recall that a Hilbert system for the modal logic K can be obtained by taking some complete set of axioms for classical propositional logic extended with the k-axiom:

$$\mathsf{k}\colon\ \Box(A \supset B) \supset (\Box A \supset \Box B) \tag{1}$$

and the rules of *modus ponens* and *necessitation*, shown below:

$$\mathsf{mp}\ \frac{A \quad A \supset B}{B} \qquad\qquad \mathsf{nec}\ \frac{A}{\Box A} \tag{2}$$

For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ we write $\mathsf{K} + \mathsf{X}$ to denote the logic obtained from K by adding the axioms in X.

Let us now turn to the deductive system defined by Brünnler in [2] using nested sequents. Nested sequents have independently also been conceived by Kashima [10] and Poggiolesi [14]. Fitting [7] observed that nested sequents have the same data structure as prefixed tableaux.

A *nested sequent* (or simply a *sequent*) is a finite multiset of formulas and *boxed sequents*; that is, expressions like $[\Gamma]$ where Γ is also a sequent. Therefore a sequent is of the form:

$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$

The *corresponding formula* of a sequent Γ, denoted by $\mathit{fm}(\Gamma)$, is defined as:

$$\mathit{fm}(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box\mathit{fm}(\Gamma_1) \vee \ldots \vee \Box\mathit{fm}(\Gamma_n)$$

Nested sequents can also be conceived as trees. For example, to the sequent $\Gamma = A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$ corresponds the tree $\mathit{tr}(\Gamma)$ defined as:

$$\begin{tikzcd} & & \{A_1, \ldots, A_m\} \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\ \mathit{tr}(\Gamma_1) & \mathit{tr}(\Gamma_2) & \cdots & \mathit{tr}(\Gamma_{n-1}) & \mathit{tr}(\Gamma_n) \end{tikzcd}$$

$$\mathsf{id}\,\frac{}{\Gamma\{a,\bar{a}\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \wedge\,\frac{\Gamma\{A\}\quad\Gamma\{B\}}{\Gamma\{A\wedge B\}}$$

$$\mathsf{c}\,\frac{\Gamma\{A,A\}}{\Gamma\{A\}} \qquad \Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

Fig. 3. System NK

$$\mathsf{d}^{\Diamond}\,\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}\,\frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}\,\frac{\Gamma\{[\Delta],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

$$\mathsf{4}^{\Diamond}\,\frac{\Gamma\{[\Diamond A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \mathsf{5}^{\Diamond}\,\frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\})\geq 1$$

Fig. 4. Modal $\Diamond$-rules for axioms $\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}$

Sometimes we will use for a sequent the vocabulary that would apply to the corresponding formula or to the corresponding tree without mentioning it.

To be able to apply inference rules deeply inside a sequent, we need the notion of context.

Definition 2.1 A *context* is a sequent with one or several holes; we distinguish *unary* context if there is exactly one hole, and *binary* context if there are exactly two. A *hole* $\{\ \}$ takes the place of a formula in the sequent but does not occur inside a formula. Finally, we write $\Gamma\{\Delta\}$ when we replace the hole in $\Gamma\{\ \}$ by Δ.

Definition 2.2 The *depth* of a unary context is defined inductively as:

$$\begin{aligned} \mathit{depth}(\{\ \}) &= 0 \\ \mathit{depth}(\Delta,\Gamma\{\ \}) &= \mathit{depth}(\Gamma\{\ \}) \\ \mathit{depth}([\Gamma\{\ \}]) &= \mathit{depth}(\Gamma\{\ \})+1 \end{aligned}$$

Example 2.3 Let $\Gamma\{\ \}\{\ \} = A,[B,\{\ \},[\{\ \}],C]$. For any sequents Δ_1 and Δ_2, we get: $\Gamma\{\Delta_1\}\{\Delta_2\} = A,[B,\Delta_1,[\Delta_2],C]$. In particular, $\Gamma\{\emptyset\}\{\Delta_2\} = A,[B,[\Delta_2],C]$ and $\Gamma\{\Delta_1\}\{\emptyset\} = A,[B,\Delta_1,[\emptyset],C]$. Moreover, we can compute $\mathit{depth}(\Gamma\{\ \}\{\Delta\}) = 1$ and $\mathit{depth}(\Gamma\{\Delta\}\{\ \}) = 2$.

The inference rules shown in Figure 3 form the *system* NK. Then, Figure 4 shows the $\Diamond$-rules for the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$. For $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ we write $\mathsf{X}^{\Diamond}$ for the corresponding subset of $\{\mathsf{d}^{\Diamond},\mathsf{t}^{\Diamond},\mathsf{b}^{\Diamond},\mathsf{4}^{\Diamond},\mathsf{5}^{\Diamond}\}$.

In the course of this paper we also need the *weakening*- and *cut-rule*, shown below:

$$\mathsf{w}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\,\frac{\Gamma\{\bar{A}\}\quad\Gamma\{A\}}{\Gamma\{\emptyset\}} \tag{3}$$

Lemma 2.4 *Let* $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. *Then the* w*-rule is height-preserving admissible for* $\mathsf{NK}\cup\mathsf{X}^{\Diamond}$. [2]

Remark 2.5 In Brünnlers original formulation [2] of system $\mathsf{NK}\cup\mathsf{X}^{\Diamond}$, contraction was not given as explicit rule, but was absorbed in the $\Diamond$-rule and the

$$\mathsf{d}^{[]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{[\Sigma],\Delta\}}$$

$$\mathsf{4}^{[]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \qquad \mathsf{5}^{[]}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\ \mathit{depth}(\Gamma\{\ \}\{[\Delta]\}) \geq 1$$

$$\mathsf{m}^{[]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[\Delta,\Sigma]\}}$$

Fig. 5. **Left:** Structural modal rules for axioms $\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}$ – **Right:** Structural medial

rules in $\mathsf{X}^{\diamond}$. It is easy to see that both formulations are equivalent. In this paper we have an explicit contraction in the system because in the presence of the structural rules (introduced in the next section), contraction is no longer admissible.

As already observed in [2], not all combinations of modal rules lead to complete cut-free systems. For example, the $\mathsf{5}$-axiom $\Diamond A \supset \Box\Diamond A$ is valid in any $\{\mathsf{b}, \mathsf{4}\}$-frame, but it is not possible to prove it in $\mathsf{NK} \cup \{\mathsf{b}^{\diamond}, \mathsf{4}^{\diamond}\}$ without cut. Therefore, to get a cut-elimination proof, Brünnler [2] introduced the notion of 45-closure.

Definition 2.6 The *45-closure* of X is defined as:

$$\hat{\mathsf{X}} = \begin{cases} \mathsf{X} \cup \{\mathsf{4}\} & \text{if } \{\mathsf{b}, \mathsf{5}\} \subseteq \mathsf{X} \text{ or if } \{\mathsf{t}, \mathsf{5}\} \subseteq \mathsf{X} \\ \mathsf{X} \cup \{\mathsf{5}\} & \text{if } \{\mathsf{b}, \mathsf{4}\} \subseteq \mathsf{X} \\ \mathsf{X} & \text{otherwise} \end{cases}$$

We say that X is *45-closed*, if $\mathsf{X} = \hat{\mathsf{X}}$.

Proposition 2.7 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *We have that* X *is 45-closed, if and only if the following two conditions hold:*

- *whenever* $\mathsf{4}$ *is derivable in* $\mathsf{K} + \mathsf{X}$, *then* $\mathsf{4} \in \mathsf{X}$, *and*
- *whenever* $\mathsf{5}$ *is derivable in* $\mathsf{K} + \mathsf{X}$, *then* $\mathsf{5} \in \mathsf{X}$.

Now we can state Brünnler's [2] main results:

Theorem 2.8 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *If a sequent* Γ *is derivable in* $\mathsf{NK} \cup \mathsf{X}^{\diamond} \cup \{\mathsf{cut}\}$ *then it is also derivable in* $\mathsf{NK} \cup \hat{\mathsf{X}}^{\diamond}$. [2]

Corollary 2.9 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *be 45-closed. Then a formula* A *is a theorem of* $\mathsf{K} + \mathsf{X}$ *if and only if it is derivable in* $\mathsf{NK} \cup \mathsf{X}^{\diamond}$. [2]

The goal of this paper is to find a way to drop the 45-closed condition.

3 Structural Rules

The first attempt to drop the 45-closed condition was made in [3] where the authors suggest to use the structural rules shown in Figure 5. For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, we write $\mathsf{X}^{[]} \subseteq \{\mathsf{d}^{[]}, \mathsf{t}^{[]}, \mathsf{b}^{[]}, \mathsf{4}^{[]}, \mathsf{5}^{[]}\}$ for the corresponding set of rules from the left of that figure.

The work in [3] claims to prove cut elimination for $\mathsf{NK} \cup \{\mathsf{m}^{[]}\} \cup \mathsf{X}^{[]}$ (the $\mathsf{m}^{[]}$-rule is shown on the right of Figure 5), in order to obtain the following:

Claim 3.1 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*. A formula is a theorem of* $\mathsf{K} + \mathsf{X}$ *if and only if it is derivable in* $\mathsf{NK} \cup \{\mathsf{m}^{[]}\} \cup \mathsf{X}^{[]}$*.*

However, there is a mistake in the proof in [3], and the claim is not correct. For example, the formula $\Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)$ is a theorem of $\mathsf{K4}$ ($= \mathsf{K}+\mathsf{4}$), and also provable in $\mathsf{NK} \cup \{\mathsf{4}^{\Diamond}\}$, but it is not provable in $\mathsf{NK} \cup \{\mathsf{m}^{[]}, \mathsf{4}^{[]}\}$. The reason is that no rule in $\mathsf{NK} \cup \{\mathsf{m}^{[]}, \mathsf{4}^{[]}\}$ can increase the modal depth of the sequent (i.e., the maximal nesting of brackets and modalities) when read bottom-up.

The mistake in the cut elimination proof of [3] is rather subtle: In the cut reduction lemma (Lemma 10), the cut is permuted up, together with a stack of structural rule instances above the two premisses of the cut. If an instance of the $\Diamond$-rule is met, this $\Diamond$-rule instance is permuted down under the structural rules, using Lemma 7 and Lemma 8 of that paper, resulting in a derivation of structural rules above a derivation of logical rules (as shown in Figure 4 above), such that all rule instances in that derivation work on the same $\Diamond$-formula as the original $\Diamond$-rule instance. This stack of logical rules is then "reflected" at the cut (using Lemma 9), resulting in a stack of structural rules above the other premise of the cut.

The problem is that this only works if that $\Diamond$-formula is the cut-formula. Otherwise, the logical $\Diamond$-rules are not reflected at the cut but move under the cut as in a commutative case. This concerns the $\mathsf{4}^{\Diamond}$-rule and the $\mathsf{5}^{\Diamond}$-rule. Thus, the cut elimination proof of [3] breaks down if the $\mathsf{4}$- or $\mathsf{5}$-axiom is present.

For the convenience of the reader, we give an example in Appendix B.

4 Modularity for Classical Modal Logics

In this section, we show how the mistake of [3] can be corrected. We show that we can drop the 45-closure condition that appears in Theorem 2.8 if we use both the logical rules from [2] and the structural rules from [3].

Theorem 4.1 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*. A formula* A *is a theorem of* $\mathsf{K} + \mathsf{X}$ *if and only if it is derivable in* $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$*.*

To be able to prove this theorem, we need to state first some lemmas. In particular, we need to show that weakening is still admissible.

Lemma 4.2 *For any* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *the rule* w *is (contraction-preserving) admissible for* $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$*.*

Proof. This is a straightforward induction on the height of the derivation. $\Box$

Lemma 4.3 *If* $\{\mathsf{t}, \mathsf{5}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *then the* $\mathsf{4}^{\Diamond}$*-rule is admissible for* $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$*.*

Proof. Any occurrence of the $\mathsf{4}^{\Diamond}$-rule can be replaced by the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{[\emptyset], [\Diamond A, \Delta]\}}\,\mathsf{w}}{\Gamma\{[\Diamond A], [\Delta]\}}\,\mathsf{5}^{\Diamond}}{\Gamma\{\Diamond A, [\Delta]\}}\,\mathsf{t}^{[]}$$

Then we apply Lemma 4.2. $\square$

Lemma 4.4 *If* $\{\mathsf{b}, \mathsf{5}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *then the* $\mathsf{4}^\diamond$*-rule is admissible for* $\mathsf{NK} \cup \mathsf{X}^\diamond \cup \mathsf{X}^{[]}$.

Proof. Any occurrence of the $\mathsf{4}^\diamond$-rule can be replaced by the following derivation:

$$\mathsf{b}^{[]}\frac{\mathsf{5}^\diamond\dfrac{\mathsf{w}\dfrac{\Gamma\{[\diamond A, \Delta]\}}{\Gamma\{[[\emptyset], \diamond A, \Delta]\}}}{\Gamma\{[[\diamond A], \Delta]\}}}{\Gamma\{\diamond A, [\Delta]\}}$$

Then we apply Lemma 4.2. $\square$

To prove the admissibility of the $\mathsf{5}^\diamond$-rule, we decompose it into three rules that only use unary contexts, and are thus are easier to handle:

$$\mathsf{5}_1^\diamond\frac{\Gamma\{[\Delta], \diamond A\}}{\Gamma\{[\Delta, \diamond A]\}} \qquad \mathsf{5}_2^\diamond\frac{\Gamma\{[\Delta], [\diamond A, \Sigma]\}}{\Gamma\{[\Delta, \diamond A], [\Sigma]\}} \qquad \mathsf{5}_3^\diamond\frac{\Gamma\{[\Delta, [\diamond A, \Sigma]]\}}{\Gamma\{[\Delta, \diamond A, [\Sigma]]\}} \tag{4}$$

Clearly, each of $\mathsf{5}_1^\diamond$, $\mathsf{5}_2^\diamond$, and $\mathsf{5}_3^\diamond$ is a special case of $\mathsf{5}^\diamond$. Conversely, we have:

Lemma 4.5 *The* $\mathsf{5}^\diamond$*-rule is derivable from* $\{\mathsf{5}_1^\diamond, \mathsf{5}_2^\diamond, \mathsf{5}_3^\diamond\}$.

Proof. As in [2], but here the situation is a bit simpler since we do not have to deal with contraction. $\square$

Lemma 4.6 *If* $\{\mathsf{4}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *then the* $\mathsf{5}_3^\diamond$*-rule is admissible for* $\mathsf{NK} \cup \mathsf{X}^\diamond \cup \mathsf{X}^{[]}$.

Proof. Any occurrence of the $\mathsf{5}_3^\diamond$-rule is an instance of the $\mathsf{4}^\diamond$-rule. $\square$

Lemma 4.7 *If* $\{\mathsf{b}, \mathsf{4}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *then the* $\mathsf{5}_2^\diamond$*-rule is (contraction-preserving) admissible for* $\mathsf{NK} \cup \mathsf{X}^\diamond \cup \mathsf{X}^{[]}$.

Proof. Any occurrence of the $\mathsf{5}_2^\diamond$-rule can be replaced by

$$\mathsf{b}^{[]}\frac{\mathsf{4}^\diamond\dfrac{\mathsf{4}^\diamond\dfrac{\mathsf{4}^{[]}\dfrac{\mathsf{4}^{[]}\dfrac{\mathsf{w}\dfrac{\Gamma\{[\Delta], [\diamond A, \Sigma]\}}{\Gamma\{[\Delta], [\emptyset], [\diamond A, \Sigma]\}}}{\Gamma\{[\Delta], [[\diamond A, \Sigma]]\}}}{\Gamma\{[\Delta, [[\diamond A, \Sigma]]]\}}}{\Gamma\{[\Delta, [\diamond A, [\Sigma]]]\}}}{\Gamma\{[\Delta, \diamond A, [[\Sigma]]]\}}}{\Gamma\{[\Delta, \diamond A], [\Sigma]\}}$$

As before, we conclude by applying Lemma 4.2. $\square$

Lemma 4.8 *If* $\{\mathsf{b}, \mathsf{4}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *then the* $\mathsf{5}_1^\diamond$*-rule is admissible for* $\mathsf{NK} \cup \mathsf{X}^\diamond \cup \mathsf{X}^{[]}$.

Proof. If $\mathsf{5} \in \mathsf{X}$ then there is nothing to prove, so assume $\mathsf{5} \notin \mathsf{X}$. There is no simple derivation that can replace $\mathsf{5}_1^\diamond$. We consider the topmost instance of $\mathsf{5}_1^\diamond$, and let π be the derivation above it. We proceed by induction on the pair

$\langle c_\pi, h_\pi \rangle$ (under lexicographic ordering), where c_π is the number of c-instances in π, and h_π is the height of π. We have to carry out a case analysis on the bottommost rule instance r of π. If r only affects the context of our $\mathsf{5}^{\Diamond}_{1}$, we speak of a *trivial* case, because we can immediately apply the induction hypothesis:

$$\mathsf{5}^{\Diamond}_{1}\,\dfrac{\mathsf{r}\,\dfrac{\Gamma'\{\Diamond A,[\Delta']\}}{\Gamma\{\Diamond A,[\Delta]\}}}{\Gamma\{[\Diamond A,\Delta]\}} \quad\rightsquigarrow\quad \mathsf{r}\,\dfrac{\mathsf{5}^{\Diamond}_{1}\,\dfrac{\Gamma'\{\Diamond A,[\Delta']\}}{\Gamma'\{[\Diamond A,\Delta']\}}}{\Gamma\{[\Diamond A,\Delta]\}}$$

- $\mathsf{r} \in \{\mathsf{id}, \wedge, \vee, \Box, \mathsf{d}^{[]}\}$: There are only trivial cases.
- $\mathsf{r} = \mathsf{c}$: There is one nontrivial case:

$$\mathsf{5}^{\Diamond}_{1}\,\dfrac{\mathsf{c}\,\dfrac{\Gamma\{\Diamond A,\Diamond A,[\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}}{\Gamma\{[\Diamond A,\Delta]\}} \quad\rightsquigarrow\quad \mathsf{c}\,\dfrac{\mathsf{5}^{\Diamond}_{1}\,\dfrac{\mathsf{5}^{\Diamond}_{1}\,\dfrac{\Gamma\{\Diamond A,\Diamond A,[\Delta]\}}{\Gamma\{\Diamond A,[\Diamond A,\Delta]\}}}{\Gamma\{[\Diamond A,\Diamond A,\Delta]\}}}{\Gamma\{[\Diamond A,\Delta]\}}$$

  We can proceed by applying the induction hypothesis twice. This is possible because the number of c-instances above both $\mathsf{5}^{\Diamond}_{1}$ has decreased. (Note that none of our cases increases the number of contractions in the proof.)

- $\mathsf{r} = \Diamond$: There are two nontrivial cases:

$$\mathsf{5}^{\Diamond}_{1}\,\dfrac{\Diamond\,\dfrac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}}{\Gamma\{[\Diamond A,\Delta]\}} \quad\rightsquigarrow\quad \mathsf{b}^{[]}\,\dfrac{\mathsf{4}^{\Diamond}\,\dfrac{\mathsf{b}^{\Diamond}\,\dfrac{\mathsf{w}\,\dfrac{\Gamma\{[A,\Delta]\}}{\Gamma\{[A,[\emptyset],\Delta]\}}}{\Gamma\{[[\Diamond A],\Delta]\}}}{\Gamma\{[\Diamond A,[\emptyset],\Delta]\}}}{\Gamma\{[\Diamond A,\Delta]\}}$$

$$\mathsf{5}^{\Diamond}_{1}\,\dfrac{\Diamond\,\dfrac{\Gamma\{[\Delta],[A,\Sigma]\}}{\Gamma\{[\Delta],\Diamond A,[\Sigma]\}}}{\Gamma\{[\Delta,\Diamond A],[\Sigma]\}} \quad\rightsquigarrow\quad \mathsf{b}^{[]}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\mathsf{b}^{\Diamond}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\mathsf{w}\,\dfrac{\Gamma\{[\Delta],[A,\Sigma]\}}{\Gamma\{[\Delta],[[\emptyset],A,\Sigma]\}}}{\Gamma\{[[\emptyset],[\Delta],A,\Sigma]\}}}{\Gamma\{[[\emptyset],[\Delta,\Diamond A],\Sigma]\}}}{\Gamma\{[[[\Delta,\Diamond A]],\Sigma]\}}}{\Gamma\{[\Delta,\Diamond A],[\Sigma]\}}$$

  In both cases, we can apply Lemma 4.2.

- $\mathsf{r} = \mathsf{d}^{\Diamond}$: There is one nontrivial case.

$$\mathsf{5}^{\Diamond}_{1}\,\dfrac{\mathsf{d}^{\Diamond}\,\dfrac{\Gamma\{[\Delta],[A]\}}{\Gamma\{[\Delta],\Diamond A\}}}{\Gamma\{[\Delta,\Diamond A]\}} \quad\rightsquigarrow\quad \mathsf{d}^{\Diamond}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\Gamma\{[\Delta],[A]\}}{\Gamma\{[\Delta,[A]]\}}}{\Gamma\{[\Delta,\Diamond A]\}}$$

- $\mathsf{r} = \mathsf{t}^{\Diamond}$: There is one nontrivial case.

$$\mathsf{5}^{\Diamond}_{1}\,\dfrac{\mathsf{t}^{\Diamond}\,\dfrac{\Gamma\{[\Delta],A\}}{\Gamma\{[\Delta],\Diamond A\}}}{\Gamma\{[\Delta,\Diamond A]\}} \quad\rightsquigarrow\quad \mathsf{b}^{\Diamond}\,\dfrac{\Gamma\{[\Delta],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

- $\mathsf{r} = \mathsf{b}^{\Diamond}$: There is one nontrivial case.

$$
\mathsf{5}_1^{\Diamond}\dfrac{\mathsf{b}^{\Diamond}\dfrac{\Gamma\{[\Sigma,[\Delta]],A\}}{\Gamma\{[\Sigma,[\Delta],\Diamond A]\}}}{\Gamma\{[\Sigma,[\Delta,\Diamond A]]\}}
\quad\rightsquigarrow\quad
\mathsf{4}^{[]}\dfrac{\mathsf{b}^{\Diamond}\dfrac{\mathsf{b}^{[]}\dfrac{\mathsf{4}^{[]}\dfrac{\mathsf{w}\dfrac{\Gamma\{[\Sigma,[\Delta]],A\}}{\Gamma\{[\Sigma,[\emptyset],[\Delta]],A\}}}{\Gamma\{[\Sigma,[[\Delta]]],A\}}}{\Gamma\{[\Sigma],[\Delta],A\}}}{\Gamma\{[\Sigma],[\Delta,\Diamond A]\}}}{\Gamma\{[\Sigma,[\Delta,\Diamond A]]\}}
$$

And we apply Lemma 4.2.

- $\mathsf{r} = \mathsf{4}^{\Diamond}$: There are two nontrivial cases.

$$
\mathsf{5}_1^{\Diamond}\dfrac{\mathsf{4}^{\Diamond}\dfrac{\Gamma\{[\Diamond A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}}{\Gamma\{[\Diamond A,\Delta]\}}
\quad\rightsquigarrow\quad
\Gamma\{[\Diamond A,\Delta]\}
$$

$$
\mathsf{5}_1^{\Diamond}\dfrac{\mathsf{4}^{\Diamond}\dfrac{\Gamma\{[\Delta],[\Diamond A,\Sigma]\}}{\Gamma\{\Diamond A,[\Delta],[\Sigma]\}}}{\Gamma\{[\Diamond A,\Delta],[\Sigma]\}}
\quad\rightsquigarrow\quad
\mathsf{5}_2^{\Diamond}\dfrac{\Gamma\{[\Delta],[\Diamond A,\Sigma]\}}{\Gamma\{[\Diamond A,\Delta],[\Sigma]\}}
$$

In the second case, we need Lemma 4.7.

- $\mathsf{r} = \mathsf{t}^{[]}$: There is one nontrivial case.

$$
\mathsf{5}_1^{\Diamond}\dfrac{\mathsf{t}^{[]}\dfrac{\Gamma\{[\Delta],[\Diamond A,\Sigma]\}}{\Gamma\{[\Delta],\Diamond A,\Sigma\}}}{\Gamma\{[\Delta,\Diamond A],\Sigma\}}
\quad\rightsquigarrow\quad
\mathsf{t}^{[]}\dfrac{\mathsf{5}_2^{\Diamond}\dfrac{\Gamma\{[\Delta],[\Diamond A,\Sigma]\}}{\Gamma\{[\Delta,\Diamond A],[\Sigma]\}}}{\Gamma\{[\Delta,\Diamond A],\Sigma\}}
$$

Again, we can apply Lemma 4.7.

- $\mathsf{r} = \mathsf{b}^{[]}$: There are three nontrivial cases.

$$
\mathsf{5}_1^{\Diamond}\dfrac{\mathsf{b}^{[]}\dfrac{\Gamma\{[\Delta,[\Diamond A,\Sigma]]\}}{\Gamma\{[\Delta],\Diamond A,\Sigma\}}}{\Gamma\{[\Delta,\Diamond A],\Sigma\}}
\quad\rightsquigarrow\quad
\mathsf{b}^{[]}\dfrac{\mathsf{4}^{\Diamond}\dfrac{\Gamma\{[\Delta,[\Diamond A,\Sigma]]\}}{\Gamma\{[\Delta,\Diamond A,[\Sigma]]\}}}{\Gamma\{[\Delta,\Diamond A],\Sigma\}}
$$

$$
\mathsf{5}_1^{\Diamond}\dfrac{\mathsf{b}^{[]}\dfrac{\Gamma\{[\Delta],[\Sigma,[\Diamond A,\Theta]]\}}{\Gamma\{[\Delta],\Diamond A,[\Sigma],\Theta\}}}{\Gamma\{[\Delta,\Diamond A],[\Sigma],\Theta\}}
\quad\rightsquigarrow\quad
\mathsf{5}_2^{\Diamond}\dfrac{\mathsf{b}^{[]}\dfrac{\mathsf{4}^{\Diamond}\dfrac{\Gamma\{[\Delta],[\Sigma,[\Diamond A,\Theta]]\}}{\Gamma\{[\Delta],[\Sigma,\Diamond A,[\Theta]]\}}}{\Gamma\{[\Delta],[\Sigma,\Diamond A],\Theta\}}}{\Gamma\{[\Delta,\Diamond A],[\Sigma],\Theta\}}
$$

$$
\mathsf{5}_1^{\Diamond}\dfrac{\mathsf{b}^{[]}\dfrac{\Gamma\{[\Sigma,[\Theta,[\Delta]]],\Diamond A\}}{\Gamma\{[\Sigma],\Theta,[\Delta],\Diamond A\}}}{\Gamma\{[\Sigma],\Theta,[\Delta,\Diamond A]\}}
\quad\rightsquigarrow\quad
\mathsf{5}_2^{\Diamond}\dfrac{\mathsf{b}^{[]}\dfrac{\mathsf{5}_1^{\Diamond}\dfrac{\Gamma\{[\Sigma,[\Theta,[\Delta]]],\Diamond A\}}{\Gamma\{[\Sigma,[\Theta,[\Delta]],\Diamond A]\}}}{\Gamma\{[\Sigma,\Diamond A],\Theta,[\Delta]\}}}{\Gamma\{[\Sigma],\Theta,[\Delta,\Diamond A]\}}
$$

In the second and third case, we need Lemma 4.7. In the last case, we also apply the induction hypothesis.

- $\mathsf{r} = \mathsf{4}^{[]}$: There is one nontrivial case.

$$\mathsf{5}_1^{\Diamond}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\Gamma\{[\Delta],[\Diamond A,\Sigma]\}}{\Gamma\{[[\Delta],\Diamond A,\Sigma]\}}}{\Gamma\{[[\Delta,\Diamond A],\Sigma]\}} \quad\rightsquigarrow\quad \mathsf{4}^{[]}\,\dfrac{\mathsf{5}_2^{\Diamond}\,\dfrac{\Gamma\{[\Delta],[\Diamond A,\Sigma]\}}{\Gamma\{[\Delta,\Diamond A],[\Sigma]\}}}{\Gamma\{[[\Delta,\Diamond A],\Sigma]\}}$$

Then we can apply Lemma 4.7. □

We can now put Lemmas 4.3–4.8 together to prove our first main result:

Proof of Theorem 4.1. All rules in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$ are sound wrt. $\mathsf{K} + \mathsf{X}$. This has already been shown in [2,3] and can easily be verified. Thus, any formula that is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$ is also a theorem of $\mathsf{K} + \mathsf{X}$. Conversely, if A is a theorem of $\mathsf{K} + \mathsf{X}$, then by Corollary 2.9 we have a proof of A in $\mathsf{NK} \cup \hat{\mathsf{X}}^{\Diamond}$. If $\hat{\mathsf{X}} = \mathsf{X}$, then a proof in $\mathsf{NK} \cup \hat{\mathsf{X}}^{\Diamond}$ is trivially a proof in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$, and we are done. Otherwise, we must have one of the following three cases:

- If $\{\mathsf{t}, \mathsf{5}\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{\mathsf{4}\}$. Then, by Lemma 4.3, we can construct a proof of A in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$.
- If $\{\mathsf{b}, \mathsf{5}\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{\mathsf{4}\}$. We can use Lemma 4.4 similarly to get a proof of A in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$.
- If $\{\mathsf{b}, \mathsf{4}\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{\mathsf{5}\}$. We can replace the $\mathsf{5}^{\Diamond}$-rule with $\mathsf{5}_1^{\Diamond}, \mathsf{5}_2^{\Diamond}, \mathsf{5}_3^{\Diamond}$ using Lemma 4.5. Then we get a proof of Γ in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$ using Lemma 4.8, Lemma 4.7 and Lemma 4.6. □

5 Nested Sequents for Intuitionistic Modal Logics

Let us now turn to intuitionistic modal logics. The set of formulas is generated by

$$A, B, \ldots ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A$$

where p stands for a propositional variable. The constant $\top$ can be recovered via $\bot \supset \bot$. Since $\Box$ and $\Diamond$ are no longer De Morgan duals, it is not enough to just add the k-axiom (1) to intuitionistic propositional logic. In fact, there have been many different proposals of what should be added, e.g., [6,15,16,13,17,1,12]. Here, we consider the variant proposed in [16,13] and studied in detail by Simpson [17]. We add the following five axioms to intuitionistic propositional logic:

$$\begin{array}{ll} \mathsf{k}_1: \ \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: \ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: \ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: \ (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & \mathsf{k}_5: \ \Diamond\bot \supset \bot \end{array} \tag{5}$$

In a classical setting the axioms k_2–k_5 would follow from k_1 and the De Morgan laws. The theorems of the intuitionistic version of K, denoted by IK, are obtained from the axioms using the rules modus ponens and necessitation (2). As in the classical case, we write $\mathsf{IK} + \mathsf{X}$ for the logic obtained by adding a set of axioms $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, shown in Figure 1.

Let us now recall how nested sequents can be used to give deductive systems for all logics in the intituionistic modal S5-cube, as done in [18]. A similar data structure is used in [8]. The sequents are essentially the same as in the

$$
\bot^\bullet \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{c}\,\frac{\Gamma\{A^\bullet, A^\bullet\}}{\Gamma\{A^\bullet\}} \qquad \mathsf{id}\,\frac{}{\Gamma\{a^\bullet, a^\circ\}}
$$

$$
\wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}
$$

$$
\vee^\bullet \frac{\Gamma\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}
$$

$$
\supset^\bullet \frac{\Gamma^{\downarrow}\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}
$$

$$
\Box^\bullet \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \qquad \Box^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}
$$

$$
\Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond^\circ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}
$$

Fig. 6. System NIK

classical case, with the difference that formulas carry a polarity—there are two polarities, *input polarity* (marked with a black dot $\bullet$) and *output polarity* (marked with a white dot $\circ$)—such that exactly one formula in the whole sequent has the output polarity. More formally, a (*full*) *nested sequent* Γ for intuitionistic modal logic has two distinct parts: an *LHS-sequent* Λ in which all formulas have input polarity and an *RHS-sequent* Π which is either an output formula or a boxed sequent: given by:

$$
\Gamma ::= \Lambda, \Pi \qquad \Lambda ::= A_1^\bullet, ..., A_m^\bullet, [\Lambda_1], ..., [\Lambda_n] \qquad \Pi ::= A^\circ \mid [\Gamma]
$$

The *corresponding formula* of a sequent Γ is now defined as:

$$
\begin{aligned}
fm(\Lambda, \Pi) &= fm(\Lambda) \supset fm(\Pi) \\
fm(A_1^\bullet, ..., A_m^\bullet, [\Lambda_1], ..., [\Lambda_n]) &= A_1 \wedge ... \wedge A_m \wedge \Diamond fm(\Lambda_1) \wedge ... \wedge \Diamond fm(\Lambda_n) \\
fm(A^\circ) &= A \\
fm([\Gamma]) &= \Box fm(\Gamma)
\end{aligned}
$$

The notion of context is here again crucial. Since there are two different polarities, we also need two types of contexts: an *input context* (resp. an *output context*) is a sequent with one or several holes that should be filled with an input formula or an LHS-sequent (resp. an output formula, an RHS-sequent or a full sequent) to give a full sequent. The *depth* of a context is defined similarly to the classical case by induction.

As only one output formula is allowed in a sequent, we need, in some inference rules, to remove the output.

Definition 5.1 For an input context $\Gamma\{\ \}$ we obtain its *output pruning* $\Gamma^{\downarrow}\{\ \}$ by removing the unique output formula from it. For an output context $\Gamma\{\ \}$ we have $\Gamma^{\downarrow}\{\ \} = \Gamma\{\ \}$.

$$\mathsf{d}^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Diamond A^\circ\}} \quad \mathsf{t}^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}} \quad \mathsf{b}^\circ \frac{\Gamma\{[\Delta], A^\circ\}}{\Gamma\{[\Delta, \Diamond A^\circ]\}} \quad \mathsf{4}^\circ \frac{\Gamma\{[\Diamond A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} \quad \mathsf{5}^\circ \frac{\Gamma\{\emptyset\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}}$$

$$\mathsf{d}^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Box A^\bullet\}} \quad \mathsf{t}^\bullet \frac{\Gamma\{A^\bullet\}}{\Gamma\{\Box A^\bullet\}} \quad \mathsf{b}^\bullet \frac{\Gamma\{[\Delta], A^\bullet\}}{\Gamma\{[\Delta, \Box A^\bullet]\}} \quad \mathsf{4}^\bullet \frac{\Gamma\{[\Box A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \quad \mathsf{5}^\bullet \frac{\Gamma\{\emptyset\}\{\Box A^\bullet\}}{\Gamma\{\Box A^\bullet\}\{\emptyset\}}$$

Fig. 7. Intuitionistic $\Diamond^\circ$- and $\Box^\bullet$-rules; $\mathsf{5}^\circ$ and $\mathsf{5}^\bullet$ have proviso $\mathit{depth}(\Gamma\{\ \}\{\emptyset\}) \geq 1$.

Example 5.2 Let $\Gamma_1\{\ \} = A^\bullet, [B^\bullet, \{\ \}]$ and $\Gamma_2\{\ \} = A^\bullet, [B^\circ, \{\ \}]$. Then $\Gamma_1^\downarrow\{\ \} = A^\bullet, [B^\bullet, \{\ \}]$ and $\Gamma_2^\downarrow\{\ \} = A^\bullet, [\{\ \}]$.

The inference rules for intuitionistic modal logic are essentially the same as for classical modal logic. But since we are in an intuitionistic framework, each connective needs to be introduced by two rules, one for the input polarity and one for the output polarity, which doubles the number of rules. The system shown in Figure 6 is called *system* NIK.

Then, Figure 7 shows the rules for the axioms $\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}$. Again, because we are intuitionistic now, the number of rules is doubled. For $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, we write X° and $\mathsf{X}^\bullet$ for the corrseponding subset of $\{\mathsf{d}^\circ, \mathsf{t}^\circ, \mathsf{b}^\circ, \mathsf{4}^\circ, \mathsf{5}^\circ\}$ and $\{\mathsf{d}^\bullet, \mathsf{t}^\bullet, \mathsf{b}^\bullet, \mathsf{4}^\bullet, \mathsf{5}^\bullet\}$, respectively.

As in the classical case, we have the rules for weakening and cut:

$$\mathsf{w} \frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda\}} \qquad \mathsf{cut} \frac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}$$

Note that in the w-rule, the Λ has to be an LHS-sequent, i.e., must not contain the output formula. In the cut-rule we use the output pruning as for $\supset^\bullet$.

Remark 5.3 As in the classical case, the original formulation of NIK in [18] had no explicit contraction, but contraction was absorbed in into the rules of $\supset^\bullet$, $\Box^\bullet$, and the rules in $\mathsf{X}^\bullet$, instead. As before, we need explicit contraction here because of the structural rules. However, as in the classical case, both formulations are equivalent.

Remark 5.4 It is easy to see that we can use the two polarities $\circ$ and $\bullet$ to present a classical system in which negation $\neg$ is a primitive, as follows:

- allowing an arbitrary number of output-formulas in a sequent, and allow "contraction on the right", i.e., also for output formulas,
- add the two negation rules to NIK:

$$\neg^\bullet \frac{\Gamma\{A^\circ\}}{\Gamma\{\neg A^\bullet\}} \qquad \neg^\circ \frac{\Gamma\{A^\bullet\}}{\Gamma\{\neg A^\circ\}} \tag{6}$$

- and drop the output pruning from the left premiss in the $\supset^\bullet$- and cut-rules.

From this classical system, one could obtain an alternative intuitionistic system by allowing *at most one* output formula in the sequent and keeping the negation rules (6). However, we think that the systems presented here are simpler.

The notion of *45-closure* is also justified in the intuitionistic case:

Proposition 5.5 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *We have that* X *is 45-closed iff*

- *whenever* $\mathsf{4}$ *is derivable in* $\mathsf{IK} + \mathsf{X}$, *then* $\mathsf{4} \in \mathsf{X}$, *and*
- *whenever* $\mathsf{5}$ *is derivable in* $\mathsf{IK} + \mathsf{X}$, *then* $\mathsf{5} \in \mathsf{X}$.

The following has been shown in [18]:

Theorem 5.6 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *If a sequent* Γ *is derivable in* $\mathsf{NIK} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ \cup \{\mathsf{cut}\}$ *then it is also derivable in* $\begin{cases} \mathsf{NIK} \cup \hat{\mathsf{X}}^\bullet \cup \hat{\mathsf{X}}^\circ & \text{if } \mathsf{d} \notin \mathsf{X} \\ \mathsf{NIK} \cup \hat{\mathsf{X}}^\bullet \cup \hat{\mathsf{X}}^\circ \cup \{\mathsf{d}^{[]}\} & \text{if } \mathsf{d} \in \mathsf{X} \end{cases}$

Remark 5.7 In the statement of Theorem 5.6, we distinguish whether d is or is not present in X rather than make use of Lemma 6.3 (ii) of [18] because it actually remains unclear how to permute the rule $\mathsf{d}^{[]}$ over the rules $\mathsf{4}^\circ$ and $\mathsf{5}^\circ$, respectively, since the contraction-rule is not available for output formulas. Furthermore, with this formulation of Theorem 5.6, we do not need to extend the notion of 45-closure to t45-closure, as done in [18].

Corollary 5.8 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, *and let* $\mathsf{Z} = \mathsf{NIK} \cup \hat{\mathsf{X}}^\bullet \cup \hat{\mathsf{X}}^\circ$ *if* $\mathsf{d} \notin \mathsf{X}$, *and let* $\mathsf{Z} = \mathsf{NIK} \cup \hat{\mathsf{X}}^\bullet \cup \hat{\mathsf{X}}^\circ \cup \{\mathsf{d}^{[]}\}$ *if* $\mathsf{d} \in \mathsf{X}$. *Then a formula* A *is a theorem of* $\mathsf{IK} + \mathsf{X}$ *iff it is derivable in* Z.

6 Modularity for Intuitionistic Modal Logics

In this section, we prove a similar result as Theorem 4.1 for the intuitionistic setting. After our preparatory work of making the intuitionistic system look almost the same as the classical system, this work now becomes almost trivial. The key observation is that the structural rules in $\mathsf{X}^{[]}$ are also sound in the intuitionistic case, independently of the position of the output formula [18].

Theorem 6.1 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *A formula* A *is a theorem of* $\mathsf{IK} + \mathsf{X}$ *if and only if it is derivable in* $\mathsf{NIK} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ \cup \mathsf{X}^{[]}$.

Lemma 6.2 *For any* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *the* w*-rule is height-preserving and contraction-preserving admissible for* $\mathsf{NK} \cup \mathsf{X}^\circ \cup \mathsf{X}^\bullet \cup \mathsf{X}^{[]}$.

Proof. This is a straightforward induction on the height of the derivation. □

Lemma 6.3 *If* $\{\mathsf{t}, \mathsf{5}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *then the rules* $\mathsf{4}^\circ$ *and* $\mathsf{4}^\bullet$ *are admissible for* $\mathsf{NIK} \cup \mathsf{X}^\circ \cup \mathsf{X}^\bullet \cup \mathsf{X}^{[]}$.

Proof. This is similar to Lemma 4.3. Any occurrence of the $\mathsf{4}^\circ$-rule (respectively the $\mathsf{4}^\bullet$-rule) can be replaced by the derivation on the left (respectively on the right) below:

$$\mathsf{t}^{[]}\,\dfrac{\mathsf{5}^\circ\,\dfrac{\mathsf{w}\,\dfrac{\Gamma\{[\Diamond A^\circ, \Delta]\}}{\Gamma\{[\emptyset], [\Diamond A^\circ, \Delta]\}}}{\Gamma\{[\Diamond A^\circ], [\Delta]\}}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} \qquad \mathsf{t}^{[]}\,\dfrac{\mathsf{5}^\bullet\,\dfrac{\mathsf{w}\,\dfrac{\Gamma\{[\Box A^\bullet, \Delta]\}}{\Gamma\{[\emptyset], [\Box A^\bullet, \Delta]\}}}{\Gamma\{[\Box A^\bullet], [\Delta]\}}}{\Gamma\{\Box A^\bullet, [\Delta]\}}$$

We then apply Lemma 6.2. □

$$\mathsf{5}_1^\circ\ \frac{\Gamma\{[\Delta],\Diamond A^\circ\}}{\Gamma\{[\Delta,\Diamond A^\circ]\}}\qquad \mathsf{5}_2^\circ\ \frac{\Gamma\{[\Delta],[\Diamond A^\circ,\Sigma]\}}{\Gamma\{[\Delta,\Diamond A^\circ],[\Sigma]\}}\qquad \mathsf{5}_3^\circ\ \frac{\Gamma\{[\Delta,[\Diamond A^\circ,\Sigma]]\}}{\Gamma\{[\Delta,\Diamond A^\circ,[\Sigma]]\}}$$

$$\mathsf{5}_1^\bullet\ \frac{\Gamma\{[\Delta],\Box A^\bullet\}}{\Gamma\{[\Delta,\Box A^\bullet]\}}\qquad \mathsf{5}_2^\bullet\ \frac{\Gamma\{[\Delta],[\Box A^\bullet,\Sigma]\}}{\Gamma\{[\Delta,\Box A^\bullet],[\Sigma]\}}\qquad \mathsf{5}_3^\bullet\ \frac{\Gamma\{[\Delta,[\Box A^\bullet,\Sigma]]\}}{\Gamma\{[\Delta,\Box A^\bullet,[\Sigma]]\}}$$

Fig. 8. Variants of the rules $\mathsf{5}^\bullet$ and $\mathsf{5}^\circ$

Lemma 6.4 *If* $\{\mathsf{b},\mathsf{5}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *then the rules* $\mathsf{4}^\circ$ *and* $\mathsf{4}^\bullet$ *are admissible for* $\mathsf{NIK} \cup \mathsf{X}^\circ \cup \mathsf{X}^\bullet \cup \mathsf{X}^{[]}$.

Proof. For the $\mathsf{4}^\circ$-rule, the proof is the same as for Lemma 4.4, and for $\mathsf{4}^\bullet$-rule we use $\mathsf{5}^\bullet$ instead of $\mathsf{5}^\circ$. □

As in the classical case, to prove the admissibility of the rules $\mathsf{5}^\circ$ and $\mathsf{5}^\bullet$, we need again to decompose them into variants asking for unary context, shown in Figure 8. The rules $\mathsf{5}_1^\circ$, $\mathsf{5}_2^\circ$, $\mathsf{5}_3^\circ$, are special cases of $\mathsf{5}^\circ$, and the rules $\mathsf{5}_1^\bullet$, $\mathsf{5}_2^\bullet$, $\mathsf{5}_3^\bullet$, are special cases of $\mathsf{5}^\bullet$.

Lemma 6.5 *The* $\mathsf{5}^\circ$*-rule is derivable from* $\{\mathsf{5}_1^\circ,\mathsf{5}_2^\circ,\mathsf{5}_3^\circ\}$*, and the* $\mathsf{5}^\bullet$*-rule is derivable from* $\{\mathsf{5}_1^\bullet,\mathsf{5}_2^\bullet,\mathsf{5}_3^\bullet\}$. [18]

Lemma 6.6 *If* $\{\mathsf{4}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *then the rules* $\mathsf{5}_3^\circ$ *and* $\mathsf{5}_3^\bullet$ *are admissible for* $\mathsf{NIK} \cup \mathsf{X}^\circ \cup \mathsf{X}^\bullet \cup \mathsf{X}^{[]}$.

Proof. Any occurrence of the $\mathsf{5}_3^\circ$-rule (resp. $\mathsf{5}_3^\bullet$-rule) is an instance of the $\mathsf{4}^\circ$-rule (resp. $\mathsf{4}^\bullet$-rule). □

Lemma 6.7 *If* $\{\mathsf{b},\mathsf{4}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *then the rules* $\mathsf{5}_2^\circ$ *and* $\mathsf{5}_2^\bullet$ *are admissible for* $\mathsf{NIK} \cup \mathsf{X}^\circ \cup \mathsf{X}^\bullet \cup \mathsf{X}^{[]}$.

Proof. For the $\mathsf{5}_2^\circ$-rule this is similar to Lemma 4.7. For the $\mathsf{5}_2^\bullet$-rule, we use $\mathsf{4}^\bullet$ instead of $\mathsf{4}^\circ$. □

Lemma 6.8 *If* $\{\mathsf{b},\mathsf{4}\} \subseteq \mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *then the rules* $\mathsf{5}_1^\circ$ *and* $\mathsf{5}_1^\bullet$ *are admissible for* $\mathsf{NIK} \cup \mathsf{X}^\circ \cup \mathsf{X}^\bullet \cup \mathsf{X}^{[]}$.

Proof. For the $\mathsf{5}_1^\circ$-rule this is similar to Lemma 4.8. For the $\mathsf{5}_1^\bullet$-rule, we use the corresponding $\Box^\bullet$-rules instead of the $\Diamond^\circ$-rules. □

Proof of Theorem 6.1. All rules in $\mathsf{NIK} \cup \mathsf{X}^\bullet \cup \mathsf{X}^\circ \cup \mathsf{X}^{[]}$ are sound wrt. $\mathsf{IK}+\mathsf{X}$ (see [18] and Appendix A). Hence, the first direction is trivial. Conversely, if A is a theorem of $\mathsf{IK}+\mathsf{X}$, then by Corollary 5.8, it is derivable in $\mathsf{NIK} \cup \hat{\mathsf{X}}^\bullet \cup \hat{\mathsf{X}}^\circ \cup \mathsf{X}^{[]}$. If $\hat{\mathsf{X}} = \mathsf{X}$, we are done. Otherwise, we have the same three cases as in the proof of Theorem 4.1, and we use Lemmas 6.3–6.8 instead of Lemmas 4.3–4.8. □

7 Future Work

We have used in this paper a combination of logical and structural rules, but for some axioms only the structural or/and only the logical rules would be sufficent, depending on the system, i.e., depending on which other axioms are present. This is a rather strange observation, and in strong contrast to what happens with substructural logics.

In order to better understand this phenomenon, we need to find a general pattern for translating axioms into structural and/or logical rules. In particular, it is an important question for future research, for which type of axioms such a translation is possible. Given the nature of nested sequents, we conjecture that this is possible for all Scott-Lemmon axioms [11], which are of the shape

$$\Diamond^h \Box^i A \supset \Box^j \Diamond^k A$$

where $h, i, j, k \geq 0$. However, for obtaining a general result, it might first be necessary to collect more evidence, as we provide it in this paper.

Another direction of future research is to investigate constructive modal logics [1], which reject axioms k_3, k_4, and k_5, shown in (5). The challenge here lies in the fact that some of the structural rules, for example $\mathsf{4}^{[]}$ and $\mathsf{5}^{[]}$, and some of the logical rules, for example $\mathsf{b}^{\bullet}$ and $\mathsf{5}^{\bullet}$, are not sound anymore.

References

[1] Bierman, G. M. and V. de Paiva, *On an intuitionistic modal logic*, Studia Logica **65** (2000), pp. 383–416.

[2] Brünnler, K., *Deep sequent systems for modal logic*, Archive for Mathematical Logic **48** (2009), pp. 551–577.

[3] Brünnler, K. and L. Straßburger, *Modular sequent systems for modal logic*, in: M. Giese and A. Waaler, editors, *Automated Reasoning with Analytic Tableaux and Related Methods, TABLEAUX'09*, LNCS **5607** (2009), pp. 152–166.

[4] Ciabattoni, A., N. Galatos and K. Terui, *From axioms to analytic rules in nonclassical logics*, in: *LICS*, 2008, pp. 229–240.

[5] Ciabattoni, A. and R. Ramanayake, *Structural extensions of display calculi: A general recipe*, in: L. Libkin, U. Kohlenbach and R. J. G. B. de Queiroz, editors, *Logic, Language, Information, and Computation – WoLLIC 2013*, LNCS **8071** (2013), pp. 81–95.

[6] Fitch, F., *Intuitionistic modal logic with quantifiers*, Portugaliae Mathematica **7** (1948), pp. 113–118.

[7] Fitting, M., *Prefixed tableaus and nested sequents*, APAL **163** (2012), pp. 291–313.

[8] Galmiche, D. and Y. Salhi, *Label-free natural deduction systems for intuitionistic and classical modal logics*, Journal of Applied Non-Classical Logics **20** (2010), pp. 373–421.

[9] Goré, R., L. Postniece and A. Tiu, *On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics*, LMCS **7** (2011).

[10] Kashima, R., *Cut-free sequent calculi for some tense logics*, Studia Logica **53** (1994), pp. 119–136.

[11] Lemmon, E. J. and D. S. Scott, "An Introduction to Modal Logic," Blackwell, 1977.

[12] Pfenning, F. and R. Davies, *A judgmental reconstruction of modal logic*, Mathematical Structures in Computer Science **11** (2001), pp. 511–540.

[13] Plotkin, G. D. and C. P. Stirling, *A framework for intuitionistic modal logic*, in: J. Y. Halpern, editor, *Theoretical Aspects of Reasoning About Knowledge*, 1986.

[14] Poggiolesi, F., *The method of tree-hypersequents for modal propositional logic*, in: D. Makinson, J. Malinowski and H. Wansing, editors, *Towards Mathematical Philosophy*, Trends in Logic **28** (2009), pp. 31–51.

[15] Prawitz, D., "Natural Deduction, A Proof-Theoretical Study," Almq. and Wiksell, 1965.

[16] Servi, G. F., *Axiomatizations for some intuitionistic modal logics*, Rend. Sem. Mat. Univers. Politecn. Torino **42** (1984), pp. 179–194.

[17] Simpson, A., "The Proof Theory and Semantics of Intuitionistic Modal Logic," Ph.D. thesis, University of Edinburgh (1994).

[18] Straßburger, L., *Cut elimination in nested sequents for intuitionistic modal logics*, in: F. Pfenning, editor, *FoSSaCS'13*, LNCS **7794** (2013), pp. 209–224.

A Soundness of the Structural Rules

The soundness of the logical rules has been shown directly in [2] and [18]. The soundness of the structural rules follows only indirectly from these papers. For the convenience of the reader we give here a direct proof of the soundness of the structural rules in Figure 5 for the intuitionistic systems. Then their soundness in the classical systems follows immediately.

For simplicity, we show soundness with respect to the Hilbert system. Let us begin with two lemmas from [18], justifying the use of deep inference:

Lemma A.1 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, let* Δ *and* Σ *be full sequents, and let* $\Gamma\{\ \}$ *be an output context. If* $\mathit{fm}(\Delta) \supset \mathit{fm}(\Sigma)$ *is a theorem of* $\mathsf{IK} + \mathsf{X}$*, then so is* $\mathit{fm}(\Gamma\{\Delta\}) \supset \mathit{fm}(\Gamma\{\Sigma\})$*.*

Lemma A.2 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, let* Δ *and* Σ *be LHS-sequents, and let* $\Gamma\{\ \}$ *be an input context. If* $\mathit{fm}(\Sigma) \supset \mathit{fm}(\Delta)$ *is a theorem of* $\mathsf{IK} + \mathsf{X}$*, then so is* $\mathit{fm}(\Gamma\{\Delta\}) \supset \mathit{fm}(\Gamma\{\Sigma\})$*.*

Both lemmas are shown by an induction on the structure of $\Gamma\{\ \}$, using the following:

Lemma A.3 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*. For any formulas* A*,* B*, and* C *we have:*

(i) *If* $A \supset B$ *is a theorem of* $\mathsf{IK} + \mathsf{X}$*, then so is* $(C \supset A) \supset (C \supset B)$*.*

(ii) *If* $A \supset B$ *is a theorem of* $\mathsf{IK} + \mathsf{X}$*, then so is* $\Box A \supset \Box B$*.*

(iii) *If* $A \supset B$ *is a theorem of* $\mathsf{IK} + \mathsf{X}$*, then so is* $(C \wedge A) \supset (C \wedge B)$*.*

(iv) *If* $A \supset B$ *is a theorem of* $\mathsf{IK} + \mathsf{X}$*, then so is* $\Diamond A \supset \Diamond B$*.*

(v) *If* $A \supset B$ *is a theorem of* $\mathsf{IK} + \mathsf{X}$*, then so is* $(B \supset C) \supset (A \supset C)$*.*

Now, for showing soundness of a rule, we have to show that for every instance of the rule, if the premiss is a theorem of $\mathsf{IK}+\mathsf{X}$, then so is the conclusion. For this, we often use the following lemma:

Lemma A.4 *For all formulas* A, B*, the following are theorems of* IK*:*

(i) $\Diamond(A \wedge B) \supset \Diamond A \wedge \Diamond B$*,*

(ii) $\Diamond A \wedge \Box B \supset \Diamond(A \wedge B)$*, and*

(iii) $(\Box A \wedge \Box B) \supset \Box(A \wedge B)$*.*

The proofs of the Lemmas A.3 and A.4 are straightforward and left to the reader. We are now ready to see the main result of this appendix:

Proposition A.5 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *and* $\mathsf{x} \in \mathsf{X}$*. The corresponding structural rule* $\mathsf{x}^{[]}$ *shown on the left of Figure 5 is sound with respect to* $\mathsf{IK} + \mathsf{X}$*.*

Proof. For each $\mathsf{x} \in \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, let $\mathsf{x}^{[]}\,\dfrac{\Gamma_1}{\Gamma_2}$ denote the corresponding structural rule. We show that $\mathit{fm}(\Gamma_1) \supset \mathit{fm}(\Gamma_2)$ is a theorem of $\mathsf{IK} + \mathsf{x}$.

- $\mathsf{x} = \mathsf{d}$: We have that $\top$ is the unit for $\wedge$ (i.e., $\top = \bigwedge \emptyset$). Therefore, we have that $\mathit{fm}([\emptyset]) = \Diamond\top$ and $\mathit{fm}(\emptyset) = \top$ while $\top \supset \Diamond\top$ is a theorem of $\mathsf{IK}+\mathsf{d}$. Thus, by applying Lemma A.2, we get that $fm(\Gamma\{[\emptyset]\}) \supset fm(\Gamma\{\emptyset\})$ is a theorem of $\mathsf{IK}+\mathsf{d}$.

- $\mathsf{x}=\mathsf{t}$: We proceed by a case analysis on the position of the output formula in the sequent (see Figure 5).
  · If the output formula is in $\Gamma\{\ \}$, then $fm([\Delta]) = \Diamond fm(\Delta)$. Since $D \supset \Diamond D$ is a theorem of $\mathsf{IK}+\mathsf{t}$, so is $fm(\Gamma\{[\Delta]\}) \supset fm(\Gamma\{\Delta\})$ by Lemma A.2.
  · If the output formula is in Δ, then $fm([\Delta]) = \Box fm(\Delta)$. Since $\Box D \supset D$ is a theorem of $\mathsf{IK}+\mathsf{t}$, so is $fm(\Gamma\{[\Delta]\}) \supset fm(\Gamma\{\Delta\})$ by Lemma A.1.
- $\mathsf{x}=\mathsf{b}$: We proceed as in the previous case by a case analysis on the position of the output formula in the sequent (see Figure 5).
  · If the output formula is in $\Gamma\{\ \}$ we use the fact that $(\Diamond S \wedge D) \supset \Diamond(S \wedge \Diamond D)$ is a theorem of $\mathsf{IK}+\mathsf{b}$, together with Lemma A.2.
  · If the output formula is in Σ we use the fact that $\Box(\Diamond D \supset S) \supset (D \supset \Box S)$ is a theorem of $\mathsf{IK}+\mathsf{b}$, together with Lemma A.1.
  · If the output formula is in Δ we use the fact that $\Box(S \supset \Box D) \supset (\Diamond S \supset D)$ is a theorem of $\mathsf{IK}+\mathsf{b}$, together with Lemma A.1.

  The three formulas can be shown using the following three derivations, where each line stands for a valid implication in $\mathsf{IK}+\mathsf{b}$:

$$\dfrac{\dfrac{\Diamond S \wedge D}{\Diamond S \wedge \Box\Diamond D}\;\mathsf{b}+A.3.(\text{iii})}{\Diamond(S \wedge \Diamond D)}\;A.4.(\text{ii}) \qquad \dfrac{\dfrac{\Box(\Diamond D \supset S)}{\Box\Diamond D \supset \Box S}\;\mathsf{k}_1}{D \supset \Box S}\;\mathsf{b}+A.3.(\text{v}) \qquad \dfrac{\dfrac{\Box(S \supset \Box D)}{\Diamond S \supset \Diamond\Box D}\;\mathsf{k}_2}{\Diamond S \supset D}\;\mathsf{b}+A.3.(\text{i})$$

- $\mathsf{x}=\mathsf{4}$: We proceed by a case analysis on the position of the output formula in the sequent (see Figure 5).
  · If the output formula is in $\Gamma\{\ \}$ we use the fact that $\Diamond(\Diamond S \wedge D) \supset (\Diamond S \wedge \Diamond D)$ is a theorem of $\mathsf{IK}+\mathsf{4}$, together with Lemma A.2.
  · If the output formula is in Δ we use the fact that $(\Diamond S \supset \Box D) \supset \Box(S \supset \Box D)$ is a theorem of $\mathsf{IK}+\mathsf{4}$, together with Lemma A.1.
  · If the output formula is in Σ we use the fact that $(\Diamond D \supset \Box S) \supset \Box(\Diamond D \supset S)$ is a theorem of $\mathsf{IK}+\mathsf{4}$, together with Lemma A.1.

  As before, we can show the three formulas by simple derivations:

$$\dfrac{\dfrac{\Diamond(\Diamond S \wedge D)}{\Diamond\Diamond S \wedge \Diamond D}\;A.4.(\text{i})}{\Diamond S \wedge \Diamond D}\;\mathsf{4}+A.3.(\text{iii}) \qquad \dfrac{\dfrac{\Diamond S \supset \Box D}{\Diamond S \supset \Box\Box D}\;\mathsf{4}+A.3.(\text{i})}{\Box(S \supset \Box D)}\;\mathsf{k}_4 \qquad \dfrac{\dfrac{\Diamond D \supset \Box S}{\Diamond\Diamond D \supset \Box S}\;\mathsf{4}+A.3.(\text{v})}{\Box(\Diamond D \supset S)}\;\mathsf{k}_4$$

- $\mathsf{x}=\mathsf{5}$: For showing soundness of $\mathsf{5}^{[]}$, we observe that it is derivable using the following three rules and show soundness for each of them individually:

$$\mathsf{5}^{[]}_1\,\dfrac{\Gamma\{[\Theta,[\Delta]]\}}{\Gamma\{[\Theta],[\Delta]\}} \qquad \mathsf{5}^{[]}_2\,\dfrac{\Gamma\{[\Theta,[\Delta]],[\Sigma]\}}{\Gamma\{[\Theta],[[\Delta],\Sigma]\}} \qquad \mathsf{5}^{[]}_3\,\dfrac{\Gamma\{[\Theta,[\Delta],[\Sigma]]\}}{\Gamma\{[\Theta,[[\Delta],\Sigma]]\}}$$

  For each of $\mathsf{5}^{[]}_1$, $\mathsf{5}^{[]}_2$, and $\mathsf{5}^{[]}_3$, we proceed by a case analysis on the position of the output formula in the sequent. The cases for $\mathsf{5}^{[]}_1$ are the following:

- If the output formula is in $\Gamma\{\ \}$ we use Lemma A.2, together with the fact that $(\Diamond T \wedge \Diamond D) \supset \Diamond(T \wedge \Diamond D)$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Θ we use Lemma A.1, together with the fact that $\Box(\Diamond D \supset T) \supset (\Diamond D \supset \Box T)$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Δ we use Lemma A.1, together with the fact that $\Box(T \supset \Box D) \supset (\Diamond T \supset \Diamond D)$ is a theorem of $\mathsf{IK}+\mathsf{5}$.

The following derivations show that the three formulas are theorems of $\mathsf{IK}+\mathsf{5}$:

$$
\dfrac{\dfrac{\Diamond T \wedge \Diamond D}{\Diamond T \wedge \Box\Diamond D}\ \mathsf{5}+A.3.(\mathrm{iii})}{\Diamond(T \wedge \Diamond D)}\ A.4.(\mathrm{ii})
\qquad
\dfrac{\dfrac{\Box(\Diamond D \supset T)}{\Box\Diamond D \supset \Box T}\ \mathsf{k}_1}{\Diamond D \supset \Box T}\ \mathsf{5}+A.3.(\mathrm{v})
\qquad
\dfrac{\dfrac{\Box(T \supset \Box D)}{\Diamond T \supset \Diamond\Box D}\ \mathsf{k}_2}{\Diamond T \supset \Box D}\ \mathsf{5}+A.3.(\mathrm{i})
$$

Let us now consider the cases for $\mathsf{5}_2^{[]}$:

- If the output formula is in $\Gamma\{\ \}$ we use Lemma A.2, together with the fact that $(\Diamond T \wedge \Diamond(\Diamond D \wedge S)) \supset (\Diamond(T \wedge \Diamond D) \wedge \Diamond S)$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Δ we use Lemma A.1 together with the fact that $(\Diamond S \supset \Box(T \supset \Box D)) \supset (\Diamond T \supset \Box(S \supset \Box D))$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Σ we use Lemma A.1 together with the fact that $(\Diamond(T \wedge \Diamond D) \supset \Box S) \supset (\Diamond T \supset \Box(\Diamond D \supset S))$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Θ we use Lemma A.1 together with the fact that $(\Diamond S \supset \Box(\Diamond D \supset T)) \supset (\Diamond(S \wedge \Diamond D) \supset \Box T)$ is a theorem of $\mathsf{IK}+\mathsf{5}$.

These formulas are shown by the following derivations:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Diamond T \wedge \Diamond(\Diamond D \wedge S)}{\Diamond T \wedge \Diamond\Diamond D \wedge \Diamond S}\ A.4.(\mathrm{i})+A.3.(\mathrm{iii})}{\Diamond T \wedge \Diamond\Box\Diamond D \wedge \Diamond S}\ \mathsf{5}+A.3.(\mathrm{iii,iv})}{\Diamond T \wedge \Box\Diamond D \wedge \Diamond S}\ \mathsf{5}+A.3.(\mathrm{iii})}{\Diamond(T \wedge \Diamond D) \wedge \Diamond S}\ A.4.(\mathrm{ii})+A.3.(\mathrm{iii})
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Diamond S \supset \Box(T \supset \Box D)}{\Diamond S \supset \Diamond T \supset \Diamond\Box D}\ \mathsf{k}_2+A.3.(\mathrm{i})}{\Diamond S \supset \Diamond T \supset \Box\Diamond\Box D}\ \mathsf{5}+A.3.(\mathrm{i})}{\Diamond S \supset \Diamond T \supset \Box\Box D}\ \mathsf{5}+A.3.(\mathrm{i,ii})}{\Diamond T \supset \Diamond S \supset \Box\Box D}}{\Diamond T \supset \Box(S \supset \Box D)}\ \mathsf{k}_4+A.3.(\mathrm{i})
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Diamond(T \wedge \Diamond D) \supset \Box S}{(\Diamond T \wedge \Box\Diamond D) \supset \Box S}\ A.4.(\mathrm{ii})+A.3.(\mathrm{v})}{(\Diamond T \wedge \Diamond\Box\Diamond D) \supset \Box S}\ \mathsf{5}+A.3.(\mathrm{iii,v})}{(\Diamond T \wedge \Diamond\Diamond D) \supset \Box S}\ \mathsf{5}+A.3.(\mathrm{iii,iv,v})}{\Diamond T \supset \Diamond\Diamond D \supset \Box B}}{\Diamond T \supset \Box(\Diamond D \supset S)}\ \mathsf{k}_4+A.3.(\mathrm{i})
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Diamond S \supset \Box(\Diamond D \supset T)}{\Diamond S \supset \Box\Diamond D \supset \Box T}\ \mathsf{k}_1+A.3.(\mathrm{i})}{(\Diamond S \wedge \Box\Diamond D) \supset \Box T}}{(\Diamond S \wedge \Diamond\Box\Diamond D) \supset \Box T}\ \mathsf{5}+A.3.(\mathrm{iii,v})}{(\Diamond S \wedge \Diamond\Diamond D) \supset \Box T}\ \mathsf{5}+A.3.(\mathrm{iii,iv,v})}{\Diamond(S \wedge \Diamond D) \supset \Box T}\ A.4.(\mathrm{i})+A.3.(\mathrm{v})
$$

Finally, let us consider the cases for $\mathsf{5}_3^{[]}$:

- If the output formula is in $\Gamma\{\ \}$, we use Lemma A.2, together with the fact that $\Diamond(T \wedge \Diamond(\Diamond D \wedge S)) \supset \Diamond(T \wedge \Diamond D \wedge \Diamond S)$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Θ, we use Lemma A.1, together with the fact that $\Box((\Diamond D \wedge \Diamond S) \supset T) \supset \Box(\Diamond(\Diamond D \wedge S) \supset T)$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Δ, we use Lemma A.1, together with the fact that $\Box((T \wedge \Diamond S) \supset \Box D) \supset \Box(T \supset \Box(S \supset \Box D))$ is a theorem of $\mathsf{IK}+\mathsf{5}$.
- If the output formula is in Σ, we use Lemma A.1, together with the fact that $\Box((T \wedge \Diamond D) \supset \Box S) \supset \Box(T \supset \Box(\Diamond D \supset S))$ is a theorem of $\mathsf{IK}+\mathsf{5}$.

Below are the derivations showing that these formulas are indeed theorems of $\mathsf{IK} + \mathsf{5}$:

$$
\begin{array}{cl}
\Diamond(T \wedge \Diamond(\Diamond D \wedge S)) & \\ \hline
\Diamond T \wedge \Diamond\Diamond(\Diamond D \wedge S) & A.4.(\mathrm{i}) \\ \hline
\Diamond T \wedge \Diamond(\Diamond\Diamond D \wedge \Diamond S) & A.4.(\mathrm{i}) + A.3.(\mathrm{iii}) \\ \hline
\Diamond T \wedge \Diamond(\Diamond\Box\Diamond D \wedge \Diamond S) & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{iv}) \\ \hline
\Diamond T \wedge \Diamond(\Box\Diamond D \wedge \Diamond S) & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{iv}) \\ \hline
\Diamond T \wedge \Diamond\Box\Diamond D \wedge \Diamond\Diamond S & A.4.(\mathrm{i}) + A.3.(\mathrm{iii}) \\ \hline
\Diamond T \wedge \Diamond\Box\Diamond D \wedge \Diamond\Box\Diamond S & \mathsf{5} + A.3.(\mathrm{iii}) \\ \hline
\Diamond T \wedge \Box\Diamond D \wedge \Box\Diamond S & \mathsf{5} + A.3.(\mathrm{iii}) \\ \hline
\Diamond(T \wedge \Diamond D) \wedge \Box\Diamond S & A.4.(\mathrm{ii}) + A.3.(\mathrm{iii}) \\ \hline
\Diamond(T \wedge \Diamond D \wedge \Diamond S) & A.4.(\mathrm{ii})
\end{array}
\qquad
\begin{array}{cl}
\Box((\Diamond D \wedge \Diamond S) \supset T) & \\ \hline
\Box(\Diamond D \wedge \Diamond S) \supset \Box T & \mathsf{k}_1 \\ \hline
(\Box\Diamond D \wedge \Box\Diamond S) \supset \Box T & A.4.(\mathrm{iii}) + A.3.(\mathrm{v}) \\ \hline
(\Diamond\Box\Diamond D \wedge \Diamond\Box\Diamond S) \supset \Box T & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{v}) \\ \hline
(\Diamond\Box\Diamond D \wedge \Diamond\Diamond S) \supset \Box T & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{v}) \\ \hline
(\Diamond\Diamond\Box\Diamond D \wedge \Diamond\Diamond S) \supset \Box T & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{v}) \\ \hline
(\Diamond\Diamond\Diamond D \wedge \Diamond\Diamond S) \supset \Box T & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{v}) \\ \hline
\Diamond(\Diamond\Diamond D \wedge \Diamond S) \supset \Box T & A.4.(\mathrm{i}) + A.3.(\mathrm{v}) \\ \hline
\Diamond\Diamond(\Diamond D \wedge S) \supset \Box T & A.4.(\mathrm{i}) + A.3.(\mathrm{v}) \\ \hline
\Box(\Diamond(\Diamond D \wedge S) \supset T) & \mathsf{k}_4
\end{array}
$$

$$
\begin{array}{cl}
\Box((T \wedge \Diamond S) \supset \Box D) & \\ \hline
\Diamond(T \wedge \Diamond S) \supset \Diamond\Box D & \mathsf{k}_2 \\ \hline
(\Diamond T \wedge \Box\Diamond S) \supset \Diamond\Box D & A.4.(\mathrm{ii}) + A.3.(\mathrm{v}) \\ \hline
(\Diamond T \wedge \Diamond\Box\Diamond S) \supset \Diamond\Box D & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{v}) \\ \hline
(\Diamond T \wedge \Diamond\Diamond S) \supset \Diamond\Box D & \mathsf{5} + A.3.(\mathrm{iv}, \mathrm{iii}, \mathrm{v}) \\ \hline
(\Diamond T \wedge \Diamond\Diamond S) \supset \Box\Diamond\Box D & \mathsf{5} + A.3.(\mathrm{i}) \\ \hline
(\Diamond T \wedge \Diamond\Diamond S) \supset \Box\Box\Diamond\Box D & \mathsf{5} + A.3.(\mathrm{i}, \mathrm{ii}) \\ \hline
(\Diamond T \wedge \Diamond\Diamond S) \supset \Box\Box\Box D & \mathsf{5} + A.3.(\mathrm{i}, \mathrm{ii}) \\ \hline
\Diamond T \supset \Diamond\Diamond S \supset \Box\Box\Box D & \\ \hline
\Diamond T \supset \Box(\Diamond S \supset \Box\Box D) & \mathsf{k}_4 + A.3.(\mathrm{i}) \\ \hline
\Diamond T \supset \Box\Box(S \supset \Box D) & \mathsf{k}_4 + A.3.(\mathrm{i}, \mathrm{ii}) \\ \hline
\Box(T \supset \Box(S \supset \Box D)) & \mathsf{k}_4
\end{array}
\qquad
\begin{array}{cl}
\Box((T \wedge \Diamond D) \supset \Box S) & \\ \hline
\Diamond(T \wedge \Diamond D) \supset \Diamond\Box S & \mathsf{k}_2 \\ \hline
(\Diamond T \wedge \Box\Diamond D) \supset \Diamond\Box S & A.4.(\mathrm{ii}) + A.3.(\mathrm{v}) \\ \hline
(\Diamond T \wedge \Box\Diamond D) \supset \Box\Diamond\Box S & \mathsf{5} + A.3.(\mathrm{i}) \\ \hline
(\Diamond T \wedge \Box\Diamond D) \supset \Box\Box S & \mathsf{5} + A.3.(\mathrm{ii}, \mathrm{i}) \\ \hline
(\Diamond T \wedge \Diamond\Box\Diamond D) \supset \Box\Box S & \mathsf{5} + A.3.(\mathrm{iii}, \mathrm{v}) \\ \hline
(\Diamond T \wedge \Diamond\Diamond\Box\Diamond D) \supset \Box\Box S & \mathsf{5} + A.3.(\mathrm{iv}, \mathrm{iii}, \mathrm{v}) \\ \hline
(\Diamond T \wedge \Diamond\Diamond\Diamond D) \supset \Box\Box S & \mathsf{5} + A.3.(\mathrm{iv}, \mathrm{iii}, \mathrm{v}) \\ \hline
\Diamond T \supset \Diamond\Diamond\Diamond D \supset \Box\Box S & \\ \hline
\Diamond T \supset \Box(\Diamond\Diamond D \supset \Box S) & \mathsf{k}_4 + A.3.(\mathrm{i}) \\ \hline
\Diamond T \supset \Box\Box(\Diamond D \supset S) & \mathsf{k}_4 + A.3.(\mathrm{i}, \mathrm{ii}) \\ \hline
\Box(T \supset \Box(\Diamond D \supset S)) & \mathsf{k}_4
\end{array}
$$

□

B Addendum to Section 3

In this appendix we use a concrete example to explain the error in [3]. The example is due to an anonymous reviewer who first observed the problem. Let us consider the formula $\Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)$, which is a theorem of $\mathsf{K4}$ (it is derivable in $\mathsf{NK} \cup \{\mathsf{4}^{\Diamond}\}$, as the reader can easily verify).

Let us now argue why this formula is not derivable in $\mathsf{NK} \cup \{\mathsf{4}^{[]}, \mathsf{m}^{[]}\}$. For this, observe that the $\mathsf{m}^{[]}$-rule becomes admissible if we replace the c-rule by

$$\hat{\mathsf{c}}\ \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}$$

which allows contraction on arbitrary sequents, and which is derivable for $\{\mathsf{c}, \mathsf{m}^{[]}\}$. Additionally, observe that the rules for $\wedge$, $\vee$, and $\Box$ are invertible and can therefore be applied eagerly. We can also apply the $\Diamond$-rule and $\mathsf{4}^{[]}$-rule eagerly, if we first apply the $\hat{\mathsf{c}}$-rule on the formula/subsequent that is moved by the $\Diamond/\mathsf{4}^{[]}$-rule. It can also be shown that there is no other need fo the $\hat{\mathsf{c}}$-rule (see [2] for a proof of admissibility of $\hat{\mathsf{c}}$ for such a system). This means we can do an exhaustive proof search without the need of backtracking. The following derivation shows our attempt to prove $\Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)$ in $\mathsf{NK} \setminus \{\mathsf{c}\} \cup \{\hat{\mathsf{c}}, \mathsf{4}^{[]}\}$:

$$
\begin{prooftree}
\AxiomC{$\Diamond\Box q, [q], [[q], q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, 4^{[]}$}
\UnaryInfC{$\Diamond\Box q, [[q], q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Diamond\Box q, [\Box q, q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, \Diamond$}
\UnaryInfC{$\Diamond\Box q, [q, p], [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, 4^{[]}$}
\UnaryInfC{$\Diamond\Box q, [[q, p], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, \Diamond$}
\UnaryInfC{$\Diamond\Box q, [[q], q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Diamond\Box q, [\Box q, q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, \Diamond$}
\UnaryInfC{$\Diamond\Box q, [q, \bar{p}, \Diamond p], [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, 4^{[]}$}
\UnaryInfC{$\Diamond\Box q, [[q, \bar{p}, \Diamond p], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, \Diamond, \Diamond$}
\UnaryInfC{$\Diamond\Box q, [[q], \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Diamond\Box q, [\Box q, \Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\hat{\mathsf{c}}, \Diamond$}
\UnaryInfC{$\Diamond\Box q, [\Diamond\bar{p}, \Diamond\Diamond p]$}
\LeftLabel{$\vee, \Box, \vee$}
\UnaryInfC{$\Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)$}
\end{prooftree}
$$

Now we are essentially stuck. If we proceed, we can only obtain copies of existing sequent nodes, and none of them contains both p and $\bar{p}$. Thus, we will never be able to conclude with the id-rule.

However, if we allow the cut-rule, we can prove our formula as follows:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$\Diamond\Box q, [[q], [q, p, \bar{p}]]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Diamond\Box q, [[q], [q, p], \Diamond\bar{p}]$}
\LeftLabel{$4^{[]}$}
\UnaryInfC{$\Diamond\Box q, [[[q, p], q], \Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Diamond\Box q, [[[q], q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Diamond\Box q, [[\Box q, q, \Diamond p], \Diamond\bar{p}]$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$[[q, \bar{q}], q, \Diamond p], [\Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[[q], \Diamond\bar{q}, q, \Diamond p], [\Diamond\bar{p}]$}
\LeftLabel{$\Box$}
\UnaryInfC{$[\Box q, \Diamond\bar{q}, q, \Diamond p], [\Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Diamond\Box q, [\Diamond\bar{q}, q, \Diamond p], [\Diamond\bar{p}]$}
\LeftLabel{$4^{[]}$}
\UnaryInfC{$\Diamond\Box q, [[\Diamond\bar{q}, q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{cut}
\BinaryInfC{$\Diamond\Box q, [[q, \Diamond p], \Diamond\bar{p}]$}
\doubleLine
\RightLabel{NK}
\UnaryInfC{$\Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)$}
\end{prooftree}
\tag{B.1}
$$

Then, [3, Lemma 8] suggests to transform the branch on the right in (B.1) as shown on the left below:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$\Diamond\Box q, [[q], [q, p, \bar{p}]]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Diamond\Box q, [[q], [q, p], \Diamond\bar{p}]$}
\LeftLabel{$4^{[]}$}
\UnaryInfC{$\Diamond\Box q, [[[q, p], q], \Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$\Diamond\Box q, [[[q], q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{$\Box$}
\UnaryInfC{$\Diamond\Box q, [[\Box q, q, \Diamond p], \Diamond\bar{p}]$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$[[q, \bar{q}], q, \Diamond p], [\Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[[q], \Diamond\bar{q}, q, \Diamond p], [\Diamond\bar{p}]$}
\LeftLabel{$\Box$}
\UnaryInfC{$[\Box q, \Diamond\bar{q}, q, \Diamond p], [\Diamond\bar{p}]$}
\LeftLabel{$4^{[]}$}
\UnaryInfC{$[[\Box q, \Diamond\bar{q}, q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[\Diamond\Box q, [\Diamond\bar{q}, q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{$4^{\Diamond}$}
\UnaryInfC{$\Diamond\Box q, [[\Diamond\bar{q}, q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{cut}
\BinaryInfC{$\Diamond\Box q, [[q, \Diamond p], \Diamond\bar{p}]$}
\doubleLine
\RightLabel{NK}
\UnaryInfC{$\Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)$}
\end{prooftree}
\quad \rightsquigarrow \quad
\begin{prooftree}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$[[q], [q, p, \bar{p}]]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[[q], [q, p], \Diamond\bar{p}]$}
\LeftLabel{$4^{[]}$}
\UnaryInfC{$[[[q, p], q], \Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[[[q], q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{$\Box$}
\UnaryInfC{$[[\Box q, q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{$\Diamond$}
\UnaryInfC{$[\Diamond\Box q, [q, \Diamond p], \Diamond\bar{p}]$}
\LeftLabel{$4^{\Diamond}$}
\UnaryInfC{$\Diamond\Box q, [[q, \Diamond p], \Diamond\bar{p}]$}
\doubleLine
\RightLabel{NK}
\UnaryInfC{$\Diamond\Box q \vee \Box(\Diamond\bar{p} \vee \Diamond\Diamond p)$}
\end{prooftree}
$$

However the $4^{\Diamond}$-rule does not apply to the cut-formula so it must be moved under the cut-rule as in a commutative case, and thus, Lemma 9 of [3] cannot be applied. This means that the proof of the reduction lemma (Lemma 10) in [3] is incorrect, and the instance of $4^{\Diamond}$ remains in the derivation after cut elimination, as shown on the right above.